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HELSON SETS OF SYNTHESIS ARE DITKIN SETS

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ABSTRACT. Let G be a locally compact group and let $A(G)$ be its Fourier algebra. A closed subset H of G is said to be a Helson set if the restriction homomorphism $\phi : A(G) \to C_0(H)$, $\phi(a) = a_{|H}$, is surjective. In this paper, under the hypothesis that G is amenable, we prove that every Helson subset H of G that is also a set of synthesis is a Ditkin set. This result is new even for $G = \mathbb{R}$.

INTRODUCTION

In this paper we present a proof of the result stated in the title. Let G be a locally compact amenable group and $A(G)$ its Fourier algebra as defined by Eymard [Ey]. The Fourier algebra $A(G)$ of G is a commutative, semisimple, regular Tauberian Banach function algebra on G with a bounded approximate identity. In the case where G is abelian and $\Gamma = \widehat{G}$ its dual group, the algebra $A(G)$ is isometrically isomorphic to the group algebra $L^1(\Gamma)$ of Γ under the Fourier transform. For the abelian groups, ample information on the algebras $A(G)$ and $L^1(\Gamma)$ can be found in the classical book [Ru] by Rudin.

To any closed subset E of G, the following two ideals are associated:

$$k(E) = \{a \in A(G) : a = 0 \text{ on } E\}$$

and

$$j(E) = \{a \in A(G) : \text{The support of } a \text{ is compact and disjoint from } E\}.$$

The closed ideals $J(E) = \overline{j(E)}$ and $k(E)$ are, respectively, the smallest and the largest closed ideals with hull E. When these two ideals coincide the set E is said to be a set of synthesis. A celebrated theorem due to Malliavin [Ma] states that every nondiscrete locally compact abelian group G contains a closed set that is not a set of synthesis for the algebra $A(G)$. The same is true in the nonabelian case too [Ka-La1]. If the following stronger condition:

$$\text{for each } a \in k(E),\ a \in \overline{aj(E)}$$

holds, then E is said to be a Ditkin set. Two outstanding unsolved problems in the subject are the following.

1. **Union Problem.** Is the union of two sets of synthesis a set of synthesis?
2. **S-Set-D-Set Problem.** Is every set of synthesis a Ditkin set?

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This paper is related, although in a very special case, to both problems. It is well known and easy to see that the union of a set of synthesis and a Ditkin set is a set of synthesis. In the rather old paper [Sa1, Theorem 4], Saeki proves that the union of a Helson set of synthesis and a set of synthesis is a set of synthesis. These two results led us to wonder whether every Helson set of synthesis is a Ditkin set. It has turned out that this is the case.

Helson sets are thin sets, but topologically they can be quite substantial. For instance, the infinite-dimensional torus $G = T^{\omega}$ contains a closed arc (a homeomorphic image of the interval $[0, 1]$), so it is a connected set, that is, a Kronecker set, and hence a Helson set [Ru, pp. 103 and 116]. On the other hand, every compact scattered set in G is a Ditkin set but need not be a Helson set [Ru, p. 117]. Both classes of sets, Ditkin sets and Helson sets, are closed under the finite union. For Helson sets this important result is due to Drury and Varopoulos ([Dr], [Va]). Moreover every closed subset of a Helson set is a Helson set. Let us mention here that a closed Helson set need not be a set of synthesis [Kö]. The main result of this paper shows that Helson sets of synthesis are hereditarily Ditkin sets. That is, every closed subset of a Helson set of synthesis is a Ditkin set. Except for the obvious class of the closed scattered sets, we do not know of any other class of hereditarily Ditkin sets.

In Section 1 we have gathered a few results and notation used in the paper. In Section 2 we present the proof of the main result. The paper is essentially self contained. Our main tool is the Arens multiplication on the Banach algebras $A(G)^{**}$ and $C_0(G)^{**}$.

1. Preliminaries

Our notation and terminology are standard. For any Banach space X, by X^* we denote its dual space. For $x \in X$ and $f \in X^*$, by $\langle x, f\rangle$ or $\langle f, x\rangle$ we denote the natural duality between X and X^*. For any subspace Y of X, by $Y^{\perp}$ we denote the annihilator of Y in X^*.

