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# HOMEOMORPHISMS OF ČECH–STONE REMAINDERS: THE ZERO-DIMENSIONAL CASE

ILIJAS FARAH AND PAUL MCKENNEY

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Abstract. We prove, using a weakening of the Proper Forcing Axiom, that any homemomorphism between Čech–Stone remainders of any two locally compact, zero-dimensional Polish spaces is induced by a homeomorphism between their cocompact subspaces.

## 1. Introduction

The Čech–Stone remainder (also known as corona) $\beta X \setminus X$ of a topological space $X$ will be denoted $X^*$. A continuous map $\varphi : X^* \to Y^*$ is called *trivial* if there is a continuous $e : X \to Y$ such that $\varphi = e^*$, where $e^* = \beta e \setminus e$ and $\beta e$ is the unique continuous extension of $e$ to $\beta X$. It follows that two remainders $X^*$ and $Y^*$ are homeomorphic via a trivial map if and only if there are cocompact subspaces of $X$ and $Y$ which are themselves homeomorphic. In this paper we prove the following (see Section 2 for the definitions).

**Theorem 1.1.** *OCA and $MA_{\aleph_1}$ together imply that every homeomorphism between Čech–Stone remainders of locally compact, zero-dimensional, Polish spaces is trivial.*

This proves a special case of the rigidity conjecture that forcing axioms imply all homeomorphisms between Čech–Stone remainders of locally compact, noncompact Polish spaces are trivial (see [10], [9], [3]). In contrast, the Continuum Hypothesis (CH), implies that Čech–Stone remainders of locally compact, noncompact, zero-dimensional Polish spaces are homeomorphic. This is a consequence of Parovičenko's topological characterization of $\omega^*$ (see, e.g., [25]). Stone duality between compact, zero-dimensional, Hausdorff spaces and Boolean algebras of their clopen sets provides a model-theoretic reformulation of this malleability phenomenon. For a locally compact, non-compact Hausdorff space $X$ let $\mathcal{C}(X)$ denote the algebra of the clopen subsets of $X$ and let $\mathcal{K}(X)$ denote its ideal of compact-open sets. If $X$ and $Y$ are in addition zero-dimensional, then continuous maps from $X^*$ to $Y^*$ functorially correspond to Boolean algebra homomorphisms from $\mathcal{C}(Y)/\mathcal{K}(Y)$ into $\mathcal{C}(X)/\mathcal{K}(X)$. All of these algebras are elementarily equivalent and (assuming CH) saturated, and therefore isomorphic (see [6] for the details and an extension to not necessarily zero-dimensional spaces).[1]

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[1]There is a deeper metamathematical explanation of the effect of CH; see [31].

Back to rigidity, Theorem 1.1 belongs to a long line of results going back to Shelah's groundbreaking construction of an oracle-cc forcing extension of the universe in which all autohomeomorphisms of $\omega^*$ are trivial ([22]). Shelah's proof was recast in terms of forcing axioms PFA and OCA+$\mathrm{MA}_{\aleph_1}$ in [23] and [27], respectively. The latter axiom also implies that homeomorphisms between Čech–Stone remainders between countable locally compact spaces, as well as their arbitrary powers, are trivial ([9, §4]) as well as strong negations of Parovičenko's theorem ([5], [7]).

The interest in quotient rigidity results was rejuvenated by the discovery that the noncommutative analogue of 'are all automorphisms of $\omega^*$ (or of $\mathcal{P}(\omega)/\operatorname{fin}$) trivial?' was a prominent open problem in the theory of operator algebras. Motivated by their work on analytic K-homology, Brown, Douglas, and Fillmore asked whether the Calkin algebra associated with the separable, infinite-dimensional, complex Hilbert space has outer automorphisms ([2]). Like its commutative analogue, this question cannot be resolved in ZFC, with CH and OCA implying the opposite answers ([21], [11]). Other rigidity results in the setting of C*-algebras were proved for reduced products of the form $\prod_n A_n/\bigoplus_n A_n$ in the case when all $A_n$ are matrix algebras ([16], [15]), separable UHF algebras ([19]), or unital separable nuclear C*-algebras ([28], [20]).

