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Abstract

We provide a new proof of a recent theorem of Ben-Yaacov, Melleray, and Tsankov. If G is a Polish group and X is a minimal, metrizable G-flow with all orbits meager, then the universal minimal flow $M(G)$ is non-metrizable. In particular, we show that given X as above, the universal highly proximal extension of X is non-metrizable.

1 Introduction

In this paper, we are concerned with actions of a topological group G on a compact space X. All groups and spaces are assumed Hausdorff. A compact space X equipped with a continuous G-action $a : G \times X \to X$ is called a *G-flow*. The action a is often suppressed in the notation, i.e. gx is written for $a(g, x)$. A G-flow X is called *minimal* if every orbit is dense. It is a fact that every topological group G admits a *universal minimal flow* $M(G)$, a minimal flow which admits a G-map onto any other minimal flow. A *G-map* is a continuous map respecting the G-action. The flow $M(G)$ is unique up to G-flow isomorphism.

We can now recall the following theorem of Ben-Yaacov, Melleray, and Tsankov [4].

Theorem 1.1. *Let G be a Polish group, and let $M(G)$ be the universal minimal flow of G. If $M(G)$ is metrizable, then $M(G)$ has a comeager orbit.*

The question of whether or not metrizability of $M(G)$ was enough to guarantee a comeager orbit was first asked by Angel, Kechris, and Lyons [5]. In [6], the current author proved Theorem 1.1 in the case when G is the automorphism group of a first-order structure. The proof given there used topological properties of the largest G-ambit $S(G)$ along with combinatorial reasoning about the structures. In [4], the authors also use topological properties of $S(G)$, but the combinatorics is replaced by the following theorem due to Rosendal; see [4] for a proof.

Theorem 1.2. *Let G be a Polish group acting continuously on a compact metric space X. Assume the action is topologically transitive. Then the following are equivalent.*

1. *G has a comeager orbit.*
2. *For any open $1 \in V \subseteq G$ and any open $B \subseteq X$, there is open $C \subseteq B$ so that for any $D \subseteq C$, the set $C \setminus VD$ is nowhere dense.*

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It is proven in [5] that comeager orbits push forward; namely, if X is a minimal G-flow, $x \in X$ is a point whose orbit is generic, and if $\pi : X \to Y$ is a surjective G-map, then $\pi(x)$ has generic orbit in Y. Theorem 1.1 then becomes equivalent to the following: whenever G is a Polish group and X is a minimal metrizable flow with all orbits meager, then G must admit some minimal, non-metrizable flow. Remarkably, neither [4] nor [6] prove Theorem 1.1 in this direct fashion.

We provide a direct proof of Theorem 1.1. For any topological group G and any G-flow X, we construct a new G-flow denoted $S_G(X)$. We then show that if X is minimal, then so is $S_G(X)$. Lastly, if G is Polish and X is metrizable and has all orbits meager, we use Theorem 1.2 to show that $S_G(X)$ is non-metrizable.

After providing our new proof of 1.1, we investigate the flow $S_G(X)$ in more detail. For any G-flow X, there is a natural map $\pi_X : S_G(X) \to X$. When X is minimal, we show that π_X is the *universal highly proximal extension* of X. The notion of a highly proximal extension was introduced by Auslander and Glasner in [2]. If X and Y are minimal G-flows, a G-map $\varphi : Y \to X$ is *highly proximal* if for any $x \in X$ and non-empty open $U \subseteq Y$, there is $g \in G$ with $g\pi^{-1}(\{x\}) \subseteq U$. Auslander and Glasner prove in [2] that for every minimal G-flow X, there is a *universal highly proximal extension* $\pi : \widehat{X} \to X$. This means that π is highly proximal, and for every other highly proximal $\varphi : Y \to X$, there is a G-map $\psi : \widehat{X} \to Y$ so that $\pi = \varphi \circ \psi$. The map π is unique up to G-flow isomorphism over X. Our construction of the flow $S_G(X)$ provides a new construction of the universal highly proximal extension of X and hints at a generalization of this notion even when X is not minimal.

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2 The flow $S_G(X)$ and proof of Theorem 1.1

All groups and spaces will be assumed Hausdorff. In this section, fix a topological group G and a G-flow X. Write $\mathcal{N}_G$ for the collection of symmetric open neighborhoods of the identity in G, and write $\mathrm{op}(X)$ for the collection of nonempty open subsets of X.

Definition 2.1. A *near filter* is any $\mathcal{F} \subseteq \mathrm{op}(X)$ so that for any $A_1, ..., A_k \in \mathcal{F}$ and any $U \in \mathcal{N}_G$, we have $UA_1 \cap \cdots \cap UA_k \neq \emptyset$. A *near ultrafilter* is a maximal near filter.

