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ON THE SEMISIMPLICITY OF THE CYCLOTOMIC QUIVER HECKE ALGEBRA OF TYPE C

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ABSTRACT. We provide criteria for the cyclotomic quiver Hecke algebras of type C to be semisimple. In the semisimple case, we construct the irreducible modules.

1. INTRODUCTION

The *quiver Hecke algebras* $\mathscr{R}_n$ were introduced by Khovanov and Lauda [KL09] and Rouquier [Rou08] to categorify the negative half of quantum groups. Kang and Kashiwara [KK12] later showed that cyclotomic quotients $\mathscr{R}_n^\Lambda$ of $\mathscr{R}_n$ categorify irreducible highest weight modules with dominant integral highest weight Λ. Motivated and propelled by an isomorphism theorem of Brundan and Kleshchev [BK09], these cyclotomic quotients have received a lot of attention in types A_∞ and $A_\ell^{(1)}$. However, in other types relatively little is known about the cyclotomic quiver Hecke algebras. Among the few results here are Ariki and Park's results on the representation type of their blocks when $\Lambda = \Lambda_0$ [AP14, AP16b, AP16a].

One of the first questions one should ask when studying a finite-dimensional algebra is whether or not it is semisimple. In this short note, we will prove semisimplicity criteria for the cyclotomic quiver Hecke algebras $\mathscr{R}_n^\Lambda$ in type C, over a field, building on previous work [APS17], in which we developed a Specht module theory in types C_∞ and $C_\ell^{(1)}$. Our result is a fundamental step in gaining a better understanding of these algebras.

Now we state our main result – see Section 2 for definitions of the notation used.

Theorem 1.1 (Main Theorem)**.** *Over a field, $\mathscr{R}_n^\Lambda$ is semisimple if and only if the following two conditions are satisfied:*

(i) *For all $i \in I$, $\langle \Lambda, \alpha_{i,n}^\vee \rangle \leqslant 1$.*
(ii) *For all $1 \leqslant j \leqslant l$, $\frac{n-1}{2} \leqslant \overline{\kappa_j} \leqslant \ell - \frac{n-1}{2}$.*

Our proof that $\mathscr{R}_n^\Lambda$ is semisimple when the above two conditions hold is inspired by an argument from Mathas's survey [Mat15] in type A. In the other direction, when the conditions fail, we explicitly construct modules that have one-dimensional submodules, which we show have no complement, thus concluding that $\mathscr{R}_n^\Lambda$ is not semisimple. In most cases, the modules we construct are in fact Specht modules, and our previous work with Ariki and Park [APS17] is crucial to our proof.

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2. Background

We begin by providing a brief summary of the necessary definitions. Throughout, we let $\mathcal{O}$ denote an arbitrary integral domain. All our modules are left modules.

2.1. **The quiver Hecke algebras.** Let $\ell \in \{2,3,\dots\} \cup \{\infty\}$, and set $I := \mathbb{Z}/(\ell+1)\mathbb{Z}$ if $\ell < \infty$ or $I = \mathbb{Z}_{\geqslant 0}$ if $\ell = \infty$. If $\ell < \infty$, we identify I with the set $\{0,1,2,\dots,\ell\}$. We adopt standard notation from [Kac90] for the root datum of type $C_\ell^{(1)}$ or C_∞. In particular, we have *simple roots* $\{\alpha_i \mid i \in I\}$, *simple coroots* $\{\alpha_i^\vee \mid i \in I\}$, and we have *fundamental weights* $\{\Lambda_i \mid i \in I\}$ in the *weight lattice* P. We let $\mathsf{Q}^+ := \bigoplus_{i\in I} \mathbb{Z}_{\geqslant 0}\alpha_i$ be the *positive cone of the root lattice* and $\mathsf{P}^+ := \{\Lambda \in \mathsf{P} \mid \langle \Lambda, \alpha_i^\vee\rangle \geqslant 0 \text{ for all } i \in I\}$ the *positive weight lattice*, where $\langle -,-\rangle$ is the natural pairing $\langle \Lambda_i, \alpha_j^\vee\rangle = \delta_{i,j}$. We say that $\beta = \sum_{i\in I} a_i\alpha_i \in \mathsf{Q}^+$ has *height* $\mathrm{ht}(\beta) = \sum_{i\in I} a_i$, and $\Lambda = \sum_{i\in I} b_i\Lambda_i \in \mathsf{P}^+$ has *level* $\sum_{i\in I} b_i$. Set $\mathsf{Q}_n^+ := \{\beta \in \mathsf{Q}^+ \mid \mathrm{ht}(\beta) = n\}$.

For any $\beta \in \mathsf{Q}_n^+$, we set $I^\beta = \{\mathbf{i} \in I^n \mid \alpha_{i_1} + \dots + \alpha_{i_n} = \beta\}$. The symmetric group $\mathfrak{S}_n$ acts on elements of I^n by place permutation.

The *quiver Hecke algebra* $\mathscr{R}_\beta$ is the unital associative $\mathcal{O}$-algebra with generators

$$\{e(\mathbf{i}) \mid \mathbf{i} \in I^\beta\} \cup \{x_1,\dots,x_n\} \cup \{\psi_1,\dots,\psi_{n-1}\},$$

subject to the following relations:

$$e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{i},\mathbf{j}}e(\mathbf{i}); \qquad x_r e(\mathbf{i}) = e(\mathbf{i})x_r;$$

$$\sum_{\mathbf{i}\in I^\beta} e(\mathbf{i}) = 1; \qquad x_r x_s = x_s x_r;$$

$$\psi_r e(\mathbf{i}) = e(s_r\mathbf{i})\psi_r; \qquad \psi_r x_s = x_s\psi_r \quad \text{if } s \neq r, r+1;$$

$$x_r\psi_r e(\mathbf{i}) = (\psi_r x_{r+1} - \delta_{i_r,i_{r+1}})e(\mathbf{i}); \qquad \psi_r\psi_s = \psi_s\psi_r \quad \text{if } |r-s| > 1;$$

$$x_{r+1}\psi_r e(i) = (\psi_r x_r + \delta_{i_r,i_{r+1}})e(\mathbf{i});$$

$$\psi_r^2 e(\mathbf{i}) = \begin{cases} (x_r + x_{r+1}^2)e(\mathbf{i}) & \text{if } (i_r,i_{r+1}) = (0,1) \text{ or if } (\ell,\ell-1); \\ (x_r^2 + x_{r+1})e(\mathbf{i}) & \text{if } (i_r,i_{r+1}) = (1,0) \text{ or if } (\ell-1,\ell); \\ (x_r + x_{r+1})e(\mathbf{i}) & \text{if } i_{r+1} = i_r \pm 1,\ i_r \neq 0 \neq i_{r+1},\ i_r \neq \ell \neq i_{r+1}; \\ 0 & \text{if } i_r = i_{r+1}; \\ e(\mathbf{i}) & \text{otherwise.} \end{cases}$$

$$\psi_{r+1}\psi_r\psi_{r+1}e(\mathbf{i}) = \begin{cases} (\psi_r\psi_{r+1}\psi_r + x_r + x_{r+2})e(\mathbf{i}) & \text{if } (i_r,i_{r+2},i_{r+1}) = (1,0,1) \\ & \qquad \text{or } (\ell-1,\ell,\ell-1); \\ (\psi_r\psi_{r+1}\psi_r + 1)e(\mathbf{i}) & \text{if } i_r = i_{r+2} = i_{r+1} \pm 1, \\ & \qquad \text{and } i_{r+1} \neq 0,\ell; \\ \psi_r\psi_{r+1}\psi_r e(\mathbf{i}) & \text{otherwise.} \end{cases}$$

The quiver Hecke algebra $\mathscr{R}_n$ is defined to be $\bigoplus_{\beta \in \mathsf{Q}_n^+} \mathscr{R}_\beta$. These algebras have cyclotomic quotients, which are our primary interest here. The *cyclotomic quiver Hecke algebra* $\mathscr{R}_\beta^\Lambda$ is the quotient of $\mathscr{R}_\beta$ by the additional *cyclotomic relations*

$$x_1^{\langle \Lambda, \alpha_{i_1}^\vee \rangle} e(\mathbf{i}) = 0 \text{ for all } \mathbf{i} \in I^\beta.$$

The *cyclotomic quiver Hecke algebra* $\mathscr{R}_n^\Lambda$ is defined to be $\bigoplus_{\beta \in \mathsf{Q}_n^+} \mathscr{R}_\beta^\Lambda$.