A general rigidity conjecture for corona C*-algebras was stated and partially verified in [3]. The model theory of coronas proved to be a bit more complex than that of Boolean algebras. While the reduced products are countably saturated ([14]), coronas possess only a modest degree of saturation ([12], [8], [30], [13]). In return, C*-algebras provided a vantage point that resulted in the construction of nontrivial autohomeomorphisms of $X^*$ for every noncompact, locally compact, metrizable manifold using CH ([29]).[2]

We note that Theorem 1.1 is not optimal. The first author's proof that all zero-dimensional, locally compact, Polish spaces satisfy the *weak extension principle* ([9, Theorem 4.10.1]) will appear elsewhere. Dow refuted the related *strong extension principle* ([9, Question 4.11.4]) by constructing a nontrivial continuous map from $\omega^*$ into $\omega^*$ (i.e., one that does not have a continuous extension to a map from $\beta\omega$ into $\beta\omega$) in ZFC ([4]). An alternative proof of our main result from a stronger assumption (PFA) is given in [14, Theorem 4.3].

In Section 2 we introduce some of the language required to prove Theorem 1.1. Section 3 treats embeddings of $\mathcal{P}(\omega)/\operatorname{fin}$ into $\mathcal{C}(X)/\mathcal{K}(X)$, and we show that under OCA +$\mathrm{MA}_{\aleph_1}$, every such embedding is trivial. Much of the proof follows the work in [27] and [26] with only minor modifications, so to avoid treading the same ground we only prove one of the ingredients going into this theorem. Section 4 completes the proof of Theorem 1.1 through an analysis of *coherent families of continuous functions.*

## 2. Notation

Our terminology is standard (see [18]). The assumption of Theorem 1.1 is a consequence of the *Proper Forcing Axiom*, PFA. OCA abbreviates the *Open Coloring Axiom* ([24]; not to be confused with the eponymous OCA of [1]), and $\mathrm{MA}_{\aleph_1}$ refers to Martin's Axiom for $\aleph_1$ dense sets.

If $E$ is a set, then $[E]^2$ will denote the set of unordered pairs from $E$. If $M \subseteq [E]^2$, then a set $H \subseteq E$ is called *$M$-homogeneous* if $[H]^2 \subseteq M$. The *Open Coloring Axiom*

[2] The only previously known case was $X = \mathbb{R}$; see [17] and [14].

states: for every separable metric space $E$ and every partition $[E]^2 = M_0 \cup M_1$ such that $M_0$ is open (here we identify $[E]^2$ with a symmetric subset of $E \times E$ minus the diagonal), either

(1) there is an uncountable $M_0$-homogeneous set, or
(2) there is a cover of $E$ by countably-many $M_1$-homogeneous sets.

We fix a zero-dimensional, locally compact and noncompact Polish space $X$. Let $\langle K_n \mid n < \omega \rangle$ be an increasing sequence of compact-open sets in $X$, such that $X = \bigcup K_n$. Then $\mathcal{K}(X)$ is generated by $\langle K_n \mid n < \omega \rangle$ since

$$K \in \mathcal{K}(X) \iff \exists n\, K \subseteq K_n.$$

It is easy to see that $\mathcal{C}(X)$ has size continuum, whereas $\mathcal{K}(X)$ is countable. When $A, B \in \mathcal{C}(X)$ are distinct, we write $\delta(A,B)$ for the least $n$ such that $A \cap X_n \neq B \cap X_n$. If

$$d(A,B) = \begin{cases} 2^{-\delta(A,B)}, & A \neq B, \\ 0, & A = B, \end{cases}$$

then $d$ is a Polish metric on $\mathcal{C}(X)$.

Let $X_0 = K_0$ and $X_{n+1} = K_{n+1} \setminus K_n$. We will often identify $\mathcal{C}(X)$ with $\prod_n \mathcal{C}(X_n)$, and $\mathcal{P}(\omega)$ with ${}^{\omega}2$. Under these identifications, $\mathcal{K}(X)$ maps to $\bigoplus_n \mathcal{C}(X_n)$ (the set of functions in $\prod_n \mathcal{C}(X_n)$ which are nonempty on only finitely many coordinates) and fin to ${}^{<\omega}2$. If $Y$ and $Z$ are zero-dimensional, locally compact Polish spaces, $\varphi : \mathcal{C}(Y)/\mathcal{K}(Y) \to \mathcal{C}(Z)/\mathcal{K}(Z)$ is a homomorphism, and $U \in \mathcal{C}(Y)$, then we write $\varphi \restriction U$ for the restriction $\varphi \restriction \mathcal{C}(U)/\mathcal{K}(U)$. When working with the quotient $\mathcal{C}(X)/\mathcal{K}(X)$ we will write $[A]$ for the equivalence class of some $A \in \mathcal{C}(X)$.