Near ultrafilters exist by an application of Zorn's lemma. Near ultrafilters on a uniform space have been considered in [1] and [3]. Two aspects of our approach are slightly different. First, the notion of nearness is not given by the natural uniform structure on the compact Hausdorff space X. Second, instead of working with a notion of nearness on $\mathcal{P}(X)$, we are more or less working with the regular open algebra on X (see item (2) in Lemma 2.2).

Let $S_G(X)$ denote the space of near ultrafilters on $\mathrm{op}(X)$.

Lemma 2.2.

1. *Let $p \in S_G(X)$, and let $A \subseteq X$ be open. If $A \notin p$, then there is some $V \in \mathcal{N}_G$ with $VA \notin p$.*

2. *Let $A \subseteq X$ be open, and let $B_1, ..., B_k \subseteq A$ be open with $B_1 \cup \cdots \cup B_k$ dense in A. If $p \in S_G(X)$ and $A \in p$, then $B_i \in p$ for some $i \leq k$.*

Proof.

1. As $A \notin p$, find $B_1, ..., B_n \in p$ and $U \in \mathcal{N}_G$ with $UA \cap UB_1 \cap \cdots \cap UB_n = \emptyset$. Let $V \in \mathcal{N}_G$ with $VV \subseteq U$. Then $V(VA) \cap VB_1 \cap \cdots \cap VB_n = \emptyset$.

2. Towards a contradiction, assume $B_i \notin p$ for each $i \leq k$. For each $i \leq k$, find $B_1^i, ..., B_{n_i}^i \in p$ and a $U \in \mathcal{N}_G$ so that $UB_i \cap UB_1^i \cap \cdots \cap UB_{n_i}^i = \emptyset$. We can take the same $U \in \mathcal{N}_G$ for each $i \leq k$ by intersecting. Let $C = \bigcap_{i \leq k} \bigcap_{j \leq n_i} UB_j^i$. Then since $A \in p$, we have $UA \cap C \neq \emptyset$. Let $ga \in UA \cap C$, where $g \in U$ and $a \in A$. Since $UA \cap C$ is open, there is open $A' \subseteq A$ with $gA' \subseteq UA \cap C$. As $B_1 \cup \cdots \cup B_k$ is dense in A, there is some $i \leq k$ and some $b \in B_i$ with $gb \in UA \cap C$. Since $gb \in UB_i$, this is a contradiction. □

Definition 2.3. If $A \in \mathrm{op}(X)$, set $N_A := \{p \in S_G(X) : A \notin p\}$. We endow $S_G(X)$ with the topology whose typical basic open neighborhood is N_A for $A \in \mathrm{op}(X)$.

Proposition 2.4. *The topology from Definition 2.3 is compact Hausdorff.*

Proof. To show that $S_G(X)$ is Hausdorff, let $p \neq q \in S_G(X)$. Find some $A \in p \setminus q$. As $A \notin q$, find some $V \in \mathcal{N}_G$ so that $VA \notin q$. Set $B = \mathrm{int}(X \setminus VA)$. Then $B \notin p$. So $p \in N_B$, $q \in N_{VA}$, and $N_{VA} \cap N_B = \emptyset$.

To show that $S_G(X)$ is compact, suppose $\mathcal{C} := \{N_{A_i} : i \in I\}$ is a collection of basic open sets without a finite subcover. Then for any $i_1, ..., i_k \in I$, we can find $p \in \bigcap_{j \leq k} S_G(X) \setminus N_{A_{i_j}}$, equivalently, with $A_{i_1}, ..., A_{i_k} \in p$. But this implies that $\{A_i : i \in I\}$ is a near filter, and can be extended to a near ultrafilter q. Therefore $\mathcal{C}$ is not an open cover. □

Definition 2.5. If $p \in S_G(X)$ and $g \in G$, we let $gp \in S_G(X)$ be defined by declaring $A \in gp$ iff $g^{-1}A \in p$ for each $A \in \mathrm{op}(X)$.

Proposition 2.6. *The action in Definition 2.5 is continuous.*

Proof. First note that for a fixed $g \in G$, the map $p \to gp$ is continuous. So let $p_i, p \in S_G(X)$ and $g_i \in G$ with $p_i \to p$ and $g_i \to 1$. Suppose $A \notin p$. Find $V \in \mathcal{N}_G$ with $VA \notin p$. So eventually $VA \notin p_i$. Also, as $g_i \to 1$, eventually we have $g_i^{-1} \in V$. Whenever $g_i^{-1}A \subseteq VA$, we must have $g_i^{-1}A \notin p_i$. So eventually $A \notin g_i p_i$. □

Up until now, no assumptions on G and X have been needed. In fact, we did not even need X to be compact to construct $S_G(X)$. We now begin adding extra assumptions to G and X to obtain stronger conclusions about $S_G(X)$.

