

A ZERO-ONE LAW FOR IMPROVEMENTS TO DIRICHLET'S THEOREM

DMITRY KLEINBOCK AND NICK WADLEIGH

(Communicated by Nimish Shah)

ABSTRACT. We give an integrability condition on a function ψ guaranteeing that for almost all (or almost no) $x \in \mathbb{R}$, the system $|qx - p| < \psi(t)$, $|q| < t$ is solvable in $p \in \mathbb{Z}$, $q \in \mathbb{Z} \setminus \{0\}$ for sufficiently large t . Along the way, we characterize such x in terms of the growth of their continued fraction entries, and we establish that Dirichlet's Approximation Theorem is sharp in a very strong sense. Higher-dimensional generalizations are discussed at the end of the paper.

1. INTRODUCTION AND MOTIVATION

The starting point for the present paper, as well as for numerous endeavors in the theory of Diophantine approximation, is the following theorem, established by Dirichlet in 1842:

Theorem 1.1 (Dirichlet's Theorem). *For any $x \in \mathbb{R}$ and $t > 1$, there exist $q \in \mathbb{Z} \setminus \{0\}$, $p \in \mathbb{Z}$ such that*

$$(1.1) \quad |qx - p| \leq \frac{1}{t} \quad \text{and} \quad |q| < t.$$

See, e.g., [Ca1, Theorem I.I] or [Sc, Theorem I.1A]. In many cases the above theorem has been applied through its corollary [Sc, Corollary I.1B], predating Dirichlet's work:

Corollary 1.2 (Dirichlet's Corollary). *For any $x \in \mathbb{R}$ there exist infinitely many $q \in \mathbb{Z}$ such that*

$$(1.2) \quad |qx - p| < \frac{1}{|q|} \quad \text{for some } p \in \mathbb{Z}.$$

The two statements above give a rate of approximation which works for all x and serve as a beginning of the *metric theory of Diophantine approximation*, which is concerned with understanding sets of x satisfying conclusions similar to those of Theorem 1.1 and Corollary 1.2 with the right hand sides of (1.1) and (1.2) replaced by faster decaying functions of t and $|q|$ respectively. Those sets are very well studied in the setting of Corollary 1.2. Indeed, for a function $\psi : [t_0, \infty) \rightarrow \mathbb{R}_+$, where $t_0 \geq 1$ is fixed, let us define $W(\psi)$, the set of ψ -approximable real numbers, to be the set of $x \in \mathbb{R}$ for which there exist infinitely many $q \in \mathbb{Z}$ such that

$$(1.3) \quad |qx - p| < \psi(|q|) \quad \text{for some } p \in \mathbb{Z}.$$

Received by the editors September 22, 2016, and, in revised form, January 24, 2017.
2010 *Mathematics Subject Classification.* Primary 11J04, 11J70; Secondary 11J13, 37A17.
The first-named author was supported by NSF grants DMS-1101320 and DMS-1600814.

In what follows we will use the notation $\psi_1(t) = 1/t$. Then Corollary 1.2 asserts that $W(\psi_1) = \mathbb{R}$. It is well known that there exists $c > 0$ such that $W(c\psi_1) \neq \mathbb{R}$. In fact, numbers which do not belong to $W(c\psi_1)$ for some $c > 0$ (equivalently, irrational numbers whose continued fraction coefficients are uniformly bounded) are called *badly approximable*. It is known that such numbers form a set of full Hausdorff dimension [J]. However the Lebesgue measure of the set of badly approximable numbers is zero; in other words, $W(c\psi_1)$ is co-null for any $c > 0$. Precise conditions for the Lebesgue measure of $W(\psi)$ to be zero or full are given by

Theorem 1.3 (Khintchine's Theorem). *Given a non-increasing ψ , the set $W(\psi)$ has zero (resp. full) measure if and only if the series $\sum_k \psi(k)$ converges (resp. diverges).*

Quite surprisingly, it seems that no such clean statement has yet been proved in the set-up of Theorem 1.1. This is the aim of the present paper.

We start by introducing the following definition: for ψ as above, let $D(\psi)$ denote the set of $x \in \mathbb{R}$ for which the system

$$(1.4) \quad |qx - p| < \psi(t) \quad \text{and} \quad |q| < t$$

has a non-trivial integer solution for all large enough t . Elements of $D(\psi)$ will be called ψ -Dirichlet. Notice that this definition arises by replacing " $\leq \psi_1(t)$ " in (1.1) with " $< \psi(t)$ " and demanding the existence of non-trivial integer solutions for all t except those belonging to a bounded set. Here are some elementary observations:

- If ψ is non-increasing, which will be our standing assumption, one can without loss of generality restrict to $t \in \mathbb{N}$: indeed, to solve (1.4) it is enough to find a solution with t replaced by $\lceil t \rceil$.
- It is not hard to see that $D(\psi_1) = \mathbb{R}$; more precisely, if $x \notin \mathbb{Q}$ (resp. if $x \in \mathbb{Q}$), the system (1.4) with $\psi = \psi_1$ has a non-zero solution for all $t > 1$ (resp. for sufficiently large t).
- Clearly $D(\psi)$ is contained in $W(\psi)$ whenever ψ is non-increasing.