## 3. Embeddings of $\mathcal{P}(\omega)/\operatorname{fin}$ into $\mathcal{C}(X)/\mathcal{K}(X)$

Let $e : X \to \omega$ be a continuous map. If $e^{-1}(n)$ is compact for every $n$, then we say $e$ is *compact-to-one*. If $e$ is compact-to-one, then the map $a \mapsto e^{-1}(a)$, from $\mathcal{P}(\omega)$ to $\mathcal{C}(X)$, induces a homomorphism $\varphi_e : \mathcal{P}(\omega)/\operatorname{fin} \to \mathcal{C}(X)/\mathcal{K}(X)$. Moreover, $\varphi_e$ is injective if and only if $e$ is bounded on compact sets. We call a homomorphism $\varphi : \mathcal{P}(\omega)/\operatorname{fin} \to \mathcal{C}(X)/\mathcal{K}(X)$ *trivial* if it is of the form $\varphi_e$ for some compact-to-one, continuous $e$.

In this section we prove

**Theorem 3.1.** *Assume OCA+MA$_{\aleph_1}$, and suppose*

$$\varphi : \mathcal{P}(\omega)/\operatorname{fin} \to \mathcal{C}(X)/\mathcal{K}(X)$$

*is an injective homomorphism. Then $\varphi$ is trivial.*

Working towards the proof of Theorem 3.1, we fix an injective homomorphism $\varphi : \mathcal{P}(\omega)/\operatorname{fin} \to \mathcal{C}(X)/\mathcal{K}(X)$ and we define

$$\mathcal{I} = \{a \subseteq \omega \mid \varphi \restriction a \text{ is trivial}\}.$$

Note that $\mathcal{I}$ is an ideal on $\omega$.

A family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is called *almost disjoint* if for all distinct $a, b \in \mathcal{A}$, $a \cap b =^* \emptyset$. Such a family $\mathcal{A}$ is called *treelike* if there is some tree $T$ on $\omega$ and a bijection $t : \omega \to {}^{<\omega}\omega$ under which each $a \in \mathcal{A}$ corresponds to a branch through $T$, and vice versa. The following lemma is proven in [27].

**Lemma 3.2.** *Assume MA$_{\aleph_1}$. Then for every uncountable almost-disjoint family $\mathcal{A}$ of subsets of $\omega$ we may find an uncountable $\mathcal{B} \subseteq \mathcal{A}$ and partitions $b = b_0 \cup b_1$ for $b \in \mathcal{B}$ such that each family $\mathcal{B}_i = \{b_i \mid b \in \mathcal{B}\}$ is treelike.*

The following three lemmas do not directly follow from the work in [27], but their proofs are nearly the same, modulo some minor modifications. Recall that an ideal $\mathcal{J} \subseteq \mathcal{P}(\omega)$ is a *P-ideal* if for each countable sequence $A_n \in \mathcal{J}$ $(n < \omega)$ there is an $A \in \mathcal{J}$ such that for all $n < \omega$, $A_n \subseteq^* A$.

**Lemma 3.3.** *Assume OCA+MA*$_{\aleph_1}$*. If $\mathcal{I}$ is a dense P-ideal, then $\varphi$ is trivial.*

**Lemma 3.4.** *Assume $\mathfrak{b} > \aleph_1$. If $\mathcal{I}$ is not a dense P-ideal, then there is an uncountable almost disjoint family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ which is disjoint from $\mathcal{I}$.*

**Lemma 3.5.** *Assume OCA. Let $\mathcal{A}$ be an uncountable, treelike, almost-disjoint family of subsets of $\omega$. Then $\mathcal{I} \setminus \mathcal{A}$ is countable.*

Theorem 3.1 now follows from a straightforward combination of Lemmas 3.2, 3.3, 3.4, and 3.5. To illustrate the kind of modifications necessary in translating from [27], we will give a proof of Lemma 3.3.