Arens product on A^{**}. Let A be a commutative Banach algebra. For $a \in A$ and $f \in A^*$, by $a.f$ we denote the functional on A defined by

$$\langle a.f, b\rangle = \langle f, ab\rangle.$$

It is immediate to see that $||a.f|| \leq ||a||.||f||$. We consider A^{**} as a Banach algebra equipped with the first Arens product, which is defined as follows. For a, b in A, f in A^*, and m, n in A^{**}, the product nm is defined in three steps by

$$\langle a, b.f\rangle = \langle ab, f\rangle, \langle a, n.f\rangle = \langle n, a.f\rangle$$

and

$$\langle mn, f\rangle = \langle m, n.f\rangle.$$

In the book [Da, Chapter 3] and the memoir [Da-La], the reader can find ample information on this notion. We note that, the algebra A being commutative, for $a \in A$ and $m \in A^{**}$, $am = ma$. We denote by Φ_A the Gelfand spectrum of A and by $\widehat{a}$ the Gelfand transform of a.

Sets of synthesis. Suppose now that the algebra A is semisimple, regular, Tauberian and, for $a \in A$, $a \in \overline{aA}$. This latter condition holds if, for instance, A has an approximate identity. The term Tauberian means that the ideal $A_c = \{a \in A : Supp(\widehat{a}) \text{ is compact}\}$ is dense in A. To any closed subset E of Φ_A, the ideals

$$k(E) = \{a \in A : \widehat{a} = 0 \text{ on } E\}$$

and

$$j(E) = \{a \in A : Supp(\widehat{a}) \text{ is compact and disjoint from } E\}$$

are associated. Let $J(E)$ be the closure in A of the ideal $j(E)$. The ideals $J(E)$ and $k(E)$ are, respectively, the smallest and the largest closed ideals with hull E. As in the case of the algebra $A(G)$, when the ideals $J(E)$ and $k(E)$ coincide, the set E is said to be a set of synthesis for the algebra A.

The spectrum (or support) of a functional f. Suppose again that the algebra A is semisimple, regular, Tauberian and is such that, for each $a \in A$, $a \in \overline{aA}$. For $f \in A^*$, the spectrum (or the support) $\sigma(f)$ of f can be defined in several ways. Below we state two of them. The set $\sigma(f)$ is a closed subset of the Gelfand spectrum of A, defined in the following equivalent ways. For more on this notion, see [Ru, Chapter 5] and [Ey, Proposition 4.4].

1. For $\gamma \in \Phi_A$, $\gamma \in \sigma(f)$ iff, for any $a \in A$, $a.f = 0$ implies that $\widehat{a}(\gamma) = 0$.
2. For $\gamma \in \Phi_A$, $\gamma \in \sigma(f)$ iff, for each neigbourhood V of γ, there is an $a \in A$ such that $Supp(\widehat{a}) \subseteq V$ and $\langle a, f\rangle \neq 0$.

The properties of the spectrum that we need are:

1. $\sigma(f) = \varnothing$ iff $f = 0$.
2. For any $a \in A$ and any $f \in A^*$, $\sigma(a.f) \subseteq \sigma(f) \cap Supp(\widehat{a})$.
3. For any closed subset E of Φ_A, $\sigma(f) \subseteq E$ iff $f \in J(E)^\perp$.
4. If E is a closed subset of Φ_A and if $(f_i)_{i\in I}$ is a weak* convergent net in A^* converging to some f, the inclusions $\sigma(f_i) \subseteq E$ for all $i \in I$ imply that $\sigma(f) \subseteq E$ too.

Let E a closed subset of Φ_A, let $a \in k(E)$, and let $f \in J(E)^\perp$. We note that, since for each $a \in A$, $a \in \overline{aA}$, the equality $a.f = 0$ implies that $\langle a, f\rangle = 0$. Conversely, if $a.f \neq 0$, then, for some $b \in A$, $\langle ab, f\rangle \neq 0$ so that $f \notin k(E)^\perp$. Thus E is a set of synthesis iff, for each $a \in k(E)$ and $f \in J(E)^\perp$, $a.f = 0$. We shall use this observation freely through the paper.