The quiver Hecke algebras and their cyclotomic quotients may be ($\mathbb{Z}$-)graded by

$$\deg e(\mathbf{i}) = 0, \qquad \deg x_r e(\mathbf{i}) = (\alpha_{i_r}, \alpha_{i_r}), \qquad \deg \psi_r e(\mathbf{i}) = (\alpha_{i_r}, \alpha_{i_{r+1}}),$$

where $(-,-)$ is the invariant symmetric bilinear form on P.

Remark. Technically, we have made a choice of certain polynomials in our definition of the quiver Hecke algebras. See [APS17, §2.1–2.2] for discussion of these polynomials and the choice we have made.

2.2. **Multipartitions and tableaux.** A *partition* λ of n is a weakly decreasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $\sum \lambda_i = n$. We write $\varnothing$ for the unique partition of 0. For any $l \geqslant 1$, an *l-multipartition* of n is an l-tuple $\lambda = (\lambda^{(1)}, \dots, \lambda^{(l)})$. We denote the set of all l-multipartitions of n by $\mathscr{P}_n^l$. For $\lambda, \mu \in \mathscr{P}_n^l$, we say that λ *dominates* μ, and write $\lambda \trianglerighteq \mu$ or $\mu \trianglelefteq \lambda$, if for all $1 \leqslant t \leqslant l$ and $r \geqslant 0$,

$$|\lambda^{(1)}| + \dots + |\lambda^{(t-1)}| + \sum_{j=1}^{r} \lambda_r^{(t)} \geqslant |\mu^{(1)}| + \dots + |\mu^{(t-1)}| + \sum_{j=1}^{r} \mu_r^{(t)}.$$

For $\lambda \in \mathscr{P}_n^l$, we define the *Young diagram* $[\lambda]$ to be the set

$$\{(r, c, t) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \times \{1, \dots, l\} \mid c \leqslant \lambda_r^{(t)}\}.$$

We call elements of $[\lambda]$ *nodes*. We draw the Young diagram of a partition using the English convention (where the first coordinate increases down the page and the second coordinate increases from left to right), and of a multipartition as a column vector of Young diagrams for each component. We say that $A \notin [\lambda]$ is an *addable node* (for λ) if $[\lambda] \cup A$ is a valid Young diagram of a multipartition.

Let p be the natural projection $p : \mathbb{Z} \to \mathbb{Z}/2\ell\mathbb{Z}$ if $\ell < \infty$, and $p = \mathrm{id}$ if $\ell = \infty$. If $\ell = \infty$, we define $f_\ell : \mathbb{Z} \to I$ by $k \mapsto |k|$. If $\ell < \infty$, we define $f_\ell : \mathbb{Z}/2\ell\mathbb{Z} \to I$ by $f_\ell(0) = 0$, $f_\ell(\ell) = \ell$, and $f_\ell(k) = f_\ell(2\ell - k) = k$ for $1 \leqslant k \leqslant \ell - 1$. Then we define $\overline{} : f_\ell \circ p : \mathbb{Z} \to I$.

Given a *multicharge* $\kappa = (\kappa_1, \dots, \kappa_l) \in \mathbb{Z}^l$, we define $\Lambda_\kappa \in \mathsf{P}^+$ by $\Lambda_\kappa = \Lambda_{\overline{\kappa_1}} + \dots + \Lambda_{\overline{\kappa_l}}$. The *residue* of a node $A = (r, c, t) \in [\lambda]$ is $\operatorname{res} A = \overline{\kappa_t + c - r}$. If $\operatorname{res} A = i$, we call A an i-node.

Example. Let $\ell = 3$, $\kappa = (1, 4)$ and $\lambda = ((8, 3, 2), (5, 3, 1))$. Then the Young diagram $[\lambda]$, along with the residue pattern, is depicted below.

1	2	3	2	1	0	1	2
0	1	2					
1	0						

2	1	0	1	2
3	2	1		
2				

A λ-*tableau* is a bijection $\mathtt{T} : [\lambda] \to \{1, \dots, n\}$. We depict $\mathtt{T}$ by filling each node (r, c, t) with $\mathtt{T}(r, c, t)$. We say that a λ-tableau is *standard* if in each component, the entries increase along each row and down each column. We denote by $\mathrm{Std}(\mathscr{P}^l_n)$ the set of all standard tableaux whose shape is an l-multipartition of n, and by $\mathrm{Std}^2(\mathscr{P}^l_n)$ the subset of $\mathrm{Std}(\mathscr{P}^l_n) \times \mathrm{Std}(\mathscr{P}^l_n)$ consisting of all pairs of standard tableaux of the same shape.

The distinguished tableau $\mathtt{T}^\lambda$ is obtained by filling nodes in order along rows, starting with the first row of $[\lambda^{(1)}]$ and working down the rows of this component before moving on to successive components.

A *Garnir node* $A = (r, c, t) \in [\lambda]$ is a node for which $(r + 1, c, t) \in [\lambda]$. The corresponding *Garnir belt* $\mathbf{B}^A$ is the set of nodes

$$\{(r, c, t), (r, c + 1, t), \dots, (r, \lambda^{(t)}_r, t)\} \cup \{(r + 1, 1, t), (r + 1, 2, t), \dots, (r + 1, c, t)\}.$$

We define the *Garnir tableau* $\mathtt{G}^A$ to be the λ-tableau which agrees with $\mathtt{T}^\lambda$ outside of $\mathbf{B}^A$, with the entries in $\mathbf{B}^A$ in order from left to right along row $r + 1$, and then row r. See [APS17, §1.4] for examples.

The *residue sequence* of a λ-tableau $\mathtt{T}$ is the sequence $\mathbf{i}^\mathtt{T} = (i_1, \dots, i_n)$, where $i_r = \mathrm{res}\,\mathtt{T}^{-1}(r)$. We define $\mathbf{i}^\lambda = \mathbf{i}^{\mathtt{T}^\lambda}$.

We denote by $\mathrm{Shp}(\mathtt{T})$ the *shape* of $\mathtt{T}$ – i.e. $\mathtt{T}$ is a $\mathrm{Shp}(\mathtt{T})$-tableau. We let $\mathtt{T}_{\downarrow m}$ denote the tableau obtained from $\mathtt{T}$ by deleting all entries greater than m. Finally, we define the dominance order on tableaux by $\mathtt{S} \trianglelefteqslant \mathtt{T}$ if $\mathrm{Shp}(\mathtt{S}_{\downarrow m}) \trianglelefteqslant \mathrm{Shp}(\mathtt{T}_{\downarrow m})$ for all $1 \leqslant m \leqslant n$.