On the other hand, one knows that $D(\psi)$ and $W(\psi)$ differ significantly for functions ψ decaying faster than ψ_1 . For example, it has been observed by Davenport and Schmidt [DS1] that the set $D(c\psi_1)$ of $c\psi_1$ -Dirichlet numbers has Lebesgue measure zero for any $c < 1$. Moreover, they showed [DS1, Theorem 1] that an irrational number belongs to $D(c\psi_1)$ for some $c < 1$ if and only if it is badly approximable. Thus $x \in \bigcup_{c < 1} D(c\psi_1)$ if and only if the continued fraction coefficients of x are uniformly bounded. This naturally motivates the following questions:

Question 1.4. Can one characterize $x \in D(\psi)$ in terms of its continued fraction expansion?

Question 1.5. Is Dirichlet's Theorem sharp in the sense that if $\psi(t) < \psi_1(t)$ for all sufficiently large t , then there exists $x \in \mathbb{R}$ which is not ψ -Dirichlet?

Question 1.6. What is a necessary and sufficient condition on ψ (presumably, expressed in the form of convergence/divergence of a certain series) guaranteeing that the set $D(\psi)$ has zero/full measure?

In this paper we answer Questions 1.4 and 1.5 in the affirmative and give an answer to Question 1.6 under an additional assumption that the function $t \mapsto t\psi(t)$

is non-decreasing. Specifically, we will prove the following:

Theorem 1.7. *If $\psi : [t_0, \infty) \rightarrow \mathbb{R}_+$ is non-increasing and $\psi(t) < \psi_1(t)$ for sufficiently large t , then $D(\psi) \neq \mathbb{R}$.*

Theorem 1.8. *Let $\psi : [t_0, \infty) \rightarrow \mathbb{R}_+$ be non-increasing, and suppose the function $t \mapsto t\psi(t)$ is non-decreasing and*

$$(1.5) \quad t\psi(t) < 1 \quad \text{for all } t \geq t_0.$$

Then if

$$(1.6) \quad \sum_n \frac{-\log(1 - n\psi(n))(1 - n\psi(n))}{n} = \infty \quad (\text{resp. } < \infty),$$

then the Lebesgue measure of $D(\psi)$ (resp. of $D(\psi)^c$) is zero.

We note that (1.5) is a natural assumption: if it is not satisfied, then $D(\psi) = D(\psi_1) = \mathbb{R}$.

As an example, taking $\psi = c\psi_1$ in Theorem 1.8 makes the sum in (1.6) equal to

$$\sum_n \frac{-c \log(1 - c)}{n}.$$

Thus we recover the aforementioned result of Davenport and Schmidt stating that $D(c\psi_1)$ has measure zero for $c < 1$. Here are two more examples:¹

- If $\psi(t) = \frac{1-at^{-k}}{t}$ for $a > 0, k \geq 0$, then the sum in (1.6) converges/diverges if and only if so does

$$\int_1^\infty \frac{-\log(t^{-k})t^{-k}}{t} dt = \int_1^\infty \frac{k \log t}{t^{k+1}} dt.$$

Thus $D(\psi)$ has full measure whenever $k > 0$.

- If $\psi(t) = \frac{1-a(\log t)^{-k}}{t}$ for $a > 0, k \geq 0$, we are led to consider

$$\int_e^\infty \frac{k \log \log t}{t(\log t)^k} dt = \int_1^\infty \frac{k \log u}{u^k} du.$$

In this case $D(\psi)$ has full measure if $k > 1$ and zero measure otherwise.

The structure of the paper is as follows. Theorem 1.7 is proved in the next section, following some lemmas expressing the ψ -Dirichlet property via continued fractions. In §3 we discuss dynamics of the Gauss map $x \mapsto \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$ in the unit interval and, following [Ph], establish a dynamical Borel-Cantelli Lemma. This Borel-Cantelli Lemma is then used to prove Theorem 1.8. In the last section of the paper we discuss possible higher-dimensional generalizations.

¹The functions below are not non-increasing, but *eventually* decreasing; clearly only the eventual behavior of ψ is relevant.

2. CONTINUED FRACTIONS

Denote by $\langle x \rangle$ the distance from x to the nearest integer. Throughout the sequel, $a_n = a_n(x)$ ($n = 1, 2, \dots$) will denote the n th entry in the continued fraction expansion of $x \in [0, 1)$, and $q_n = q_n(x)$ will refer to the denominator of the n th convergent to x . That is,

$$[a_1(x), a_2(x), \dots, a_n(x)] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots + \frac{1}{a_n}}}} = \frac{p_n}{q_n}$$

with $p_n, q_n \in \mathbb{N}$ coprime. If we take $q_0 = 1$, $\{q_n\}_{n=0}^\infty$ may be defined as the increasing sequence of positive integers with the property $\langle q_n x \rangle < \langle qx \rangle$ for all positive integers $q < q_n$. The sequences $\{a_n\}, \{q_n\}$ are related by the recurrence

$$(2.1) \quad q_n = a_n q_{n-1} + q_{n-2}.$$

We refer the reader to [Kh] or the first chapter of [Ca] for background on the theory of continued fractions.²

We prefer to work with x for which the sequences $q_n(x), a_n(x)$ do not terminate; that is, we exclude the case $x \in \mathbb{Q}$. Since all the properties that concern us are invariant under translation by \mathbb{Z} , we will only consider $x \in [0, 1) \setminus \mathbb{Q}$.