*Proof of Lemma* 3.3. For each $a \in \mathcal{I}$, we fix $Z_a \in \mathcal{C}(X)$ and a continuous, compact-to-one map $e_a : Z_a \to a$ such that $\varphi([a]) = [Z_a]$ and for all $b \subseteq a$, $\varphi([b]) = [e_a^{-1}(b)]$. We define $f_a : \omega \to \mathcal{C}(X)$ by

$$f_a(n) = e_a^{-1}(\{n\}).$$

Define a partition $[\mathcal{I}]^2 = M_0 \cup M_1$ by placing $\{a, b\} \in M_0$ if and only if there is some $n \in a \cap b$ such that $f_a(n) \neq f_b(n)$. Then $M_0$ is open when $\mathcal{I}$ is given the topology obtained by identifying $a \in \mathcal{I}$ with $(a, f_a) \in \mathcal{P}(\omega) \times {}^{\omega}\mathcal{C}(X)$.

*Claim* 3.6. There is no uncountable, $M_0$-homogeneous subset $H$ of $\mathcal{I}$.

*Proof.* Assume $H$ is such a set, and that $|H| = \aleph_1$. Since $\mathcal{I}$ is a P-ideal, there is a set $\bar{H} \subseteq \mathcal{I}$ such that for every $a \in H$ there is some $b \in \bar{H}$ with $a \subseteq^* b$, and moreover $\bar{H}$ is a chain of order-type $\omega_1$ with respect to $\subseteq^*$. By OCA, there is an uncountable subset of $\bar{H}$ which is homogeneous for one of the two colors $M_0$ and $M_1$; hence, by passing to this subset, we may assume $\bar{H}$ is either $M_0$ or $M_1$ homogeneous.

Say $\bar{H}$ is $M_1$-homogeneous. Put $\bar{a} = \bigcup \bar{H}$, and $\bar{f} = \bigcup_{a \in \bar{H}} f_a$. Then $\bar{f} : \bar{a} \to \mathcal{C}(X)$, and for all $a \in H$ we have $a \subseteq^* \bar{a}$ and $f_{\bar{a}} \restriction (a \cap \bar{a}) =^* f_a \restriction (a \cap \bar{a})$. Choose $n$ so that for uncountably many $a \in H$, we have $a \setminus n \subseteq \bar{a}$, and $f_{\bar{a}} \restriction a \setminus n = f_a \restriction a \setminus n$. Then if $a, b \in H$ are such, and $f_a \restriction n = f_b \restriction n$, we have $\{a, b\} \in M_1$, a contradiction.

So $\bar{H}$ is $M_0$-homogeneous. Define a poset $\mathbb{P}$ as follows. Put $p \in \mathbb{P}$ if and only if $p = (A_p, m_p, H_p)$ where $m_p < \omega$, $A_p \in \mathcal{C}(K_{m_p})$, and $H_p \in [\bar{H}]^{<\omega}$, and for all distinct $a, b \in H_p$, there is an $n \in a \cap b$ such that

$$\neg(f_a(n) \cap A_p = \emptyset \iff f_b(n) \cap A_p = \emptyset).$$

That is, one of $f_a(n)$, $f_b(n)$ is disjoint from $A_p$, and the other isn't. Put $p \leq q$ if and only if $m_p \geq m_q$, $A_p \cap K_{m_q} = A_q$, and $H_p \supseteq H_q$.

First we must show that $\mathbb{P}$ is ccc. Suppose $\mathcal{X}$ is an uncountable subset of $\mathbb{P}$. We may assume without loss of generality that for some fixed $m$ and $A \in \mathcal{C}(K_m)$, and for all $p \in \mathcal{X}$, $m_p = m$ and $A_p = A$, and moreover that $H_p$ is the same size for all $p \in \mathcal{X}$. Let $a_p$ be the minimal element of $H_p$ under $\subseteq^*$, for each $p \in \mathcal{X}$. Find $n_p$ so that for all $a \in H_p$,

$$f_{a_p} \restriction (a_p \setminus n_p) \subseteq f_a, \qquad e''_{a_p} K_m \subseteq n_p.$$

We may assume that for some fixed $n$, we have $n_p = n$ for all $p \in \mathcal{X}$. Find $p, q \in \mathcal{X}$ with $f_{a_p} \restriction n = f_{a_q} \restriction n$. Since $\{a_p, a_q\} \in M_0$, there is some $k \in a_p \cap a_q$ such that

$f_{a_p}(k) \neq f_{a_q}(k)$. Then $k \geq n$, and so $f_{a_p}(k) \cap K_m = f_{a_q}(k) \cap K_m = \emptyset$. At least one of $f_{a_p}(k) \setminus f_{a_q}(k)$ and $f_{a_q}(k) \setminus f_{a_p}(k)$ must be nonempty; whichever one it is, call it $B$. Put $A_r = A \cup B$ and $H_r = H_p \cup H_q$, and choose $m_r$ large enough that $A_r \subseteq K_{m_r}$. Then $r = (A_r, m_r, H_r) \in \mathbb{P}$, and $r \leq p, q$.