For each $w \in \mathfrak{S}_n$, we fix a preferred reduced expression $w = s_{i_1} \dots s_{i_r}$. For $\mathtt{T}$ a λ-tableau, we define $w^\mathtt{T} \in \mathfrak{S}_n$ to be the permutation such that $w^\mathtt{T}\mathtt{T}^\lambda = \mathtt{T}$, where $\mathfrak{S}_n$ acts on tableaux by permuting entries. If $w^\mathtt{T} = s_{i_1} \dots s_{i_r}$ is our preferred reduced expression for $w^\mathtt{T}$, we define the element $\psi_{w^\mathtt{T}} = \psi_{i_1} \dots \psi_{i_r} \in \mathscr{R}_n$.

2.3. **Specht modules.** We will briefly recall the definition of the (graded) Specht modules from [APS17]. The reader should refer to [APS17] for a more thorough treatment of Specht modules, and of the graded module categories of $\mathscr{R}^\Lambda_n$.

Fix a multicharge $\kappa \in \mathbb{Z}^l$ and let $\lambda \in \mathscr{P}^l_n$. For each Garnir node $A \in [\lambda]$ we may define the *Garnir element* $\mathtt{g}^A \in \mathscr{R}_n$. See [APS17, §3.2] for the definition of $\mathtt{g}^A$.

The graded Specht module $\mathcal{S}^\lambda_\kappa$ is the unital $\mathscr{R}_n$-module with generator z^λ of degree $\deg \mathtt{T}^\lambda$ (see [APS17, §1.3]) subject to the relations

(i) $e(\mathbf{i})z^\lambda = z^\lambda$;
(ii) $x_r z^\lambda = 0$ for all $1 \leqslant r \leqslant n$;
(iii) $\psi_r z^\lambda = 0$ whenever r and $r + 1$ lie in the same row of $\mathtt{T}^\lambda$;
(iv) $\mathtt{g}^A z^\lambda = 0$ for all Garnir nodes $A \in [\lambda]$.

For each λ-tableau $\mathtt{T}$, we define $v^\mathtt{T} = \psi_{w^\mathtt{T}} z^\lambda \in \mathcal{S}^\lambda_\kappa$.

Theorem 2.1 ([APS17, Theorem 3.12]). *The Specht module $\mathcal{S}^\lambda_\kappa$ is a graded $\mathscr{R}^\Lambda_n$-module and is generated by $\{v^\mathtt{T} \mid \mathtt{T} \in \mathrm{Std}(\lambda)\}$ as an $\mathcal{O}$-module.*

In type C_∞, [APS17, Theorem 3.19] tells us that the generating set in Theorem 2.1 is in fact a (homogeneous) basis, and we conjectured in [APS17, Conjecture 5.3] that the same is true in type $C^{(1)}_\ell$.

For us, it will suffice to note that $\mathtt{g}^A = \psi_{w^{\mathtt{G}^A}}$ for all Specht modules we will consider. Indeed, we have [APS17, Equation 3.3]:

$$\mathtt{g}^A = \psi_{w^{\mathtt{G}^A}} + \sum_w a_w \psi_w \quad \text{for some } a_w \in \mathcal{O},$$

where the sum is taken over $\{w \in \mathfrak{S}_n \mid w < w^{\mathtt{G}^A}, \mathbf{i}^{w\mathtt{T}^\lambda} = \mathbf{i}^{\mathtt{G}^A}, w\mathtt{T}^\lambda \text{ is } \textit{row-strict}\}$. In every Specht module we will consider in this paper, there is no row-strict λ-tableau which dominates $\mathtt{G}^A$ and has the same residue sequence as $\mathtt{G}^A$, for any Garnir node A. In general, however, $\mathtt{g}^A$ will also include these terms indexed by more dominant tableaux with the same residue sequence.

The following lemma will be useful to us later.

Lemma 2.2. *Let $\lambda \in \mathscr{P}^l_n$. Then we have the following actions of the generators of $\mathscr{R}_n$ on the $\mathcal{O}$-generating set for $\mathcal{S}^\lambda$ in Theorem* 2.1.

(i) *Let $\mathtt{T} \in \mathrm{Std}(\lambda)$ and $1 \leqslant r \leqslant n$. Then*

$$x_r v^{\mathtt{T}} = \sum_{\substack{\mathtt{S} \in \mathrm{Std}(\lambda) \\ \mathbf{i}^{\mathtt{S}} = \mathbf{i}^{\mathtt{T}} \\ \mathtt{S} \rhd \mathtt{T}}} a_{\mathtt{S}} v^{\mathtt{S}} \quad \textit{for some } a_{\mathtt{S}} \in \mathcal{O}.$$

(ii) *Let $\mathtt{T} \in \mathrm{Std}(\lambda)$ and $1 \leqslant r < n$. Then*

$$\psi_r v^{\mathtt{T}} = \sum_{\substack{\mathtt{S} \in \mathrm{Std}(\lambda) \\ \mathbf{i}^{\mathtt{S}} = \mathbf{i}^{s_r\mathtt{T}} \\ \mathtt{S} \rhd \mathtt{T}}} a_{\mathtt{S}} v^{\mathtt{S}} \quad \textit{for some } a_{\mathtt{S}} \in \mathcal{O},$$

unless $s_r\mathtt{T} \in \mathrm{Std}(\lambda)$ and $s_r w^{\mathtt{T}}$ is a reduced expression of length $\ell(w^{\mathtt{T}}) + 1$.

Proof. This is identical to [BKW11, Lemmas 4.8 and 4.9] and [FS16, Lemma 2.14]. □

3. Semisimplicity of $\mathscr{R}^\Lambda_n$

Let $\ell \in \{2, 3, \dots\} \cup \{\infty\}$, $\Lambda \in \mathsf{P}^+$ be a dominant weight of level $l \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}_{>1}$ so that we have the cyclotomic quiver Hecke algebra $\mathscr{R}^\Lambda_n$. Let $\kappa \in \mathbb{Z}^l$ be any multicharge such that $\Lambda = \Lambda_\kappa$.

For $i \in I$ and $k \in \mathbb{Z}_{>0}$, we set $\alpha^\vee_{i,k} = \alpha^\vee_i + \alpha^\vee_{i+1} + \cdots + \alpha^\vee_{i+k-1}$, where the indices are taken modulo $\ell + 1$.

The following two conditions will be key in our semisimplicity arguments, and we will refer back to them frequently:

(SS1) For all $i \in I$, $\langle \Lambda, \alpha^\vee_{i,n} \rangle \leqslant 1$.
(SS2) For all $1 \leqslant j \leqslant l$, $\frac{n-1}{2} \leqslant \overline{\kappa_j} \leqslant \ell - \frac{n-1}{2}$.

Remark. The following observations are the driving force for this paper, and will be used frequently:

(1) Suppose that (SS1) holds, and let $\lambda \in \mathscr{P}^l_m$ for some $0 \leqslant m < n$. Then for any $i \in I$, μ has at most one component with addable i-nodes. Informally, we may think of (SS1) as ensuring that for any $\lambda \in \mathscr{P}^l_n$, nodes in distinct components of $[\lambda]$ must have distinct residues.
(2) Suppose that (SS2) holds, and let $\lambda \in \mathscr{P}^l_n$. For a given residue $i \in I$, there is either only one possible diagonal of residue i which may appear in the Young diagram of some partition, or there are two diagonals which, in any multipartition, may contain at most a single node each (in which case both nodes lie in the same row or the same column of the multipartition, and the residue is either 1 or $\ell - 1$).

3.1. **The semisimple case.** First, we will handle the case when $\mathscr{R}^\Lambda_n$ is semisimple. This subsection mirrors the corresponding type A arguments of [Mat15, §2.4], which we have adapted to fit the type C case.