Lemma 2.1. *Let $\psi : [t_0, \infty) \rightarrow \mathbb{R}_+$ be non-increasing. Then $x \in [0, 1) \setminus \mathbb{Q}$ is ψ -Dirichlet if and only if $\langle q_{n-1}x \rangle < \psi(q_n)$ for sufficiently large n .*

Proof. Suppose $x \in [0, 1) \setminus \mathbb{Q}$ is ψ -Dirichlet. Then for sufficiently large n there exists a positive integer, q , with $\langle qx \rangle < \psi(q_n)$, $q < q_n$. Since $\langle q_{n-1}x \rangle \leq \langle qx \rangle$ whenever $q < q_n$, we have $\langle q_{n-1}x \rangle < \psi(q_n)$ for sufficiently large n . Conversely, suppose $\langle q_{n-1}x \rangle < \psi(q_n)$ for $n \geq N$. Then for a real number $t > q_N$, write $q_{n-1} < t \leq q_n$. The inequality $\langle q_{n-1}x \rangle < \psi(t)$ follows since ψ is non-increasing. Thus x is ψ -Dirichlet. □

Lemma 2.1 is one step toward rephrasing the ψ -Dirichlet property of x in terms of the growth of the continued fraction entries, $a_n(x)$. For fixed $x = [a_1, a_2, \dots]$, consider the sequences

$$\theta_{n+1} = [a_{n+1}, a_{n+2}, \dots], \quad \phi_n = [a_n, a_{n-1}, \dots, a_1].$$

These are related to the sequences $q_n, \langle q_{n-1}x \rangle$ by the identity

$$(2.2) \quad (1 + \theta_{n+1} \phi_n)^{-1} = q_n \langle q_{n-1}x \rangle$$

(see [Ca1, §II.2]).³ This is our device for passing from Lemma 2.1 to continued fractions, allowing us to answer Question 1.4.

Lemma 2.2. *Let $x \in [0, 1) \setminus \mathbb{Q}$, and let $\psi : [t_0, \infty) \rightarrow \mathbb{R}_+$ be non-increasing with $t\psi(t) < 1$ for all $t \geq t_0$. Then*

- (i) x is ψ -Dirichlet if $a_{n+1}a_n \leq \frac{1}{4} \left((q_n \psi(q_n))^{-1} - 1 \right)^{-1}$ for all sufficiently large n .

²Note however that Cassels' definition of the sequence $\{q_n\}$ differs from that of Khintchine by one index. That is, Cassels has $q_1 = 1, q_2 = a_1, \dots$. We have adopted Khintchine's notation.

³In truth, Cassels' formula reads $(1 + \theta_{n+1} \phi_n)^{-1} = q_{n+1} \langle q_n x \rangle$ because his q_n 's are shifted by one index, as we have already noted.

(ii) x is not ψ -Dirichlet if $a_{n+1}a_n > \left((q_n\psi(q_n))^{-1} - 1 \right)^{-1}$ for infinitely many n .

Proof. Fix $x \in [0, 1] \setminus \mathbb{Q}$. Using (2.2), Lemma 2.1 becomes

$$(2.3) \quad x \in D(\psi) \text{ if and only if } (1 + \theta_{n+1}\phi_n)^{-1} < q_n\psi(q_n) \text{ for all large } n.$$

Since $(a_{n+1} + \frac{1}{a_{n+2}})(a_n + \frac{1}{a_{n-1}}) \leq 4a_{n+1}a_n$, we have

$$\begin{aligned} \left(1 + \frac{1}{a_{n+1}} \cdot \frac{1}{a_n} \right)^{-1} &< (1 + \theta_{n+1}\phi_n)^{-1} \\ &< \left(1 + \frac{1}{a_{n+1} + \frac{1}{a_{n+2}}} \cdot \frac{1}{a_n + \frac{1}{a_{n-1}}} \right)^{-1} \leq \left(1 + \frac{1}{4a_{n+1}a_n} \right)^{-1}. \end{aligned}$$

Hence from (2.3), $x \in D(\psi)$ if

$$\left(1 + \frac{1}{4a_{n+1}a_n} \right)^{-1} \leq q_n\psi(q_n)$$

for sufficiently large n . We get the first assertion of the lemma by solving for $a_n a_{n+1}$. Similarly, $x \notin D(\psi)$ if

$$\left(1 + \frac{1}{a_{n+1}a_n} \right)^{-1} > q_n\psi(q_n)$$

for an unbounded set of n . Solving for $a_n a_{n+1}$ gives the second assertion of the lemma. □

Remark 2.3. Lemma 2.2 generalizes the aforementioned result of Davenport and Schmidt [DS1, Theorem 1] stating that $x \in D(c\psi_1)$ for some $c < 1$ if and only if the sequence $a_n(x)$ is uniformly bounded.

Now we can answer Question 1.5 and exhibit real numbers which are not ψ -Dirichlet for any non-increasing ψ with $\psi(t) < \psi_1(t)$ for sufficiently large t .

Proof of Theorem 1.7. By the recurrence (2.1), q_n depends only on a_1, \dots, a_n . Since $t\psi(t) < 1$ for large enough t , we may construct $x = [a_1, a_2, \dots]$ by successively choosing a_{n+1} so that part (ii) of Lemma 2.2 is satisfied. □

Remark 2.4. We point out that for a given ψ , the proof of Theorem 1.7 is entirely constructive, since each q_n is determined recursively by the preceding choice of a_n . Also note that the proof constructs x such that the system (1.4) is insoluble when $t = q_n$ for all sufficiently large n – not just for infinitely many q_n .

3. BOREL-CANTELLI LEMMAS

For almost every x , we have reduced the ψ -Dirichlet property of x to the growth of its continued fraction entries. The *Gauss map*,

$$(3.1) \quad T : [0, 1] \setminus \mathbb{Q} \rightarrow [0, 1] \setminus \mathbb{Q}, \quad x \mapsto x^{-1} - [x^{-1}],$$

has the convenient property $T([a_1, a_2, a_3, \dots]) = [a_2, a_3, a_4, \dots]$, and it preserves the *Gauss measure*,

$$(3.2) \quad \mu(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x} dx.$$

We will use two results of Philipp [Ph] related to the mixing rate of T and the divergence case of the Borel-Cantelli Lemma.