Proposition 2.7. *Suppose X is a minimal G-flow. Then so is $S_G(X)$.*

Proof. Let $p \in S_G(X)$, and let $A \in op(X)$ with $N_A \neq \emptyset$. Find some $V \in \mathcal{N}_G$ with $N_{VA} \neq \emptyset$. Then $B := \text{int}(X \setminus VA) \neq \emptyset$. As X is minimal, find $g_1, ..., g_k$ with $X = \bigcup_{i \leq k} g_i B$. For some $i \leq k$, we must have $g_i B \in p$. Then $B \in g_i^{-1}p$, so we must have $A \notin g_i^{-1}p$, and the orbit of p is dense as desired. □

Before proving Theorem 1.1, we need a sufficient criterion for when $S_G(X)$ is non-metrizable.

Proposition 2.8. *Suppose there are $\{A_n : n < \omega\} \subseteq \text{op}(X)$ and $V \in \mathcal{N}_G$ so that the collection $\{VA_n : n < \omega\}$ is pairwise disjoint. Then $S_G(X)$ is non-metrizable.*

Proof. If $S \subseteq \omega$, let $A_S = \bigcup_{n \in S} A_n$, and let $Y = \{p \in S_G(X) : A_\omega \in p\}$. Then $Y \subseteq S_G(X)$ is a closed subspace. To show that $S_G(X)$ is non-metrizable, we will exhibit a continuous surjection $\pi : Y \to \beta\omega$. First note that if $S \subseteq \omega$, then $VA_S \cap VA_{\omega \setminus S} = \emptyset$. Therefore, if $p \in Y$, p contains exactly one of A_S or $A_{\omega \setminus S}$ for each $S \subseteq \omega$. We let $\pi : Y \to \beta\omega$ be defined so that for $S \subseteq \omega$, $S \in \pi(p)$ iff $A_S \in p$. It is immediate that π is continuous. To see that π is surjective, let $q \in \beta\omega$. Then $\{A_S : S \in q\}$ is a near filter; any near ultrafilter p extending it is a member of Y with $\pi(p) = q$. □

Proof of Theorem 1.1. We now fix a Polish group G and a minimal G-flow X whose orbits are all meager. Then by Theorem 1.2, there is $U \in \mathcal{N}_G$ and open $B \subseteq X$ so that for any open $C \subseteq B$, there is open $D \subseteq C$ with $C \setminus UD$ somewhere dense (since C and UD are open, this is the same as $C \setminus UD$ having nonempty interior).

Let $V \in \mathcal{N}_G$ with $VV \subseteq U$. We now produce $\{A_n : n < \omega\} \subseteq \text{op}(X)$ with $\{VA_n : n < \omega\}$ pairwise disjoint. First set $B_0 = B$. As $B_0 \subseteq B$, there is $A_0 \subseteq B_0$ so that $B_0 \setminus UA_0$ has nonempty interior. Suppose open sets $B_0, ..., B_{n-1}$ and $A_0, ..., A_{n-1}$ have been produced so that $A_i \subseteq B_i$ and $\text{int}(B_i \setminus UA_i) \neq \emptyset$. We continue by setting $B_n = \text{int}(B_{n-1} \setminus UA_{n-1})$. As $B_n \subseteq B$, there is $A_n \subseteq B_n$ so that $B_n \setminus UA_n$ has nonempty interior. Notice that for any $m \leq n$, we also have $A_n \subseteq B_m$. It follows that if $m < n$, we have $UA_m \cap A_n = \emptyset$. This implies that $VA_m \cap VA_n = \emptyset$ as desired. We can now apply Proposition 2.8 to conclude that $S_G(X)$ is not metrizable. □

3 Universal highly proximal extensions

Let $\varphi : Y \to X$ be a G-map between minimal flows. There are several equivalent definitions which all say that φ is *highly proximal*. The definition we will use here is that φ is highly proximal iff every non-empty open $B \subseteq Y$ contains a fiber $\varphi^{-1}(\{x\})$ for some $x \in X$. Define the *fiber image* of B to be the set $\varphi_{fib}(B) := \{x \in X : \varphi^{-1}(\{x\}) \subseteq B\}$. Notice that $\varphi_{fib}(B)$ is open, and φ is highly proximal iff $\varphi_{fib}(B) \neq \emptyset$ for every non-empty open $B \subseteq Y$. It follows that this definition is the same as the one given in the introduction.