Lemma 3.1. *Suppose that conditions* (SS1) *and* (SS2) *hold, and let* $\mathtt{T}, \mathtt{S} \in \mathrm{Std}(\mathscr{P}^l_n)$. *Then* $\mathtt{T} = \mathtt{S}$ *if and only if* $\mathbf{i}^{\mathtt{T}} = \mathbf{i}^{\mathtt{S}}$.

Proof. If $i \in I$ and $\lambda \in \mathscr{P}^l_m$ for some $0 \leqslant m < n$, then by the above remark, $[\lambda]$ has at most one addable i-node, and the result follows by induction on n. □

Let $I^n_\Lambda = \{\mathbf{i}^{\mathtt{T}} \mid \mathtt{T} \in \mathrm{Std}(\mathscr{P}^l_n)\}$.

Corollary 3.2. *Suppose that conditions* (SS1) *and* (SS2) *hold, and let* $\mathbf{i} \in I^n_\Lambda$ *such that* $i_{r+1} = i_r \pm 1$. *Then* $s_r\mathbf{i} \notin I^n_\Lambda$.

Proof. By the above remark, if $\mathbf{i} = \mathbf{i}^{\mathtt{T}}$ for some $\mathtt{T} \in \mathrm{Std}(\mathscr{P}^l_n)$, and $i_{r+1} = i_r \pm 1$, then r and $r+1$ must lie in adjacent diagonals of $\mathtt{T}$. In particular, they must lie in either the same row or the same column of $\mathtt{T}$. Given that the number of residues is equal to i_r and i_{r+1} in $(i_1, \dots, i_{r-1})$ is unchanged, we deduce that r and $r+1$ must occupy the same two pair of nodes in $\mathtt{T}$ as in any standard tableau with residue sequence $s_r\mathbf{i}$. But this is a contradiction, as such a tableau cannot be standard. □

Recall that we have fixed a multicharge $\kappa \in \mathbb{Z}^l$ such that $\Lambda = \Lambda_\kappa$.

Theorem 3.3. *Suppose that* $\mathcal{O} = \mathbb{F}$ *is a field, and that conditions* (SS1) *and* (SS2) *hold. Then for each* $\lambda \in \mathscr{P}^l_n$ *there is an irreducible graded* $\mathscr{R}^\Lambda_n$*-module* $\mathrm{S}^\lambda_\kappa$ *with homogeneous basis* $\{v^{\mathtt{T}} \mid \mathtt{T} \in \mathrm{Std}(\lambda)\}$ *such that* $\deg v^{\mathtt{T}} = 0$ *for all* $\mathtt{T} \in \mathrm{Std}(\lambda)$, *and the* $\mathscr{R}^\Lambda_n$*-action is given by*

$$e(\mathbf{i})v^{\mathtt{T}} = \delta_{\mathbf{i},\mathbf{i}^{\mathtt{T}}} v^{\mathtt{T}}, \quad x_r v^{\mathtt{T}} = 0, \quad \psi_r v^{\mathtt{T}} = v^{s_r\mathtt{T}},$$

where we set $v^{s_r\mathtt{T}} = 0$ *if* $s_r\mathtt{T}$ *is not standard.*

Proof. We first check that the relations above really define an $\mathscr{R}^\Lambda_n$-module. Almost all the defining relations for $\mathscr{R}^\Lambda_n$ are trivially satisfied, thanks to Lemma 3.1 and Corollary 3.2. We must check that the ψ generators satisfy the braid relations and the quadratic relations when acting on basis elements. Let $\lambda \in \mathscr{P}^l_n$, $\mathtt{T} \in \mathrm{Std}(\lambda)$ and set $\mathbf{i} = \mathbf{i}^{\mathtt{T}} = (i_1, \dots, i_n)$.

For the braid relations, (SS1) and (SS2) ensure that we never have $i_r = i_{r+2} = i_{r+1} \pm 1$ with $i_{r+1} \neq 0, \ell$. To see this, we again invoke our remark made after introducing conditions (SS1) and (SS2). Since we can only have a single diagonal of any residue besides 1 and ℓ, it is not possible for the (arbitrarily chosen) standard tableau $\mathtt{T}$ to have consecutive residues $i, i\pm1, i$, except for $(1,0,1)$ and $(\ell-1, \ell, \ell-1)$. Finally, if $(i_r, i_{r+1}, i_{r+2}) = (1,0,1)$ or $(\ell-1, \ell, \ell-1)$, then we have $\psi_{r+1}\psi_r\psi_{r+1}v^{\mathtt{T}} = \psi_r\psi_{r+1}\psi_r v^{\mathtt{T}} = 0$.

Since $i_{r+1} \neq i_r$ for any r and $i_{r+1} = i_r \pm 1$ if and only if r and $r+1$ are in the same row or column of $\mathtt{T}$, it follows from Corollary 3.2 that $\psi_r^2 v^{\mathtt{T}} = 0$ when $i_{r+1} = i_r \pm 1$.

These residue conditions also tell us that $\deg \psi_r e(\mathbf{i}) = 0$ whenever $s_r\mathtt{T} \in \mathrm{Std}(\lambda)$ (and if $s_r\mathtt{T} \notin \mathrm{Std}(\lambda)$, $\psi_r e(\mathbf{i}) = 0$ by Corollary 3.2). Thus setting $\deg v^{\mathtt{T}} = 0$ gives a grading on $\mathrm{S}^\lambda_\kappa$.

Finally, we show that $\mathrm{S}^\lambda_\kappa$ is irreducible. If $\mathtt{S},\mathtt{T} \in \mathrm{Std}(\lambda)$, then $\mathtt{S} = w^{\mathtt{S}}\mathtt{T}^\lambda = w^{\mathtt{S}}(w^{\mathtt{T}})^{-1}\mathtt{T}$. So $v^{\mathtt{S}} = \psi_{w^{\mathtt{S}}}\psi_{(w^{\mathtt{T}})^{-1}}v^{\mathtt{T}}$. Take a non-zero element $v = \sum_{\mathtt{T}\in\mathrm{Std}(\lambda)} a_{\mathtt{T}}v^{\mathtt{T}} \in \mathrm{S}^\lambda_\kappa$. If $a_{\mathtt{T}} \neq 0$, then, by Lemma 3.1, $v^{\mathtt{T}} = \frac{1}{a_{\mathtt{T}}}e(\mathbf{i}^{\mathtt{T}})v$, and therefore for any $\mathtt{S} \in \mathrm{Std}(\lambda)$, $v^{\mathtt{S}} \in \mathscr{R}^\Lambda_n v$. It follows that $\mathrm{S}^\lambda_\kappa$ is irreducible. □

Remark. The modules $\mathrm{S}^\lambda_\kappa$ are easily seen to be isomorphic to the Specht modules $\mathcal{S}^\lambda_\kappa$ constructed in Subsection 2.3, providing evidence for the importance of the Specht modules constructed in [APS17]. Indeed, as remarked after Theorem 2.1, we know that $\mathtt{g}^A = \psi_{w^{\mathtt{G}^A}}$, and this is sufficient to prove that $\mathcal{S}^\lambda_\kappa$ has a basis indexed by standard tableaux (the elements constructed in [APS17, Theorem 3.12 and Corollary 3.13]), showing that the dimensions match. By the definition of $\mathcal{S}^\lambda_\kappa$, the cyclic generator z^λ satisfies the same relations as the element $v^{\mathtt{T}^\lambda}$ constructed in Theorem 3.3, so that we have an isomorphism $\mathrm{S}^\lambda_\kappa \to \mathcal{S}^\lambda_\kappa$ determined by $v^{\mathtt{T}^\lambda} \mapsto z^\lambda$.