Theorem 3.1 ([Ph, Theorem 2.3]). *Let $E_n, n \geq 1$, be a sequence of measurable sets in a probability space (X, ν) . Denote by $A(N, x)$ the number of integers $n \leq N$ such that $x \in E_n$. Put*

$$\phi(N) = \sum_{n \leq N} \nu(E_n).$$

Suppose that there exists a convergent series $\sum_{j \geq 1} C_j$ with $C_j \geq 0$ such that for all integers $m > n$ we have

$$(3.3) \quad \nu(E_n \cap E_m) \leq \nu(E_n)\nu(E_m) + \nu(E_m)C_{m-n}.$$

Then for any $\epsilon > 0$ one has

$$A(N, x) = \phi(N) + O_\epsilon(\phi^{1/2}(N) \log^{3/2+\epsilon} \phi(N))$$

for almost all x .

Remark 3.2. Since $|\nu(E_n \cap E_m) - \nu(E_n)\nu(E_m)| \leq \nu(E_m)$, Theorem 3.1 can be trivially strengthened: given any $\ell > 0$, the conclusion of the theorem holds provided the inequality (3.3) holds whenever $m > n + \ell$.

Theorem 3.3 ([Ph, Theorem 3.2]). *There exist constants $c_0 > 0$ and $0 < \gamma < 1$ with the following property. Fix $\mathbf{r} = (r_1, \dots, r_k) \in \mathbb{N}^k$, and write*

$$E_{\mathbf{r}} := \{x \in [0, 1] \setminus \mathbb{Q} : a_1(x) = r_1, a_2(x) = r_2, \dots, a_k(x) = r_k\}.$$

Let $F \subset [0, 1]$ be any measurable set. Then for all $n \geq 0$,

$$(3.4) \quad \left| \mu(E_{\mathbf{r}} \cap T^{-n-k}F) - \mu(E_{\mathbf{r}})\mu(F) \right| \leq c_0 \mu(E_{\mathbf{r}})\mu(F)\gamma^{\sqrt{n}}.$$

As Philipp observed, this estimate admits passing to unions:

Corollary 3.4. *Let c_0 and γ be as in Theorem 3.3. Let $F \subset [0, 1]$ be any measurable set. Fix $k \in \mathbb{N}$, and let $\mathcal{R} \subset \mathbb{N}^k$. Then (3.4) holds for all $n \geq 0$ when $E_{\mathbf{r}}$ is replaced with $\cup_{\mathbf{r} \in \mathcal{R}} E_{\mathbf{r}}$.*

Proof. We have

$$\begin{aligned} & \left| \mu \left(\bigcup_{\mathbf{r} \in \mathcal{R}} E_{\mathbf{r}} \cap T^{-n-k}F \right) - \mu \left(\bigcup_{\mathbf{r} \in \mathcal{R}} E_{\mathbf{r}} \right) \mu(F) \right| \\ &= \left| \sum_{\mathbf{r} \in \mathcal{R}} \mu(E_{\mathbf{r}} \cap T^{-n-k}F) - \mu(E_{\mathbf{r}})\mu(F) \right| \\ &\leq \sum_{\mathbf{r} \in \mathcal{R}} c_0 \mu(E_{\mathbf{r}})\mu(F)\gamma^{\sqrt{n}} = c_0 \mu \left(\bigcup_{\mathbf{r} \in \mathcal{R}} E_{\mathbf{r}} \right) \mu(F)\gamma^{\sqrt{n}}. \end{aligned}$$

□

We now combine the above statements to establish a quite general dynamical Borel-Cantelli Lemma:

Lemma 3.5. *Fix $k \in \mathbb{N}$. Suppose A_n ($n \in \mathbb{N}$) is a sequence of sets such that each A_n is a union of sets of the form $E_{\mathbf{r}}$, $\mathbf{r} \in \mathbb{N}^k$ ($E_{\mathbf{r}}$ as defined in Theorem 3.3). If $\sum_n \mu(A_n) = \infty$ (resp. $< \infty$), then for almost every (resp. almost no) $x \in [0, 1]$ one has $T^n(x) \in A_n$ for infinitely many n .*

Proof. The convergence case follows from the Borel-Cantelli Lemma and the fact that μ is T -invariant. Suppose $\sum_n \mu(A_n) = \infty$. For $m \geq n + k$ write

$$\begin{aligned} \mu(T^{-n}A_n \cap T^{-m}A_m) &= \mu(A_n \cap T^{-(m-n)}A_m) \\ &\leq \mu(A_n)\mu(A_m) + c_0\mu(A_m)\mu(A_n)\gamma^{\sqrt{m-n-k}} \\ &\leq \mu(A_n)\mu(A_m) + \mu(A_m)c_0\gamma^{\sqrt{m-n-k}} \\ &= \mu(T^{-n}A_n)\mu(T^{-m}A_m) + \mu(T^{-m}A_m)c_0\gamma^{\sqrt{m-n-k}} \end{aligned}$$

for c_0, γ as in Theorem 3.3. The sets $T^{-n}A_n$ therefore satisfy the condition of Theorem 3.1 (in light of Remark 3.2). By that theorem, $\sum_n \mu(T^{-n}A_n) = \infty$ guarantees that almost all x lie in $T^{-n}A_n$ for infinitely many n . \square

The above lemma can now be applied to describe real numbers which belong to infinitely many sets of the form $\{x : a_1(x)a_2(x) > \Psi(n)\}$:

Theorem 3.6. *Let $\Psi : \mathbb{N} \rightarrow [1, \infty)$ be any function with $\lim_{n \rightarrow \infty} \Psi(n) = \infty$. If*

$$\sum_n \frac{\log(\Psi(n))}{\Psi(n)} < \infty \text{ (resp. } = \infty),$$

then almost every (resp. almost no) $x \in [0, 1] \setminus \mathbb{Q}$ has

$$(3.5) \quad a_{n+1}(x)a_n(x) \leq \Psi(n)$$

for sufficiently large n .