Now let X be a G-flow, and form $S_G(X)$. We define the map $\pi_X : S_G(X) \to X$ as follows. For each $p \in S_G(X)$, there is a unique $x_p \in X$ so that every neighborhood of x_p is in p. The existence of such a point is an easy consequence of the compactness of X and the second item of 2.2. For uniqueness, notice that if $x \neq y \in X$, we can find open

$A \ni x$, $B \ni y$ and $U \in \mathcal{N}_G$ with $UA \cap UB = \emptyset$. We set $\pi_X(p) = x_p$. This map clearly respects the G-action. To check continuity, one can check that if $K \subseteq X$ is closed, then $\pi_X^{-1}(K) = \{p \in S_G(X) : A \in p \text{ for every open } A \supseteq K\}$, and this is a closed condition.

Proposition 3.1. *Let X be minimal. Then the map $\pi_X : S_G(X) \to X$ is highly proximal.*

Proof. By 2.7, $S_G(X)$ is a minimal flow. So let $N_A \subseteq S_G(X)$ be a nonempty basic open neighborhood. This implies that $\text{int}(X \setminus A) \neq \emptyset$. Let $x \in \text{int}(X \setminus A)$. Then there are open $B \ni x$ and $U \in \mathcal{N}_G$ with $UB \cap A = \emptyset$. It follows that any $p \in S_G(X)$ containing B cannot contain A. In particular, we have $\pi_X^{-1}(\{x\}) \subseteq N_A$. □

Theorem 3.2. *Let X be minimal. Then the map $\pi_X : S_G(X) \to X$ is the universal highly proximal extension of X.*

Proof. Fix a highly proximal extension $\varphi : Y \to X$. For each $y \in Y$, let $\mathcal{F}_y := \{\varphi_{fib}(B) : B \ni y \text{ open}\}$. Then $\mathcal{F}_y \subseteq \text{op}(X)$ is a filter of open sets, so in particular it is a near filter. We will show that for each $p \in S_G(X)$, there is a unique $y \in Y$ with $\mathcal{F}_y \subseteq p$. This will define the map $\psi : S_G(X) \to Y$.

We first show that for each $p \in S_G(X)$, there is at least one such $y \in Y$. To the contrary, suppose for each $y \in Y$, there were $B_y \ni y$ open so that $\varphi_{fib}(B_y) \notin p$. Find $y_1, ..., y_k$ so that $\{B_{y_1}, ..., B_{y_k}\}$ is a finite subcover. Let $A_i = \varphi_{fib}(B_{y_i})$. Each A_i is open, so we will reach a contradiction once we show that $\bigcup_{i \leq k} A_i$ is dense. Let $A \subseteq X$ be open. Then $C := B_{y_i} \cap \varphi^{-1}(A) \neq \emptyset$ for some $i \leq k$. As C is open, $\varphi_{fib}(C) \neq \emptyset$, and $\varphi_{fib}(C) \subseteq A \cap A_i$.

Now we consider uniqueness. Let $p \in S_G(X)$, and consider $y \neq z \in Y$. Find open $B \ni y$ and $C \ni z$ and some $V \in \mathcal{N}_G$ so that $VB \cap VC = \emptyset$. It follows that $\varphi_{fib}(VB) \cap \varphi_{fib}(VC) = \emptyset$. Now notice that $V\varphi_{fib}(B) \subseteq \varphi_{fib}(VB)$, and likewise for C. Hence p cannot contain both $\mathcal{F}_y$ and $\mathcal{F}_z$.

The map ψ clearly respects the G-action and satisfies $\pi_X = \varphi \circ \psi$. To show continuity, let $K \subseteq Y$ be closed. Let $\mathcal{F}_K := \{\varphi_{fib}(B) : B \supseteq K \text{ open}\}$. We will show that $\psi(p) \in K$ iff $\mathcal{F}_K \subseteq p$. From this it follows that $\psi^{-1}(K)$ is closed. One direction is clear. For the other, suppose $\psi(p) = y \notin K$. Find open sets $B \ni y$, $C \supseteq K$, and $V \in \mathcal{N}_G$ with $VB \cap VC = \emptyset$. As in the proof of uniqueness, p cannot contain both $\mathcal{F}_y$ and $\mathcal{F}_K$. □

By combining the main results of the previous two sections, we obtain the following.

Corollary 3.3. *Let G be a Polish group, and let X be a minimal, metrizable G-flow with all orbits meager. Then the universal highly proximal extension of X is non-metrizable.*

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