If $\mathbf{i} = (i_1, \dots, i_n) \in I^n$, we set $\mathbf{i}_{\downarrow r} = (i_1, \dots, i_r)$.

Lemma 3.4. *Suppose that conditions* (SS1) *and* (SS2) *hold, and let* $\mathbf{i} \in I^n$. *Then* $\mathbf{i} \in I^n_\Lambda$ *if and only if* $\mathbf{i}$ *satisfies the following three conditions:*

(i) $\langle \Lambda, \alpha^\vee_{i_1}\rangle \neq 0$.

(ii) *If* $1 < r \leqslant n$ *and* $\langle \Lambda, \alpha^\vee_{i_r}\rangle = 0$, *then* $\{\overline{i_r - 1}, \overline{i_r + 1}\} \cap \{i_1, \dots, i_{r-1}\} \neq \emptyset$.

(iii) *Let* $1 \leqslant s < r \leqslant n$. *If* $i_r = i_s \neq 1, \ell - 1$, *then* $\{\overline{i_r - 1}, \overline{i_r + 1}\} \subseteq \{i_{s+1}, \dots, i_{r-1}\}$. *If* $i_r = i_s = 1$, *then* $0 \in \{i_{s+1}, \dots, i_{r-1}\}$. *If* $i_r = i_s = \ell - 1$, *then* $\ell \in \{i_{s+1}, \dots, i_{r-1}\}$.

Proof. Let $\mathtt{T} \in \mathrm{Std}(\mathscr{P}^l_n)$ with $\mathbf{i}^{\mathtt{T}} = \mathbf{i}$. We prove by induction on r that $\mathbf{i}_{\downarrow r} \in I^r_\Lambda$ satisfies all three conditions as claimed. By definition, $i_1 = \overline{\kappa_j}$ for some j, so *(i)* holds. By induction, we assume that $\mathbf{i}_{\downarrow r-1}$ satisfies *(i)–(iii)*. If $\langle \Lambda, \alpha^\vee_{i_r}\rangle = 0$, then r is not in the $(1,1)$ node of any component of $\mathtt{T}$, so $\mathtt{T}$ has an entry directly above or to the left of r, so *(ii)* holds. Now suppose that $i_r = i_s \neq 1, \ell - 1$ are as in the first part of *(iii)*. Condition (SS1) ensures that residues in different components are distinct, so that r and s must be in the same component of $\mathtt{T}$. Condition (SS2) ensures that r and s are on the same diagonal, so that r is not in the first row or first column of the component, so *(iii)* holds. Finally, suppose that $i_r = i_s = 1$ or $\ell - 1$. Then we have r and s both appearing in the first row or both appearing in the first column of $\mathtt{T}$, so that $\{i_{s+1}, \dots, i_{r-1}\}$ contains 0 if $i_r = 1$, or ℓ if $i_r = \ell - 1$, proving the second and third statements in *(iii)*.

Conversely, suppose that $\mathbf{i} \in I^n$ satisfies conditions *(i)–(iii)*. We show by induction on r that $\mathbf{i}_{\downarrow r} \in I^r_\Lambda$ for $1 \leqslant r \leqslant n$. If $r = 1$, *(i)* implies that $\mathbf{i}_{\downarrow r} \in I^r_\Lambda$. So suppose by the induction hypothesis that for some $1 < r < n$, $\mathbf{i}_{\downarrow r} = \mathbf{i}^{\mathtt{S}}$ for some $\mathtt{S} \in \mathrm{Std}(\mathscr{P}^l_r)$. Let $\lambda = \mathrm{Shp}(\mathtt{S})$. From the proof of Lemma 3.1, we know that for any $i \in I$, $[\lambda]$ has at most one addable i-node.

If $\langle \Lambda, \alpha^\vee_{i_{r+1}}\rangle = 0$, then by *(ii)*, λ contains either an $(\overline{i_{r+1} - 1})$-node or $(\overline{i_{r+1} + 1})$-node (or both). Thus either there is an addable i_{r+1}-node in the first row or first column of the corresponding component of λ, or else there is some $1 \leqslant s < r + 1$ such that $i_s = i_{r+1}$. By (SS2), if there is no addable i_{r+1}-node in the first row or column, then $1 < i_{r+1} < \ell - 1$ in this case, and condition *(iii)* tells us that there is again an addable i_{r+1}-node.

If $\langle \Lambda, \alpha^\vee_{i_{r+1}}\rangle = 1$, then either the $(1,1)$ node of some component of $[\lambda]$ is an addable i_{r+1}-node, or $[\lambda]$ already contains a $(1,1)$ node which has residue i_{r+1}. In the latter case, *(iii)* implies that $[\lambda]$ has an addable i_{r+1}-node.

Thus we know that $[\lambda]$ has precisely one addable i_{r+1}-node, which we shall denote by A. Then $\mathbf{i}_{\downarrow r+1} = \mathbf{i}^{\mathtt{T}}$ where $\mathtt{T}$ is the unique standard tableau satisfying $\mathtt{T}_{\downarrow r} = \mathtt{S}$ and $\mathtt{T}(A) = r+1$. Hence $\mathbf{i}_{\downarrow r+1} \in I_\Lambda^{r+1}$ and the proof is complete. □

The following lemma follows easily from the rank formula for $e(\mathbf{i})\mathscr{R}_n^\Lambda e(\mathbf{i})$ given in [APS17, Theorem 2.5], and does not require that (SS1) and (SS2) are satisfied.

Lemma 3.5. *If $\mathbf{i} \in I^n \setminus I_\Lambda^n$, then $e(\mathbf{i}) = 0$ in $\mathscr{R}_n^\Lambda$.*

Lemma 3.6. *Let $1 \leqslant m < n$ and suppose that* (SS1) *and* (SS2) *hold if n is replaced with m. Then $x_1 = \cdots = x_m = 0$.*

Proof. Using the defining relations of $\mathscr{R}_n^\Lambda$, we will prove by induction on r that $x_r e(\mathbf{i}) = 0$ for all $\mathbf{i} \in I_\Lambda^n$ and $1 \leqslant r \leqslant m$, from which the result will follow by Lemma 3.5.

When $r = 1$, the result follows immediately from the cyclotomic relations. So we will assume that $x_1 = \cdots = x_{r-1} = 0$, and show that $x_r e(\mathbf{i}) = 0$ whenever $\mathbf{i}_{\downarrow r} \in I_\Lambda^r$.