Proof. Define

$$\begin{aligned} A_n &:= \{x : a_1(x)a_2(x) > \Psi(n)\} \\ &= \bigcup_{a=1}^{\infty} \bigcup_{b=\lfloor \frac{\Psi(n)}{a} \rfloor + 1}^{\infty} \left(\frac{1}{a + \frac{1}{b}}, \frac{1}{a + \frac{1}{b+1}} \right) = \bigcup_{a=1}^{\infty} \left(\frac{1}{a + \frac{1}{\lfloor \frac{\Psi(n)}{a} \rfloor + 1}}, \frac{1}{a} \right). \end{aligned}$$

Clearly $x \in [0, 1] \setminus \mathbb{Q}$ has $a_{n+1}a_n > \Psi(n)$ if and only if $T^{n-1}(x) \in A_n$, where T denotes the Gauss map (3.1). By Lemma 3.5, it suffices to show

$$c^{-1}\mu(A_n) \leq \frac{\log \Psi(n)}{\Psi(n)} \leq c\mu(A_n)$$

for some $c > 0$ for all large n . In fact, since $\frac{\lambda}{\log 2} \leq \mu \leq \frac{2\lambda}{\log 2}$, where λ is a Lebesgue measure on $[0, 1]$, it suffices to show

$$c^{-1}\lambda(A_n) \leq \frac{\log \Psi(n)}{\Psi(n)} \leq c\lambda(A_n).$$

We have

$$\begin{aligned} A_n &\subset \left(\bigcup_{a \leq \Psi(n)} \left(\frac{1}{a + \frac{a}{\Psi(n)}}, \frac{1}{a} \right) \right) \cup \left(\bigcup_{a > \Psi(n)} \left(\frac{1}{a+1}, \frac{1}{a} \right) \right) \\ &\subset \left(\bigcup_{a \leq \Psi(n)} \left(\frac{1}{a + \frac{a}{\Psi(n)}}, \frac{1}{a} \right) \right) \cup \left(0, \frac{1}{\Psi(n)} \right) \\ &= \left(\frac{1}{1 + \frac{1}{\Psi(n)}}, 1 \right) \cup \left(\bigcup_{a=2}^{\lfloor \Psi(n) \rfloor} \left(\frac{1}{a + \frac{a}{\Psi(n)}}, \frac{1}{a} \right) \right) \cup \left(0, \frac{1}{\Psi(n)} \right). \end{aligned}$$

So

$$\begin{aligned} \lambda(A_n) &\leq 1 - \frac{1}{1 + \frac{1}{\Psi(n)}} + \int_1^{\Psi(n)} \left(\frac{1}{a} - \frac{1}{a + \frac{a}{\Psi(n)}} \right) da + \frac{1}{\Psi(n)} \\ &= 1 - \frac{1}{1 + \frac{1}{\Psi(n)}} + \log \Psi(n) \left(1 - \frac{1}{1 + \frac{1}{\Psi(n)}} \right) + \frac{1}{\Psi(n)} \\ &= \frac{1}{\Psi(n) + 1} + \frac{\log \Psi(n)}{1 + \Psi(n)} + \frac{1}{\Psi(n)} \asymp \frac{\log \Psi(n)}{\Psi(n)}. \end{aligned}$$

To see the asymptotic lower bound, we start with

$$A_n \supset \bigcup_{a=1}^{\lfloor \Psi(n) \rfloor} \left(\frac{1}{a + \frac{1}{\Psi(n)+1}}, \frac{1}{a} \right).$$

Then

$$\begin{aligned} \lambda(A_n) &\geq \int_1^{\Psi(n)} \left(\frac{1}{a} - \frac{1}{a + \frac{1}{\Psi(n)+1}} \right) da = \int_1^{\Psi(n)} \frac{1}{a(a + \Psi(n) + 1)} da \\ &= \int_1^{\Psi(n)} \left(\frac{(\Psi(n) + 1)^{-1}}{a} - \frac{(\Psi(n) + 1)^{-1}}{\Psi(n) + a + 1} \right) da \\ &= \frac{\log \Psi(n)}{\Psi(n) + 1} + \frac{\log(\frac{\Psi(n)+2}{2\Psi(n)+1})}{\Psi(n) + 1} \asymp \frac{\log \Psi(n)}{\Psi(n)}. \end{aligned}$$

□

Comparing Theorem 3.6 with Lemma 2.2, one can see that in order to answer Question 1.6, one would need to replace the right hand side of (3.5) with a function depending on q_n . This can be easily achieved using known facts about the growth of $q_n(x)$ for almost all x .