Fourier algebra $A(G)$. Concerning the Fourier algebra $A(G)$ and its dual $VN(G)$, our main references are Eymard's paper [Ey] and [Ka-La2]. Let G be a locally compact group and $A(G)$ its Fourier algebra. The dual space $VN(G)$ of $A(G)$ is the von Neumann algebra of G. This is the closure in the weak operator topology of the operator algebra $B(L^2(G))$ of the subspace generated by left translations operators $\ell_t : L^2(G) \to L^2(G)$, defined by $\ell_t(f)(s) = f(t^{-1}s)$, for $t \in G$. We shall denote the elements of $A(G)$ by the letters a, b and those of $A(G)^* = VN(G)$ by f, g. The Gelfand spectrum of $A(G)$ is (homeomorphic to) G. For $t \in G$, we shall denote the corresponding evaluation functional by ρ_t. Thus, for $a \in A(G)$, $\langle \rho_t, a\rangle = a(t)$.

Helson sets. A closed subset H of G is said to be a Helson set if the space $A(G)_{|H} = \{a_{|H} : a \in A(G)\}$ is the whole of the space $C_0(H)$, the space of the continuous functions $\varphi : H \to \mathbb{C}$ that vanish at infinity. The origin of this notion goes back to Helson's paper [He]. In the case where G is abelian, in Rudin's book [Ru, Chapter 5] the reader can find some examples of Helson sets. For instance, every Kronecker set is a Helson set. Certain (but not all) Cantor sets (the sets that are compact, metrizable, perfect, and totally disconnected) are also Helson sets [Ru, p. 100].

If H is a Helson, the restriction homomorphism $\phi : A(G) \to C_0(H)$, $\phi(a) = a_{|H}$, being onto, there is constant $\beta > 0$, a Helson constant of H, such that for any $\varphi \in C_0(H)$ there is an $a \in A(G)$ such that $a_{|H} = \varphi$ and $||a|| \leq \beta||\varphi||_\infty$.

In connection with the Helson constant, we recall the following result. Let X, Y be two Banach spaces and $T : X \to Y$ be a bounded onto linear operator. As a consequence of the Open Mapping Theorem, there is a constant $\beta > 0$ such that, given any $y \in Y$, there is an $x \in X$ such that $||x|| \leq \beta||y||$ and $T(x) = y$. The second adjoint T^{**} of T is also onto and, as one can easily see, given any $n \in Y^{**}$, there is an $m \in X^{**}$ such that $||m|| \leq \beta||n||$ (with the same constant β) and $T^{**}(m) = n$. In the proof of Lemma 2.2 below we use this fact.

In the paper [Sa2] the reader can find a characterization of the compact Helson sets of synthesis. Every closed subset of a Helson set of synthesis is a Helson set of synthesis; so is the union of the finitely many Helson sets of synthesis [Sa1]. As proved in [Sa3], the compact extremally disconnected subsets of G are Helson sets of synthesis. In [Sa2] and [Sa3] these are proved for abelian groups, but the proofs are also valid for the Fourier algebra $A(G)$ of a nonabelian group G. In connection with these results, see Corollary 2.6 below.

Throughout the paper G will be a locally compact amenable group.

Finally, we recall that, as proved by Leptin [Le], the algebra $A(G)$ has a bounded approximate identity if and only if the group G is amenable. We note that, the group G being amenable and the algebra $A(G)$ being Tauberian, the algebra $A(G)$ has a bounded approximate identity consisting of functions with compact supports.

2. Helson sets of synthesis are hereditary Ditkin

In this section we present the proof of the main result.