If $i_{r-1} = i_r \pm 1$ and neither i_{r-1} nor i_r are 0 or ℓ, then by induction we have

$$x_r e(\mathbf{i}) = (x_r + x_{r-1})e(\mathbf{i}) = \psi_{r-1}^2 e(\mathbf{i}) = \psi_{r-1} e(s_{r-1}\mathbf{i})\psi_{r-1} = 0,$$

where the last equality follows from Corollary 3.2. Similarly, if $(i_{r-1}, i_r) = (1,0)$ or $(\ell-1, \ell)$, then

$$x_r e(\mathbf{i}) = (x_r + x_{r-1}^2)e(\mathbf{i}) = \psi_{r-1}^2 e(\mathbf{i}) = \psi_{r-1} e(s_{r-1}\mathbf{i})\psi_{r-1} = 0.$$

If $(i_{r-1}, i_r) = (0,1)$ or $(\ell, \ell-1)$, then by (SS2) and Lemma 3.1, $\mathbf{i}$ is the residue sequence of some standard tableau $\mathtt{T}$ of shape λ, and $[\lambda]$ has exactly one other 1-node (resp. $(\ell-1)$-node) besides $\mathtt{T}^{-1}(r)$, and $\mathtt{T}^{-1}(r-1)$ is the only 0-node (resp. ℓ-node) of $[\lambda]$. Moreover, the other 1-node (resp. $(\ell-1)$-node) is $\mathtt{T}^{-1}(u)$ for some $1 \leqslant u < r$, and $\mathtt{T}^{-1}(v) \neq 2$ (resp. $\ell - 2$) for any $u < v < r$. Thus we have

$$\begin{aligned} e(\mathbf{i}) &= \psi_u^2 e(\mathbf{i}) = \psi_u e(s_u\mathbf{i})\psi_u = \psi_u\psi_{u+1} e(s_{u+1}s_u\mathbf{i})\psi_{u+1}\psi_u \\ &= \cdots = \psi_u\psi_{u+1}\dots\psi_{r-3} e(s_{r-3}\dots s_u\mathbf{i})\psi_{r-3}\dots\psi_u, \end{aligned}$$

so that $((s_{r-3}\dots s_u\mathbf{i})_{r-2}, (s_{r-3}\dots s_u\mathbf{i})_{r-1}, (s_{r-3}\dots s_u\mathbf{i})_r) = (1,0,1)$ or $(\ell-1, \ell, \ell-1)$ and

$$\begin{aligned} &x_r e(\mathbf{i}) \\ &= \psi_u\psi_{u+1}\dots\psi_{r-3} x_r e(s_{r-3}\dots s_u\mathbf{i})\psi_{r-3}\dots\psi_u \\ &= \psi_u\psi_{u+1}\dots \\ &\qquad \psi_{r-3}(\psi_{r-1}\psi_{r-2}\psi_{r-1} - \psi_{r-2}\psi_{r-1}\psi_{r-2} - x_{r-2})e(s_{r-3}\dots s_u\mathbf{i})\psi_{r-3}\dots\psi_u \\ &= 0 \end{aligned}$$

by the induction hypothesis and the fact that $s_{r-2}s_{r-3}\dots s_u\mathbf{i}, s_{r-1}s_{r-3}\dots s_u\mathbf{i} \notin I_\Lambda^n$ by Corollary 3.2.

Finally, if $i_{r-1} \neq i_r \pm 1$, then since we know that $i_{r-1} \neq i_r$ by Lemma 3.4,

$$x_r e(\mathbf{i}) = x_r \psi_{r-1}^2 e(\mathbf{i}) = x_r \psi_{r-1} e(s_{r-1}\mathbf{i})\psi_{r-1} = 0,$$

which completes the proof. □

Definition 3.7. Let $(\mathtt{S}, \mathtt{T}) \in \mathrm{Std}^2(\mathscr{P}_n^l)$. Then we define the element $e_{\mathtt{ST}} \in \mathscr{R}_n^\Lambda$ to be $e_{\mathtt{ST}} = \psi_{(w^{\mathtt{S}})^{-1}} e(\mathbf{i}^\lambda)\psi_{w^{\mathtt{T}}}$.

By Lemma 3.1, the elements $e_{\mathtt{ST}}$ do not depend on the choice of reduced expression.

Theorem 3.8. *Suppose that conditions* (SS1) *and* (SS2) *hold. Then* $\mathscr{R}_n^\Lambda$ *is a graded cellular algebra with graded cellular basis*

$$\mathscr{B} = \{e_{\mathtt{ST}} \mid (\mathtt{S},\mathtt{T}) \in \operatorname{Std}^2(\mathscr{P}_n^l)\}$$

with $\deg e_{\mathtt{ST}} = 0$ *for all* $\mathtt{S},\mathtt{T}$.

Proof. By Lemma 3.4, if $\mathbf{i} \in I_\Lambda^n$, $\mathbf{i}$ cannot contain a subsequence of the form $(i, i \pm 1, i)$ for any $i \in I$, except possibly $(1,0,1)$ or $(\ell-1,\ell,\ell-1)$. Lemmas 3.5 and 3.6 imply that (even in the degenerate cases above) the ψ generators satisfy the braid relations for $\mathfrak{S}_n$. Therefore $\mathscr{R}_n^\Lambda$ is spanned by the elements $\{\psi_v e(\mathbf{i})\psi_w \mid v, w \in \mathfrak{S}_n,\ \mathbf{i} \in I_\Lambda^n\}$. Since $\psi_v e(\mathbf{i})\psi_w = e(v\mathbf{i})\psi_v e(\mathbf{i})\psi_w = 0$ if $v\mathbf{i} \notin I_\Lambda^n$, $\mathscr{R}_n^\Lambda$ it is in fact spanned by the elements of $\mathscr{B}$. It follows from the rank formula [APS17, Theorem 2.5] that $\mathscr{B}$ is a basis for $\mathscr{R}_n^\Lambda$.

The orthogonality relations on the idempotents $e(\mathbf{i})$ imply that $e_{\mathtt{ST}}e_{\mathtt{UV}} = \delta_{\mathtt{T},\mathtt{U}}e_{\mathtt{SV}}$, so that $\mathscr{B}$ is in fact a basis of matrix units, and

$$\mathscr{R}_n^\Lambda = \bigoplus_{\lambda \in \mathscr{P}_n^l} \operatorname{Mat}_{\dim \mathrm{S}_\kappa^\lambda}(\mathcal{O}).$$

It follows that this basis is a cellular basis. As in the proof of Theorem 3.3, we have that $\deg \psi_r e(\mathbf{i}) = 0$ for all $1 \leqslant r < n$ and $\mathbf{i} \in I_\Lambda^n$, so all elements of $\mathscr{B}$ are homogeneous of degree 0. □

In the proof of the above theorem, we showed that if conditions (SS1) and (SS2) hold, $\mathscr{R}_n^\Lambda$ is a direct sum of matrix algebras. We obtain the main result of this subsection as a corollary of this fact.

Corollary 3.9. *Suppose that* $\mathcal{O} = \mathbb{F}$ *is a field and that conditions* (SS1) *and* (SS2) *hold. Then* $\mathscr{R}_n^\Lambda$ *is semisimple.*

3.2. **The non-semisimple case.** In this section, we will assume throughout that $\mathcal{O} = \mathbb{F}$ is a field and prove the following converse to Corollary 3.9.

Theorem 3.10. *Suppose that* $\mathcal{O} = \mathbb{F}$ *is a field, and that at least one of the conditions* (SS1) *and* (SS2) *fails. Then* $\mathscr{R}_n^\Lambda$ *is not semisimple.*

We break the proof into several lemmas. First we will look at the case where (SS2) fails. We begin with separate treatment of the case where $\overline{\kappa_j} = 0$ or ℓ for some $1 \leqslant j \leqslant l$.

Lemma 3.11. *Suppose* $\overline{\kappa_j} = 0$ *or* ℓ *for some* $1 \leqslant j \leqslant l$. *If* $n > 1$, *then* $\mathscr{R}_n^\Lambda$ *is not semisimple.*

Proof. For any $n > 1$ we construct an explicit two-dimensional uniserial $\mathscr{R}_n^\Lambda$-module. Let $\lambda \in \mathscr{P}_n^l$ be the multipartition such that every component is empty except for component j, with $\lambda^{(j)} = (n)$, andlet $\mathbf{i} = \mathbf{i}^\lambda$.