Corollary 3.7. *Let $\Psi : \mathbb{N} \rightarrow [1, \infty)$ be a non-decreasing function with $\lim_{n \rightarrow \infty} \Psi(n) = \infty$. If*

$$(3.6) \quad \sum_n \frac{\log \Psi(n)}{n\Psi(n)} < \infty \text{ (resp. } = \infty),$$

then almost every (resp. almost no) $x \in [0, 1] \setminus \mathbb{Q}$ has $a_{n+1}(x)a_n(x) \leq \Psi(q_n(x))$ for sufficiently large n .

Proof. There exists $b > 1$ such that

$$(3.7) \quad \text{for every } x \notin \mathbb{Q}, \quad b^n \leq q_n(x) \text{ for all } n \geq 2,$$

(see [Kh, §4]). There also exists $B > b$ such that

$$(3.8) \quad \text{for almost every } x, \quad q_n(x) \leq B^n \text{ for all large enough } n,$$

(see [Kh, §14]). By using Cauchy’s condensation argument it is straightforward to see that

$$(3.9) \quad \sum_n \frac{\log \Psi(n)}{n\Psi(n)} = \infty \iff \sum_n \frac{\log \Psi(b^n)}{\Psi(b^n)} = \infty \iff \sum_n \frac{\log \Psi(B^n)}{\Psi(B^n)} = \infty.$$

Thus if the sum in (3.6) converges, Theorem 3.6 implies that almost every $x \in [0, 1] \setminus \mathbb{Q}$ has

$$a_{n+1}(x)a_n(x) \leq \Psi(b^n) \stackrel{(3.7)}{\leq} \Psi(q_n(x))$$

for sufficiently large n . Conversely, if the sum in (3.6) diverges, (3.9) and Theorem 3.6 imply that for almost every $x \in [0, 1]$, one has

$$a_{n+1}(x)a_n(x) > \Psi(B^n) \stackrel{(3.8)}{\geq} \Psi(q_n(x))$$

for infinitely many n . □

Remark 3.8. The proof of Corollary 3.7 also shows that, modulo a null set, it is possible to describe ψ -Dirichlet points in a way similar to Lemma 2.2, but with the bounds on $a_{n+1}a_n$ depending on n and not on q_n . Namely, with b, B as above, almost every x is ψ -Dirichlet if $a_{n+1}a_n \leq \frac{1}{4} \cdot ([b^n\psi(b^n)]^{-1} - 1)^{-1}$ for all sufficiently large n , and is not ψ -Dirichlet if $a_{n+1}a_n > ([B^n\psi(B^n)]^{-1} - 1)^{-1}$ for infinitely many n . Here we use the hypothesis of Theorem 1.8 that $t \mapsto t\psi(t)$ is non-decreasing.

We are now ready to characterize ψ such that $D(\psi)$ has zero/full measure.

Proof of Theorem 1.8. If $t\psi(t)$ is bounded away from 1, $D(\psi)$ is null since $D(c\psi_1)$ is null for any $c < 1$ [DS1]. We therefore assume $t\psi(t) \rightarrow 1$ as $t \rightarrow \infty$ (recall $t\psi(t)$ is assumed non-decreasing). Let us write $\Psi(t) := (1 - t\psi(t))^{-1}$. The sum in Theorem 1.8 becomes

$$(3.10) \quad \sum_n \frac{\log \Psi(n)}{n\Psi(n)}.$$

Note that this sum converges if and only if it converges when $\Psi(n)$ is replaced with $c\Psi(n)$ for any $c > 0$. Also note that $\Psi(n)$ is asymptotic to the function that appears in Lemma 2.2. That is,

$$(3.11) \quad \Psi(n) \asymp \left((n\psi(n))^{-1} - 1 \right)^{-1} \text{ as } n \rightarrow \infty.$$

Suppose the sum (3.10) converges. Then by Corollary 3.7, for any $\varepsilon > 0$, almost every x has

$$a_{n+1}(x)a_n(x) \leq \varepsilon\Psi(q_n(x))$$

for all large enough n . Thus Lemma 2.2(i) and the limit (3.11) imply that $D(\psi)$ has full measure. Conversely, suppose that (3.10) diverges. Then for any $M > 0$, almost every x has

$$a_{n+1}(x)a_n(x) > M\Psi(q_n(x))$$

for infinitely many n . Therefore Lemma 2.2(ii) and the limit (3.11) imply that $D(\psi)$ has measure zero. □

4. GENERALIZATIONS TO HIGHER DIMENSIONS

Let m, n be positive integers, and denote by $M_{m,n}$ the space of $m \times n$ matrices with real entries. The following is the general form of Dirichlet's Theorem on

simultaneous Diophantine approximation (see, e.g., [Ca1, §I.5] or [Sc, Theorem II.1E]):

Theorem 4.1. *For any $Y \in M_{m,n}$ and $t > 1$ there exist $\mathbf{q} = (q_1, \dots, q_n) \in \mathbb{Z}^n \setminus \{0\}$ and $\mathbf{p} = (p_1, \dots, p_m) \in \mathbb{Z}^m$ satisfying the following system of inequalities:*

$$(4.1) \quad \|Y\mathbf{q} - \mathbf{p}\| \leq t^{-n/m} \quad \text{and} \quad \|\mathbf{q}\| < t.$$

Here $\|\cdot\|$ stands for the norm on \mathbb{R}^k given by $\|\mathbf{x}\| = \max_{1 \leq i \leq k} |x_i|$.