Let H be a locally compact space and let $C_0(H)$ be the commutative C^*-algebra of the complex-valued continuous functions on the locally compact space H vanishing at infinity. The second dual $C_0(H)^{**}$ of $C_0(H)$ is supposed to be equipped with the Arens product as defined in the preceding section. We note that, for $\mu \in M(H) = C_0(H)^*$, the spectrum of μ is the same as the support of the measure μ (i.e., $\sigma(\mu) = Supp(\mu)$). For any closed subset F of H, we regard the characteristic function χ_F of F as an element of $C_0(H)^{**}$. If $\mu \in M(H)$, then $\chi_F.\mu$ is just the pointwise product defined, for $\varphi \in C_0(H)$, by

$$\langle \varphi, \chi_F.\mu \rangle = \langle \varphi\chi_F, \mu \rangle = \int_\Omega \chi_F(t)\varphi(t)d\mu.$$

In particular, if $F = Supp(\mu)$, then $\chi_F.\mu = \mu$.

For later use, we record this as a lemma.

Lemma 2.1. *Let H be a locally compact space, let $\mu \in M(H)$ be a given measure and let $F = \sigma(\mu)$. Then, $\chi_F.\mu = \mu$.*

We need an analogue of this lemma for the algebra $A(G)$ in the case where F is a subset of a Helson set H. We first prove the following lemma.

Lemma 2.2. *Let $H \subseteq G$ be a Helson set with the Helson constant β. Then, for any closed subset F of H, there is an $m \in A(G)^{**}$ such that*

$$(\clubsuit) \qquad ||m|| \le \beta \text{ and, for } t \in H,\ \langle m, \rho_t\rangle = 1 \text{ if } t \in F; \quad \text{and } \langle m, \rho_t\rangle = 0 \text{ if } t \in H\backslash F.$$

Proof. Let $\phi : A(G) \to C_0(H)$ be the restriction homomorphism defined by $\phi(a) = a_{|H}$. Then $\ker(\varphi) = k(H)$ and the second adjoint ϕ^{**} maps $A(G)^{**}$ onto $C_0(H)^{**}$. Let χ_F be the characteristic function of the set F considered as a function on H. As χ_F is in $C_0(H)^{**}$ and $||\chi_F||_{C_0(H)^{**}} = 1$, there is an $m \in A(G)^{**}$ (unique modulo $k(H)^{**}$) such such $||m|| \le \beta$ and $\phi^{**}(m) = \chi_F$. Since for $t \in H$, $\phi^*(\rho_t) \in k(H)^{\perp}$, we have

$$\langle m, \rho_t\rangle = 1 \text{ if } t \in F \text{ and } \langle m, \rho_t\rangle = 0 \text{ if } t \in H\backslash F.$$

□

For any $m \in A(G)^{**}$ and any subset E of $G = \Phi_{A(G)}$, we write below

"$m = 1$ on E" instead of "for each $t \in E, \langle m, \rho_t\rangle = 1$".

Corollary 2.3. *Let $H \subseteq G$ be a Helson set of synthesis and let $f \in VN(G)$ be a functional with $F = \sigma(f) \subseteq H$. Then, for any $m \in A(G)^{**}$ such that $m = 1$ on F and $m = 0$ on $H\backslash F$, we have $m.f = f$.*

Proof. We first note that, H being a set of synthesis, $f \in k(H)^{\perp}$. Let $m \in A(G)^{**}$ be such that $m = 1$ on F and $m = 0$ on $H\backslash F$. As above, let $\phi : A(G) \to C_0(H)$ be the restriction homomorphism, $\phi(a) = a_{|H}$. The kernel of ϕ is $k(H)$ so that $\phi^*(M(H)) = k(H)^{\perp}$. Then, ϕ^* being one-to-one and $\phi^*(M(H)) = k(H)^{\perp}$, there is a unique $\mu \in M(H)$ such that $\phi^*(\mu) = f$. This implies that $\sigma(\mu) = F$; see [Ka-Ü, Proposition 4.1] for a more general result. The element $\phi^{**}(m)$ of $C_0(H)^{**}$, as a function on H, is just the characteristic function of the set F. So, as noted in Lemma 2.1, $\phi^{**}(m).\mu = \mu$. Hence, applying ϕ^* to the equality $\mu = \phi^{**}(m).\mu$ and using the fact that ϕ is a homomorphism, we get that

$$f = \phi^*(\mu) = \phi^*(\phi^{**}(m).\mu) = m.f.$$

Hence $m.f = f$. □

Next we "reduce" the spectrum of $f \in k(H)^{\perp}$.