Define M to be the $\mathscr{R}^\Lambda_n$-module with generators u, v subject to the following relations:

$$\begin{aligned} e(\mathbf{i})u &= u, \\ e(\mathbf{i})v &= v, \\ \psi_r u = \psi_r v &= 0 \text{ for all } r, \\ x_r u &= 0 \text{ for all } r, \\ x_r v &= 0 \text{ if } r \equiv 1 \mod \ell, \\ x_{2k\ell+r} v &= (-1)^r u \text{ for all } k \text{ and all } 2 \leqslant r \leqslant \ell, \\ x_{2k\ell+r} v &= (-1)^{r+1} u \text{ for all } k \text{ and all } \ell+2 \leqslant r \leqslant 2\ell. \end{aligned}$$

Then M is a two-dimensional vector space over $\mathbb{F}$, and we must show it is an $\mathscr{R}^\Lambda_n$-module, whence the result follows since u generates a proper submodule of M, while v generates the whole of M. So we must check that the defining relations of $\mathscr{R}^\Lambda_n$ hold when acting on M. For most of the relations, the result is trivial – if ψ generators appear in every term or for the idempotent relations, or the product of two x generators. By definition of $\mathbf{i}$, there are no error terms in the relations pushing x generators past ψ generators, so these are also trivial. This leaves the quadratic and braid relations.

First, we deal with the quadratic relations. If $r \equiv 1 \mod \ell$, then $(i_r, i_{r+1}) = (0,1)$ or $(\ell, \ell-1)$, so that $\psi_r^2 e(\mathbf{i}) = (x_r + x_{r+1}^2)e(\mathbf{i})$. In both cases, ψ_r, x_r and x_{r+1}^2 each kill both u and v, so the relation holds. If $(i_r, i_{r+1}) = (1,0)$ or $(\ell-1, \ell)$, then $\psi_r^2 e(\mathbf{i}) = (x_r^2 + x_{r+1})e(\mathbf{i})$, and again each of ψ_r, x_r^2 and x_{r+1} kills both u and v, so the relation holds. Finally, suppose that $i_r, i_{r+1} \neq 0$ or ℓ. Then $\psi_r^2 e(\mathbf{i}) = (x_r + x_{r+1})e(\mathbf{i})$, with the left-hand side killing u and v, x_r and x_{r+1} each killing u, and $x_r v = -x_{r+1} v$, so that this relation always holds.

Next, we check the braid relations. Since $\psi_r u = \psi_r v = 0$ for all r, we only have to worry about the braid relations which yield error terms. With our chosen $\mathbf{i}$, this only happens for the relations $(\psi_{r+1}\psi_r\psi_{r+1} - \psi_r\psi_{r+1}\psi_r)e(\mathbf{i}) = (x_r + x_{r+2})e(\mathbf{i})$ for $r \equiv 0 \mod \ell$. Now we have that $x_r v = -x_{r+2} v$ by the final two defining relations for M. □

Next, we will handle the case where (SS2) fails and $\overline{\kappa_j} \neq 0, \ell$ for any $1 \leqslant j \leqslant l$. Recall that we have fixed a multicharge $\kappa = (\kappa_1, \dots, \kappa_l) \in \mathbb{Z}^l$ such that $\Lambda = \Lambda_\kappa$. Define $\overline{\kappa} = (\overline{\kappa_1}, \dots, \overline{\kappa_l}) \in I^l$ and $\hat{\kappa} = (2\ell - \overline{\kappa_1}, \dots, 2\ell - \overline{\kappa_l}) \in I^l$.

Lemma 3.12. *Suppose that $\overline{\kappa_j} \neq 0, \ell$ for all $1 \leqslant j \leqslant l$, and* (SS2) *fails. Then $\mathscr{R}^\Lambda_n$ is not semisimple.*

Proof. We fix $1 \leqslant j \leqslant l$ such that either $\frac{n-1}{2} > \overline{\kappa_j}$ or $\ell - \frac{n-1}{2} < \overline{\kappa_j}$. Set $\mu \in \mathscr{P}^l_n$ to be the multipartition such that every component is empty except component j, with $\mu^{(j)} = (1^n)$.

If $\frac{n-1}{2} > \overline{\kappa_j}$, we set $\lambda \in \mathscr{P}^l_n$ to be the multipartition such that every component is empty except component j, with $\lambda^{(j)} = (n - 2\overline{\kappa_j}, 1^{2\overline{\kappa_j}})$. We will show that for $\mathtt{T}$ the least dominant standard λ-tableau, the homogeneous basis element $v^{\mathtt{T}} = \psi_{w^{\mathtt{T}}} z^\lambda$ generates a one-dimensional submodule of $\mathcal{S}^\lambda_{\overline{\kappa}}$ isomorphic to $\mathcal{S}^\mu_{\overline{\kappa}}$.

If $\ell - \frac{n-1}{2} < \overline{\kappa_j}$, we may instead set $\lambda \in \mathscr{P}^l_n$ to be the multipartition such that every component is empty except $\lambda^{(j)} = (2(\ell - \overline{\kappa_j}), 1^{n-2(\ell-\overline{\kappa_j})})$. A similar argument shows that for $\mathtt{T}$ the least dominant standard λ-tableau, $v^{\mathtt{T}}$ generates a one-dimensional submodule of $\mathcal{S}^{\lambda}_{\hat{\kappa}}$ isomorphic to $\mathcal{S}^{\mu}_{\hat{\kappa}}$, so we will focus on the former case, leaving the latter as an exercise.

We have that $e(\mathbf{i}^{\mathtt{T}})v^{\mathtt{T}} = v^{\mathtt{T}}$, where $\mathbf{i}^{\mathtt{T}} = (\overline{\kappa_j}, \overline{\kappa_j} - 1, \dots, 1, 0, 1, \dots, \overline{\kappa_j}, \overline{\kappa_j} + 1, \dots)$. We will show that all x and ψ generators of $\mathscr{R}^{\Lambda}_n$ annihilate $v^{\mathtt{T}}$. First, let $1 \leqslant r \leqslant n$. Then by Lemma 2.2(i),

$$x_r v^{\mathtt{T}} = \sum_{\substack{\mathtt{S}\in\mathrm{Std}(\lambda)\\ \mathbf{i}^{\mathtt{S}}=\mathbf{i}^{\mathtt{T}}\\ \mathtt{S}\rhd\mathtt{T}}} a_{\mathtt{S}} v^{\mathtt{S}}.$$

However, it is clear that $\mathtt{T}$ is the only standard λ-tableau with residue sequence $\mathbf{i}$, so that $x_r v^{\mathtt{T}} = 0$. Now suppose that $1 \leqslant r < n$. Then since $\mathtt{T}$ is the least dominant standard λ-tableau, we see by Lemma 2.2(ii) that

$$\psi_r v^{\mathtt{T}} = \sum_{\substack{\mathtt{S}\in\mathrm{Std}(\lambda)\\ \mathbf{i}^{\mathtt{S}}=s_r\mathbf{i}^{\mathtt{T}}\\ \mathtt{S}\rhd\mathtt{T}}} a_{\mathtt{S}} v^{\mathtt{S}}.$$

However, there is no standard λ-tableau with residue sequence $s_r\mathbf{i}^{\mathtt{T}}$, so that $\psi_r v^{\mathtt{T}} = 0$.