Let $\psi : [t_0, \infty) \rightarrow \mathbb{R}_+$ be non-increasing. In analogy with the definition for $m = n = 1$, let us say that $Y \in M_{m,n}$ is ψ -Dirichlet, and write $Y \in D(\psi)$, if for every sufficiently large t one can find $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ and $\mathbf{p} \in \mathbb{Z}^m$ with

$$(4.2) \quad \|Y\mathbf{q} - \mathbf{p}\|^m < \psi(t) \quad \text{and} \quad \|\mathbf{q}\|^n < t.$$

Note that the sharpness result of Davenport and Schmidt mentioned in the introduction also holds in higher dimensions: the Lebesgue measure of $D(c\psi_1)$ is zero for any $c < 1$. See [DS2, Theorem 1] for the case $\min(m, n) = 1$, and [KWe, Theorem 4] for further generalizations. This naturally motivates higher-dimensional analogues of Questions 1.5 and 1.6:

Question 4.2. Is Theorem 4.1 sharp in the sense that if ψ is non-increasing and $\psi(t) < \psi_1(t)$ for all sufficiently large t , then there exists $Y \in M_{m,n}$ which is not ψ -Dirichlet?

Question 4.3. For fixed $m, n \in \mathbb{N}$, what is a necessary and sufficient condition on a non-increasing ψ (presumably, expressed in the form of convergence/divergence of a certain series) guaranteeing that the set $D(\psi) \subset M_{m,n}$ has zero/full measure?

In higher dimensions the machinery of continued fractions is no longer available. It is nonetheless still possible to restate the problem in terms of a shrinking target phenomenon in a dynamical system. This approach is based on ideas from [DS2] and [Da], and, in a more explicit form, on [KM, §8], where the Khintchine-Groshev Theorem (the natural higher-dimensional analogue of Theorem 1.3) is proved using a dynamical Borel-Cantelli Lemma for a diagonal flow on the space of unimodular lattices in \mathbb{R}^{m+n} . The starting point for the reduction is the ‘‘Dani Correspondence’’:

Lemma 4.4 ([KM, Lemma 8.3]). *Fix $m, n \in \mathbb{N}$ and $t_0 \geq 1$, and let $\psi : [t_0, \infty) \rightarrow \mathbb{R}_+$ be a continuous, non-increasing function. Then there exists a unique continuous function*

$$r : [s_0, \infty) \rightarrow \mathbb{R}, \quad \text{where } s_0 = \frac{m}{m+n} \log t_0 - \frac{m}{m+n} \log \psi(t_0),$$

such that the function $s \mapsto s - nr(s)$ is strictly increasing and unbounded, the function $s \mapsto s + mr(s)$ is non-decreasing, and

$$(4.3) \quad \psi(e^{s-nr(s)}) = e^{-s-mr(s)} \quad \text{for all } s \geq s_0.$$

Denote by X the space of unimodular lattices in \mathbb{R}^{m+n} , and define

$$\Delta : X \rightarrow \mathbb{R}, \quad \Lambda \mapsto -\log \inf_{\mathbf{v} \in \Lambda \setminus \{0\}} \|\mathbf{v}\|.$$

$X \cong \text{SL}_{m+n}(\mathbb{R})/\text{SL}_{m+n}(\mathbb{Z})$ is a non-compact homogeneous space. According to Mahler’s Compactness Criterion, a subset K of X is relatively compact if and only

if the restriction of Δ to K is bounded from above. Also, in view of Minkowski's Lemma, Δ is always bounded from below by 0. Furthermore,

$$(4.4) \quad K_0 := \Delta^{-1}(0)$$

is a union of finitely many compact submanifolds of X , whose structure is explicitly described by the Hajós-Minkowski Theorem (see [Ca2, §XI.1.3] or [Sh, Theorem 2.3]).

For $Y \in M_{m,n}$, define

$$\Lambda_Y := \begin{pmatrix} I_m & Y \\ 0 & I_n \end{pmatrix} \mathbb{Z}^{m+n} \in X.$$

Finally, define

$$g_s := \text{diag}(e^{s/m}, \dots, e^{s/m}, e^{-s/n}, \dots, e^{-s/n}),$$

where there are m copies of $e^{s/m}$ and n copies of $e^{-s/n}$. We may now rephrase the ψ -Dirichlet property of $Y \in M_{m,n}$ as a statement about the orbit of Λ_Y in the dynamical system (X, g_s) :

Proposition 4.5. *Fix positive integers m, n , and let $\psi : [t_0, \infty) \rightarrow \mathbb{R}_+$ be continuous, non-increasing and such that $\psi(t) < 1$ for large enough t . Let $r = r_\psi$ be as in Lemma 4.4. Then $Y \in D(\psi)$ if and only if*

$$\Delta(g_s \Lambda_Y) > r_\psi(s)$$

for all sufficiently large s

Proof. Recall that $Y \in D(\psi)$ if and only if for large enough t the system (4.2) has a solution (\mathbf{p}, \mathbf{q}) with $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ and $\mathbf{p} \in \mathbb{Z}^m$. If $\psi(t) < 1$, all solutions $(\mathbf{p}, \mathbf{q}) \neq 0$ to this system will have $\mathbf{q} \neq 0$. Since ψ is eventually less than 1, $Y \in D(\psi)$ if and only if (4.2) is solvable in $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^{m+n} \setminus \{0\}$ for sufficiently large t .