By $\mathrm{MA}_{\aleph_1}$, there is a set $A \in \mathcal{C}(X)$ and an uncountable $H^* \subseteq \bar{H}$ such that for all distinct $a, b \in H^*$,

$$\exists n \in a \cap b, \quad \neg(f_a(n) \cap A = \emptyset \iff f_b(n) \cap A = \emptyset).$$

Fix $x \subseteq \omega$ such that $F(x) = A$. Then for all $a \in H^*$, $e_a^{-1}(x \cap a)\Delta(A \cap F(a))$ is compact; hence there are $k_a$ and $m_a$ such that

$$e_a^{-1}(x \cap a \setminus k_a) = (A \cap F(a)) \setminus K_{m_a} \qquad \text{and} \qquad e_a^{-1}(a \setminus k_a) = F(a) \setminus K_{m_a}.$$

Then, for all $n \in a \setminus k_a$, $n \in x$ implies $f_a(n) \subseteq A$, and $n \notin x$ implies $f_a(n) \cap A = \emptyset$. Fix distinct $a, b \in H^*$ with $k_a = k_b = k$, and $f_a \restriction k = f_b \restriction k$. Then,

$$\forall n \in a \cap b \; (f_a(n) \cap A = \emptyset \iff f_b(n) \cap A = \emptyset).$$

This contradicts the choice of $A$. □

By OCA, there is a cover of $\mathcal{I}$ by countably many sets $\mathcal{I}_n$, each of which is $M_1$-homogeneous. Since $\mathcal{I}$ is a P-ideal, at least one of the $\mathcal{I}_n$'s must be cofinal in $\mathcal{I}$ with respect to $\subseteq^*$. Choose such an $\mathcal{I}_n$, and let $f = \bigcup\{f_a \mid a \in \mathcal{I}_n\}$. Then $f$ is a function from some subset of $\omega$ to $\mathcal{C}(X)$. Setting $e(x) = n$ if and only if $x \in f(n)$, we get a function $e : X \to \omega$, and since $\mathcal{I}$ is dense and $\mathcal{I}_n$ cofinal in $\mathcal{I}$, $a \mapsto e^{-1}(a)$ witnesses that $\varphi$ is trivial. □

## 4. Coherent families of continuous functions

**Theorem 4.1.** *Assume OCA+$\mathrm{MA}_{\aleph_1}$. Let $X$ and $Y$ be zero-dimensional, locally compact Polish spaces, and let $\varphi : \mathcal{C}(Y)/\mathcal{K}(Y) \to \mathcal{C}(X)/\mathcal{K}(X)$ be an isomorphism. Then there are compact-open $K \subseteq X$ and $L \subseteq Y$, and a homeomorphism $e : X \setminus K \to Y \setminus L$, such that for all $A \in \mathcal{C}(Y \setminus L)$, $\varphi([A]) = [e^{-1}(A)]$.*

By Stone duality, a homeomorphism $\varphi : X^* \to Y^*$ induces an isomorphism $\hat{\varphi} : \mathcal{C}(Y)/\mathcal{K}(Y) \to \mathcal{C}(X)/\mathcal{K}(X)$, and any map $e$ as in the conclusion to Theorem 4.1 will in this case be a witness to the triviality of $\varphi$. Hence Theorem 4.1 implies Theorem 1.1. Before proving Theorem 4.1 we note a corollary involving definable isomorphisms.

**Corollary 4.2.** *Suppose $X$ and $Y$ are zero-dimensional, locally compact, Polish spaces, and $\varphi : \mathcal{C}(Y)/\mathcal{K}(Y) \to \mathcal{C}(X)/\mathcal{K}(X)$ is an isomorphism such that the set*

$$\Gamma = \{(A, B) \in \mathcal{C}(Y) \times \mathcal{C}(X) \mid \varphi([A]) = [B]\}$$