Lemma 2.4. *Let $H \subseteq G$ be a Helson set of synthesis with Helson constant β, let $f \in k(H)^{\perp}$ and let $F \subseteq \sigma(f)$ be a nonempty closed set. Then, there is an $m \in A^{**}$ satisfying ($\clubsuit$) of Lemma 2.2 such that we have $\sigma(m.f) \subseteq F$.*

Proof. Let $\phi : A(G) \to C_0(H)$ be the restriction homomorphism. Then, for a uniquely determined $\mu \in M(H)$, $f = \phi^*(\mu)$. As seen above, $\sigma(f) = \sigma(\mu) = Supp(\mu)$. We first assume that F is compact. Let $(V_\alpha)_{\alpha\in I}$ be a downward directed (i.e., directed by the reverse inclusion) family of neighborhood system of F in G such that $\bigcap_{\alpha\in I} V_\alpha = \bigcap_{\alpha\in I} \overline{V_\alpha} = F$. Then $(V_\alpha \cap H)_{\alpha\in I}$ is a directed family of neighborhood system of F in H. For each $\alpha \in I$, let $\varphi_\alpha \in C_0(H)$ be a function such that $0 \le \varphi_a \le 1$ on H, $\varphi_\alpha = 1$ on F, and $Supp(\varphi_\alpha) \subseteq V_\alpha \cap H$. As H is a Helson set with Helson constant β, for each $\alpha \in I$, there is an a_α in $A(G)$ with

$||a_\alpha|| \le \beta$ such that $a_\alpha = \varphi_\alpha$ on H. In particular, $a_\alpha = 1$ on F. These, taking into account the facts that $f = \phi^*(\mu)$ and $a_{\alpha|H} = \varphi_\alpha$, imply that

$$\sigma(a_\alpha.f) \subseteq \sigma(f) \cap Supp(a_{\alpha|H}) \subseteq \sigma(f) \cap V_\alpha \subseteq V_\alpha.$$

Now fix an $\alpha_0 \in I$ arbitrarily. Since, for all $\alpha \ge \alpha_0$, $V_\alpha \subseteq V_{\alpha_0}$, the inclusion $\sigma(a_\alpha.f) \subseteq V_{\alpha_0}$ holds. Passing to a subnet, we can assume that the net $(a_\alpha)_{\alpha\in I}$ converges in the weak* topology of $A(G)^{**}$ to some m. This m satisfies (♣) and, since $a_\alpha.f \to m.f$ in the weak* topology of $A(G)^*$, $\sigma(m.f) \subseteq \overline{V_{\alpha_0}}$. This being valid for each $\alpha_0 \in I$ and $\bigcap_{\alpha\in I} V_\alpha = \bigcap_{\alpha\in I} \overline{V_\alpha} = F$, we conclude that $\sigma(m.f) \subseteq F$.

If the set F is not compact, let $(e_i)_{i\in I}$ be a bounded approximate identity such that the support of each e_i is compact. Let $F_i = F \cap Supp(e_i)$. Then $F_i \subseteq F \subseteq \sigma(f)$. So, as in the preceding paragraph, for each $i \in I$, we can find an $m_i \in A(G)^{**}$ such that $||m_i|| \le \beta$,

$$\text{for } t \in H,\ \langle m_i, \rho_t\rangle = 1 \text{ if } t \in F_i;\ \text{and } \langle m_i, \rho_t\rangle = 0 \text{ if } t \in H\backslash F,$$

and $\sigma(m_i.f) \subseteq F_i \subseteq F$. Let m be a weak* cluster point of the net $(m_i)_{i\in I}$ in $A(G)^{**}$. Then, since $||m|| \le \liminf ||m_i|| \le \beta$ and since $\bigcup_{i\in I} Supp(e_i) = G$, m satisfies (♣) and $\sigma(m.f) \subseteq F$. □