Since the function $s \mapsto s - nr(s)$ is increasing and unbounded, $Y \in D(\psi)$ if and only if for large enough s ,

$$\|Y\mathbf{q} - \mathbf{p}\|^m < \psi(e^{s-nr(s)}) = e^{-s-mr(s)} \quad \|\mathbf{q}\|^n < e^{s-nr(s)}$$

for some $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^{m+n} \setminus \{0\}$. This is equivalent to

$$e^{s/m} \|Y\mathbf{q} - \mathbf{p}\| < e^{-r(s)} \quad e^{-s/n} \|\mathbf{q}\| < e^{-r(s)},$$

which is the same as $\Delta(g_s \Lambda_Y) > r_\psi(s)$. □

Thus $Y \notin D(\psi)$ if and only if $g_s \Lambda_Y \in \Delta^{-1}([0, r_\psi(s)])$ for an unbounded set of $s \in \mathbb{R}_+$. For example, the choice $\psi = c\psi_1$ for $c < 1$ in view of (4.3) yields

$$r(s) \equiv r_c := \frac{1}{m+n} \log(1/c),$$

a constant function. That is, $Y \notin D(c\psi_1)$ if and only if $g_s \Lambda_Y \in \Delta^{-1}([0, r_c])$ for an unbounded set of $s \in \mathbb{R}_+$. Therefore the aforementioned fact that $D(c\psi_1)$ is null for any $c < 1$ follows from the ergodicity of the g_s -action on X and the set $\{\Lambda_Y : Y \in M_{m,n}\}$ being an unstable leaf for this action.

In general, the targets $\Delta^{-1}([0, r])$ are neighborhoods of the set K_0 as in (4.4). We are thus interested in whether these shrinking targets are hit at an unbounded set of times by trajectories of a measure-preserving flow. There are some technical obstructions, perhaps surmountable, to this approach to Questions 4.2 and 4.3. However, in a forthcoming paper [KWa] we use a similar approach to solve an

analogous inhomogeneous problem. Specifically, we establish a dynamical Borel-Cantelli Lemma for the flow g_s on the space of *affine* unimodular lattices in \mathbb{R}^{m+n} , and go on to prove the following result:

Theorem 4.6. *Let $\psi : [t_0, \infty) \rightarrow \mathbb{R}_+$ be non-increasing. If*

$$\sum_k \frac{1}{k^2 \psi(k)} < \infty \quad (\text{resp. } = \infty),$$

then for almost all (resp. almost no) pairs $Y \in M_{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, the system

$$\|Y\mathbf{q} + \mathbf{b} - \mathbf{p}\|^m < \psi(t) \quad \|\mathbf{q}\|^n < t$$

is solvable in integer vectors $\mathbf{q} \in \mathbb{Z}^n$ and $\mathbf{p} \in \mathbb{Z}^m$ for sufficiently large t .

ACKNOWLEDGMENTS

The authors are grateful to Mumtaz Hussain and Bao-Wei Wang for helpful discussions and to an anonymous referee for useful comments.

REFERENCES

- [Ca1] J. W. S. Cassels, *An introduction to Diophantine approximation*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 45, Cambridge University Press, New York, 1957. MR0087708
- [Ca2] J. W. S. Cassels, *An introduction to the geometry of numbers*, Die Grundlehren der mathematischen Wissenschaften, Band 99, Springer-Verlag, Berlin-New York, 1971.
- [Da] S. G. Dani, *Divergent trajectories of flows on homogeneous spaces and Diophantine approximation*, J. Reine Angew. Math. **359** (1985), 55–89, DOI 10.1515/crll.1985.359.55. MR794799
- [DS1] H. Davenport and Wolfgang M. Schmidt, *Dirichlet's theorem on diophantine approximation*, Symposia Mathematica, Vol. IV (INDAM, Rome, 1968/69), Academic Press, London, 1970, pp. 113–132. MR0272722
- [DS2] H. Davenport and W. M. Schmidt, *Dirichlet's theorem on diophantine approximation. II*, Acta Arith. **16** (1969/1970), 413–424. MR0279040
- [J] V. Jarník, *Zur metrischen Theorie der diophantischen Approximationen*, Prace Mat.-Fiz. (1928–1929), 91–106.
- [Kh] A. Ya. Khinchin, *Continued fractions*, The University of Chicago Press, Chicago, Ill.-London, 1964. MR0161833
- [KM] D. Y. Kleinbock and G. A. Margulis, *Logarithm laws for flows on homogeneous spaces*, Invent. Math. **138** (1999), no. 3, 451–494, DOI 10.1007/s002220050350. MR1719827
- [KWa] D. Kleinbock and N. Wadleigh, *An inhomogeneous Dirichlet's theorem via shrinking targets*, preprint, arXiv:1709:04082.
- [KWe] Dmitry Kleinbock and Barak Weiss, *Dirichlet's theorem on Diophantine approximation and homogeneous flows*, J. Mod. Dyn. **2** (2008), no. 1, 43–62. MR2366229
- [Ph] Walter Philipp, *Some metrical theorems in number theory*, Pacific J. Math. **20** (1967), 109–127. MR0205930
- [Sc] W.M. Schmidt, *Diophantine approximation*, Lecture Notes in Mathematics, vol. 785, Springer-Verlag, Berlin, 1980.
- [Sh] Nimish A. Shah, *Expanding translates of curves and Dirichlet-Minkowski theorem on linear forms*, J. Amer. Math. Soc. **23** (2010), no. 2, 563–589, DOI 10.1090/S0894-0347-09-00657-2. MR2601043

DEPARTMENT OF MATHEMATICS, BRANDEIS UNIVERSITY, WALTHAM, MASSACHUSETTS 02454-9110

Email address: kleinboc@brandeis.edu

DEPARTMENT OF MATHEMATICS, BRANDEIS UNIVERSITY, WALTHAM, MASSACHUSETTS 02454-9110

Email address: wadleigh@brandeis.edu