*is Borel. Then $\varphi$ is trivial.*

*Proof of Corollary* 4.2. The fact that $\varphi$ is an isomorphism between $\mathcal{C}(Y)/\mathcal{K}(Y)$ and $\mathcal{C}(X)/\mathcal{K}(X)$ can be written as a $\mathbf{\Pi}^1_2$ statement using $\Gamma$; hence by Schoenfield absoluteness, if $V^{\mathbb{P}}$ is a forcing extension satisfying OCA+$\mathrm{MA}_{\aleph_1}$ (see [24]), then in $V^{\mathbb{P}}$ the map $\bar{\varphi} : \mathcal{C}(Y)/\mathcal{K}(Y) \to \mathcal{C}(X)/\mathcal{K}(X)$, defined from the reinterpretation of $\Gamma$ in $V^{\mathbb{P}}$, is also an isomorphism. By Theorem 4.1, then, we have in $V^{\mathbb{P}}$ that

$$\exists e \in C(X, Y) \, \forall A \in \mathcal{C}(Y) \; \bar{\varphi}([A]) = [e^{-1}(A)],$$

where $C(X,Y)$ denotes the space of continuous maps from $X$ to $Y$. This can be written as a $\mathbf{\Sigma}^1_2$ statement and so by Schoenfield absoluteness again, it must be true in $V$ with $\varphi$ replacing $\bar{\varphi}$. $\square$

Before the proof of Theorem 4.1 we set down some more notation. Fix $X, Y$ and $\varphi$ as in the statement of the theorem. Let $L_n$ be an increasing sequence of compact subsets of $Y$, with union $Y$, and let $Y_{n+1} = L_{n+1} \setminus L_n$ and $Y_0 = L_0$. Let $\mathcal{B}$ be a countable base for $Y$ consisting of compact-open sets, such that

- for all $U \in \mathcal{B}$, the set of $V \in \mathcal{B}$ with $V \supseteq U$ is finite and linearly ordered by $\subseteq$, and
- for all $U \in \mathcal{B}$ and all $n < \omega$, either $U \subseteq Y_n$ or $U \cap Y_n = \emptyset$.

It follows that for all $U, V \in \mathcal{B}$, either $U \cap V = \emptyset$, $U \subseteq V$, or $V \subseteq U$. Let $\mathbb{P}$ be the poset of all partitions of $Y$ into elements of $\mathcal{B}$, ordered by refinement;

$$P \prec Q \iff \forall U \in P\ \exists V \in Q \quad U \subseteq V.$$

We also use $\prec^*$ to denote *eventual refinement*;

$$P \prec^* Q \iff \forall^\infty U \in P\ \exists V \in Q \quad U \subseteq V.$$

When $P \prec^* Q$ we let $\Gamma(P,Q)$ be the least $n$ such that every $U \in P$ disjoint from $L_n$ is contained in some element of $Q$.

For a given $P \in \mathbb{P}$, let $s_P : Y \to P$ be the unique function satisfying $x \in s_P(x)$ for all $x \in Y$; similarly, when $P, Q \in \mathbb{P}$ and $P \prec Q$ we let $s_{PQ} : P \to Q$ be the unique function satisfying $U \subseteq s_{PQ}(U)$ for all $U \in P$. These maps induce embeddings $\sigma_P : \mathcal{P}(P)/\operatorname{fin} \to \mathcal{C}(Y)/\mathcal{K}(Y)$ and $\sigma_{PQ} : \mathcal{P}(Q)/\operatorname{fin} \to \mathcal{P}(P)/\operatorname{fin}$ in the usual way.

Finally, we need to prove a uniqueness result for maps $e : Z \to \omega$ inducing the same map $\mathcal{P}(\omega)/\operatorname{fin} \to \mathcal{C}(Z)/\mathcal{K}(Z)$.

**Lemma 4.3.** *Suppose $Z \in \mathcal{C}(X)$ and $e, f : Z \to \omega$ are continuous, compact-to-one maps, such that $e^{-1}(a) \Delta f^{-1}(a)$ is compact for every $a \subseteq \omega$. Then $\{x \in Z \mid e(x) \neq f(x)\}$ is compact.*

*Proof.* Suppose not; then for some infinite set $I \subseteq \omega$ and all $n \in I$, there is a point $x_n \in Z \cap X_n$ such that $e(x_n) \neq f(x_n)$. Since $e$ and $f$ are compact-to-one, we may assume also that $m \neq n$ implies $e(x_m) \neq e(x_n)$ and $f(x_m) \neq f(x_n)$. Now define a coloring $F : [I]^2 \to 3$ by

$$F(\{m < n\}) = \begin{cases} 0, & e(x_m) \neq f(x_n) \wedge f(x_m) \neq e(x_n), \\ 1, & e(x_m) = f(x_n) \wedge f(x_m) \neq e(x_n), \\ 2, & e(x_m) \neq f(x_n) \wedge f(x_m) = e(x_n). \end{cases}$$