Before the main result, we would like to note that, since every finite (and also every closed scattered) subset of G is a Ditkin set and since the union of finitely many Ditkin sets is a Ditkin set, every closed subset F of G can be written as a union of an upward directed family of Ditkin sets. That is,

$$F = \bigcup_{\alpha\in I} D_\alpha,\ \text{each } D_\alpha \text{ is a Ditkin set, and, for } \alpha \le \beta, D_\alpha \subseteq D_\beta.$$

The next result is the main result of this paper.

Theorem 2.5. *Every Helson set of synthesis H in G is a hereditarily Ditkin set.*

Proof. Let H be a Helson set of synthesis with Helson constant β. We have to prove that, for any closed subset E of H and any $a \in k(E)$, we have

$$a \in \overline{aJ(E)}.$$

For a contradiction, suppose that, for a closed subset E of H and for some $a \in k(E)$, we have $a \notin \overline{aJ(E)}$. Then, there is an $f \in A(G)^*$ such that $\langle a, f\rangle \ne 0$ and $\sigma(a.f) \subseteq E$. Let $F = \sigma(a.f)$. We write F as a union of an upward directed family of nonempty Ditkin sets: $F = \bigcup_{\alpha\in I} D_\alpha$. Since, for each $\alpha \in I$, $D_\alpha \subseteq \sigma(a.f) \subseteq E \subseteq H$, by the preceding lemma, for any $\alpha \in I$, there is an element $m_\alpha \in A(G)^{**}$ such that $||m_\alpha|| \le \beta$,

$$\text{for } t \in H, \langle m_\alpha, \rho_t\rangle = 1 \text{ if } t \in D_\alpha;\ \text{and } \langle m_\alpha, \rho_t\rangle = 0 \text{ if } t \in H\backslash D_\alpha,$$

and $\sigma(m_\alpha a.f) \subseteq D_\alpha$. Since D_α is a Ditkin set and $a \in k(D_\alpha)$, there is a sequence $(b_n)_{n\in\mathbb{N}}$ in $j(D_\alpha)$ such that $||a - ab_n|| \to 0$. Then, as $b_n m_\alpha a.f = 0$, we conclude that $m_\alpha a.f = 0$. Let m be a weak* cluster point of the net $(m_\alpha)_{\alpha\in I}$ in $A(G)^{**}$. Then $ma.f = 0$ and m satisfies (♣). Since, by Corollary 2.3, $ma.f = a.f$, we see that $a.f = 0$. This implies that $\langle a, f\rangle = 0$. This contradiction proves the theorem. □

The following corollary is now obvious. In the case where G is abelian, see [Sa1] and [Sa3] for the assertions (c) and (d), respectively.

Corollary 2.6.

a) *If the boundary of a closed set $E \subseteq G$ is a Helson set of synthesis, then E is a Ditkin set.*

b) *A Helson set H is a set of synthesis iff, for each $a \in k(H)$ and $f \in J(H)^{\perp}$, the set $\sigma(a.f)$ is a set of synthesis.*

c) *The union of a set of synthesis and a Helson set of synthesis is a set of synthesis.*

d) *Every compact extremally disconnected subset of G is an hereditarily Ditkin set.*

Remark 2.7. The hypothesis that G is amenable is used only at two places: 1. in the proof of Lemma 2.4 to pass from the compact case to the noncompact case; 2. in the proof of Theorem 2.5 to deduce from $a.f = 0$ that $\langle a, f\rangle = 0$. If, instead of amenability, we assume that, for each $a \in A(G)$, $a \in \overline{aA(G)}$, then from $a.f = 0$ we can deduce that $\langle a, f\rangle = 0$. It seems that no group G is known for which this last condition fails. So, if we assume that H is compact and, for each $a \in A(G)$, $a \in \overline{aA(G)}$, we can drop the amenability hypothesis.

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