To see that $\mathcal{S}^{\mu}_{\overline{\kappa}}$ has no complement in $\mathcal{S}^{\lambda}_{\overline{\kappa}}$ (i.e. is not a direct summand), it suffices to note that the residue sequence of the unique standard μ-tableau is different from the residue sequence $\mathbf{i}^{\lambda}$ of the initial λ-tableau $\mathtt{T}^{\lambda}$, so that there is no non-zero homomorphism $\mathcal{S}^{\lambda}_{\overline{\kappa}} \to \mathcal{S}^{\mu}_{\overline{\kappa}}$. □

Remark. Our choice of multicharge defining the Specht modules in Lemma 3.12 ensures (since $\overline{\kappa_j} \neq \ell$) that $\mathrm{res}(1,2,j) = \overline{\kappa_j} + 1$. Similarly, in the case left as an exercise, $\mathrm{res}(2,1,j) = \overline{\kappa_j} + 1$. Thanks to the symmetry in the type C residue pattern, this suffices to prove that $\mathscr{R}^{\Lambda}_n$ is not semisimple. A different choice of multicharge κ' satisfying $\Lambda = \Lambda_{\overline{\kappa'}}$ would also do the trick, but would need a slightly different choice of multipartition λ.

We now turn our attention to the case where condition (SS1) fails.

Lemma 3.13. *Suppose that condition* (SS2) *holds, but $\overline{\kappa_j} = \overline{\kappa_{j'}}$ for some $1 \leqslant j \neq j' \leqslant l$. Then $\mathscr{R}^{\Lambda}_n$ is not semisimple.*

Proof. The proof is similar to the proof of Lemma 3.11. We let $\lambda \in \mathscr{P}^l_n$ be the multipartition such that every component is empty except for component j, with $\lambda^{(j)} = (n)$, and let $\mathbf{i} = \mathbf{i}^{\lambda}$.

Define M to be the $\mathscr{R}^{\Lambda}_n$-module with generators u, v subject to the following relations:

$$\begin{aligned}
e(\mathbf{i})u &= u,\\
e(\mathbf{i})v &= v,\\
\psi_r u &= \psi_r v = 0 \text{ for all } r,\\
x_r u &= 0 \text{ for all } r,\\
x_{\ell-\overline{\kappa_j}+1} v &= 0,\\
(-1)^{r+1} x_r v &= u \text{ for all } 1 \leqslant r < \ell - \overline{\kappa_j} + 1,\\
(-1)^{r} x_r v &= u \text{ for all } \ell - \overline{\kappa_j} + 1 < r \leqslant n.
\end{aligned}$$

Then M is a two-dimensional vector space over $\mathbb{F}$, and we proceed to show that it is an $\mathscr{R}_n^\Lambda$-module. For most of the relations, the result is trivial, so we check the quadratic and braid relations. We note that since (SS2) holds, $\mathbf{i}$ is a prefix of $(\overline{\kappa_j}, \overline{\kappa_j}+1, \dots, \ell-1, \ell, \ell-1, \dots, \overline{\kappa_j})$, so that there is only a single non-trivial braid relation to check, corresponding to $(i_{\ell-\overline{\kappa_j}}, i_{\ell-\overline{\kappa_j}+1}, i_{\ell-\overline{\kappa_j}+2}) = (\ell-1, \ell, \ell-1)$.

First, we deal with the quadratic relations. For $1 \leqslant r < \ell - \overline{\kappa_j}$, we have $\psi_r^2 e(\mathbf{i}) = (x_r + x_{r+1}) e(\mathbf{i})$, and both sides kill u and v. Next, $\psi_{\ell-\overline{\kappa_j}}^2 e(\mathbf{i}) = (x_{\ell-\overline{\kappa_j}}^2 + x_{\ell-\overline{\kappa_j}+1}) e(\mathbf{i})$, and both sides again kill u and v. Similarly, both sides of $\psi_{\ell-\overline{\kappa_j}+1}^2 e(\mathbf{i}) = (x_{\ell-\overline{\kappa_j}+1} + x_{\ell-\overline{\kappa_j}+2}^2) e(\mathbf{i})$ kill u and v. Finally, for $\ell - \overline{\kappa_j} + 1 < r < n$, we have $\psi_r^2 e(\mathbf{i}) = (x_r + x_{r+1}) e(\mathbf{i})$ and both sides kill u and v.

Finally, we check the non-trivial braid relation, which is only present if $n > \ell - \overline{\kappa_j} + 2$. We have

$$(\psi_{\ell-\overline{\kappa_j}+1}\psi_{\ell-\overline{\kappa_j}}\psi_{\ell-\overline{\kappa_j}+1} - \psi_{\ell-\overline{\kappa_j}}\psi_{\ell-\overline{\kappa_j}+1}\psi_{\ell-\overline{\kappa_j}}) e(\mathbf{i}) = (x_{\ell-\overline{\kappa_j}} + x_{\ell-\overline{\kappa_j}+2}) e(\mathbf{i}).$$

Both sides of the above equation kill u and v, which completes our proof, as M is uniserial. □

Lemma 3.14. *Suppose that condition* (SS2) *holds,* $\overline{\kappa_j}$ *are distinct, but for some* $i \in I$, $\langle \Lambda, \alpha_{i,n}^\vee \rangle > 1$. *Then* $\mathscr{R}_n^\Lambda$ *is not semisimple.*

Proof. In spirit, the proof is the same as that of Lemma 3.12. Since we have assumed that condition (SS2) holds, $\ell \geqslant n-1$ and we may assume that $i = \overline{\kappa_j}$ for some $1 \leqslant j \leqslant l$, and for some $1 \leqslant j' \leqslant l$ and $1 \leqslant k \leqslant \ell - i - \frac{n-1}{2}$, $\overline{\kappa_{j'}} = i + k$.

We consider two cases – either $j < j'$ or $j > j'$. As in the proof of Lemma 3.12, we will in each case define a multipartition $\lambda \in \mathscr{P}_n^l$ and let $\mathtt{T}$ denote the least dominant standard λ-tableau, and will show that $v^\mathtt{T} = \psi_{w^\mathtt{T}} z^\lambda = \psi_1\psi_2 \dots \psi_{n-1} z^\lambda$ generates a one-dimensional submodule of $\mathcal{S}_\kappa^\lambda$.

First suppose that $j < j'$. Then we define $\lambda \in \mathscr{P}_n^l$ to be the multipartition with all components empty except components j and j', with $\lambda^{(j)} = (1^{n-k})$ and $\lambda^{(j')} = (1^k)$. Note that the standard λ-tableaux are uniquely determined by their residue sequences, by Lemma 3.1. Now it follows from Lemma 2.2 that all x and ψ generators except possibly ψ_k annihilate $v^\mathtt{T}$. We note that $\psi_{w^\mathtt{T}}$ is fully commutative, and has an expression starting with ψ_k. Let $\mathtt{S}$ denote the tableau $s_k\mathtt{T}$, so that $v^\mathtt{T} = \psi_k \psi_{w^\mathtt{S}} z^\lambda$. Then

$$\psi_k v^\mathtt{T} = \psi_k^2 \psi_{w^\mathtt{S}} z^\lambda = (x_k + x_{k+1}) \psi_{w^\mathtt{S}} z^\lambda = 0,$$

where the last equality follows from Lemma 2.2(i). We have proved that $v^\mathtt{T}$ generates a one-dimensional submodule of $\mathcal{S}_\kappa^\lambda$. As in the proof of Lemma 3.12, examining residues yields that this module is not a direct summand of $\mathcal{S}_\kappa^\lambda$.

Finally, if $k > 0$ and $j > j'$, we define $\lambda \in \mathscr{P}_n^l$ to be the multipartition with all components empty except components j and j', with $\lambda^{(j)} = (k)$ and $\lambda^{(j')} = (n-k)$. This case is almost identical to the other, and we leave the details to the reader. □

Combining Corollary 3.9 and Lemmas 3.11 to 3.14, we have proved our Main Theorem, Theorem 1.1.

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