By Ramsey's theorem, there is an infinite set $a \subseteq I$ which is homogeneous for this coloring. Suppose first that $a$ is 1-homogeneous, and let $m < n < k$ be members of $a$. Then

$$e(x_m) = f(x_n) \quad \text{and} \quad e(x_m) = f(x_k) \quad \text{and} \quad e(x_n) = f(x_k)$$

which implies $e(x_n) = f(x_n)$, a contradiction. Similarly, $a$ cannot be 2-homogeneous.

Now suppose $a$ is 0-homogeneous. Let $a = a_0 \cup a_1$ be a partition of $a$ into two infinite sets, and put $W_i = \{x_n \mid n \in a_i\}$ and $W = \{x_n \mid n \in a\} = W_0 \cup W_1$. From the homogeneity of $a$, it follows that $e''W \cap f''W = \emptyset$, and hence (as $e$ and $f$ are injective on $W$)

$$W \cap e^{-1}((e''W_0) \cup (f''W_1)) = W_0 \quad \text{and} \quad W \cap f^{-1}((e''W_0) \cup (f''W_1)) = W_1.$$

So, if $b = e''W_0 \cup f''W_1$, we have $W \subseteq e^{-1}(b)\Delta f^{-1}(b)$. But $W$ is not compact, so this is a contradiction. □

*Proof of Theorem* 4.1. For each $P \in \mathbb{P}$, let $\varphi_P = \varphi \circ \sigma_P$. Then $\varphi_P$ is an embedding of $\mathcal{P}(P)/\operatorname{fin}$ into $\mathcal{C}(X)/\mathcal{K}(X)$. By Theorem 4.1, there is a continuous map $e_P : X \to P$ such that $a \mapsto e_P^{-1}(a)$ lifts $\varphi_P$. Note that if $P, Q \in \mathbb{P}$ and $P \prec^* Q$, then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{P}(P)/\operatorname{fin} & \xrightarrow{\varphi_P} & \mathcal{C}(X)/\mathcal{K}(X) \\ {\scriptstyle \sigma_{PQ}}\big\uparrow & \nearrow{\scriptstyle \varphi_Q} & \\ \mathcal{P}(Q)/\operatorname{fin} & & \end{array}$$

So by Lemma 4.3, the set $\{x \in X \mid s_{PQ}(e_P(x)) \neq e_Q(x)\}$ is compact. Now let $[\mathbb{P}]^2 = M_0 \cup M_1$ be the partition defined by

$$\{P, Q\} \in M_0 \iff \exists x \in X \quad s_{P,P\vee Q}(e_P(x)) \neq s_{Q,P\vee Q}(e_Q(x)).$$

Here $P \vee Q$ is the finest partition coarser than both $P$ and $Q$. If we define $f_P : \mathcal{B} \to \mathcal{C}(X)$ by

$$f_P(U) = \{x \in X \mid e_P(x) \subseteq U\},$$

then we have

$$\{P, Q\} \in M_0 \iff \exists U \in \mathcal{B}, \quad f_P(U) \neq f_Q(U),$$

and it follows that $M_0$ is open in the topology on $\mathbb{P}$ obtained by identifying $P$ with $f_P$.

*Claim* 4.4. There is no uncountable, $M_0$-homogeneous subset of $\mathbb{P}$.

*Proof.* Suppose $H$ is such, and has size $\aleph_1$. Using $MA_{\aleph_1}$ with a simple modification of Hechler forcing, we see that there is some $\bar{P} \in \mathbb{P}$ such that $P \succ^* \bar{P}$ for all $P \in H$. By thinning out $H$ and refining a finite subset of $\bar{P}$, we may assume that $P \succ \bar{P}$ for all $P \in H$, and moreover that there is an $\bar{n}$ such that for all $P \in H$,

$$\left\{x \in X \;\middle|\; s_{\bar{P},P}(e_{\bar{P}}(x)) \neq e_P(x)\right\} \subseteq K_{\bar{n}}.$$

Now fix $P, Q \in H$ such that $e_P \restriction K_{\bar{n}} = e_Q \restriction K_{\bar{n}}$. Then $s_{P,P\vee Q} \circ e_P = s_{Q,P\vee Q} \circ e_Q$, contradicting the fact that $\{P, Q\} \in M_0$. □

By OCA, there is a countable cover of $\mathbb{P}$ by $M_1$-homogeneous sets; since $\mathbb{P}$ is countably directed under $\succ^*$, it follows that one of them, say $\mathbb{Q}$, is cofinal in $\mathbb{P}$. It follows moreover that for some $n$, we have

$$\forall P \in \mathbb{P}\; \exists Q \in \mathbb{Q} \quad \Gamma(Q, P) \leq n$$

That is, $\mathbb{Q}$ is cofinal in $\mathbb{P}$ under $\succ^n$ defined by

$$P \prec^n Q \iff \forall U \in P \;\; (U \cap L_n = \emptyset \implies \exists V \in Q\; U \subseteq V).$$

*Claim* 4.5. There is a compact set $K \subseteq X$ and a unique continuous map $e : X \setminus K \to Y$ satisfying

$$\forall x \in X \setminus K \quad e(x) \in \bigcap_{P \in \mathcal{Q}} e_P(x).$$

*Proof.* Fix $x \in X$. If $P, Q \in \mathbb{Q}$, then by $M_1$-homogeneity of $\mathbb{Q}$ we have

$$s_{P,P\vee Q}(e_P(x)) = s_{Q,P\vee Q}(e_Q(x)).$$

Then, the unique member of $P \vee Q$ containing $e_P(x)$ is the same as the unique member of $P \vee Q$ containing $e_Q(x)$. It follows that $e_P(x) \cap e_Q(x) \neq \emptyset$, and so either $e_P(x) \subseteq e_Q(x)$ or vice versa. Then the collection $\{e_P(x) \mid P \in \mathbb{Q}\}$ is a chain, and hence by compactness has nonempty intersection.

Now let

$$K = \{x \in X \mid \forall P \in \mathbb{Q}\ e_P(x) \subseteq L_n\} \subseteq \bigcap_{P\in\mathbb{Q}} e_P^{-1}(P \cap \mathcal{C}(L_n)).$$

Then $K$ is contained in a compact set. If $x \in X \setminus K$ and $P \in \mathbb{Q}$, then $e_P(x)$ is disjoint from $L_n$. Then for any $x \in X \setminus K$ and $\epsilon > 0$, there is some $P \in \mathbb{Q}$ such that $e_P(x)$ has diameter less than $\epsilon$ (since $\mathbb{Q}$ is cofinal in $\mathbb{P}$ under $\succ^n$). Thus $e$, as defined above, is unique.

To see that $e$ is continuous, note that for any open $U \subseteq X$,

$$x \in e^{-1}(U) \iff \exists P \in \mathbb{Q} \quad e_P(x) \subseteq U.$$

□

*Claim* 4.6. The map $U \mapsto e^{-1}(U)$ lifts $\varphi$.

*Proof.* Fix $P \in \mathbb{Q}$, and let $U \in P$. Then clearly, for all $x \in X \setminus K$, $e_P(x) = U$ if and only if $e(x) \in U$. Since there are only finitely many $U \in P$ such that one of $e_P^{-1}(\{U\})$ or $e^{-1}(U)$ meets $K$, it follows that

$$\forall^\infty U \in P\ e_P^{-1}(\{U\}) = e^{-1}(U).$$

Then $U \mapsto e^{-1}(U)$ lifts $\varphi_P$.

Now fix $A \in \mathcal{C}(Y)$. Then there is some $P \in \mathbb{P}$ such that $A$ can be written as a union of a subset of $P$. Find $Q \in \mathbb{Q}$ with $Q \prec^* P$; then, up to a compact set, $A$ can be written as a union of some subset $a$ of $Q$. Hence,

$$\varphi[A] = \varphi_Q[a] = [e^{-1}(A)].$$

□

□

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Department of Mathematics and Statistics, York University, 4700 Keele Street, North York, Ontario M3J 1P3, Canada
*Email address*: ifarah@mathstat.yorku.ca
*URL*: http://www.math.yorku.ca/∼ifarah

Department of Mathematics, Miami University, 501 E. High St., Oxford, Ohio 45056
*Email address*: mckennp2@miamioh.edu
*URL*: http://users.miamioh.edu/mckennp2