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# QUIVER GRASSMANNIANS FOR WILD ACYCLIC QUIVERS

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ABSTRACT. A famous result of Zimmermann-Huisgen, Hille and Reineke asserts that any projective variety occurs as a quiver Grassmannian for a suitable representation of some wild acyclic quiver. We show that this happens for *any* wild acyclic quiver.

Let $k$ be an algebraically closed field, and $Q$ a finite acyclic quiver. The modules which we consider are the (finite-dimensional) $kQ$-modules, where $kQ$ is the path algebra of $Q$, thus the (finite-dimensional) representations of $Q$ (with coefficients in $k$). We denote by $\operatorname{mod} kQ$ the corresponding module category.

Let $M$ be a representation of $Q$ and let $\mathbf{d}$ be a dimension vector for $Q$. The *quiver Grassmannian* $\mathbb{G}_{\mathbf{d}}(M)$ is the set of submodules of $M$ with dimension vector $\mathbf{dim}\, M = \mathbf{d}$; this is a projective variety. A famous result of Zimmermann-Huisgen, Hille and Reineke asserts that any projective variety occurs as the quiver Grassmannian for a representation of **some** wild acyclic quiver $Q$; see for example [3]. We are going to show:

**Theorem.** *Let $Q$ be* **any** *wild acyclic quiver. Any projective variety occurs as a quiver Grassmannian $\mathbb{G}_{\mathbf{d}}(M)$ for some representation $M$ of $Q$ and some dimension vector $\mathbf{d}$.*

Typical wild acyclic quivers are the Kronecker quivers $Q = K(n)$ with $n \geq 3$ (the Kronecker quiver $K(n)$ has two vertices 1 and 2 and $n$ arrows pointing from 2 to 1). A representation of $K(n)$ will be said to be *reduced* provided $N$ has no simple injective direct summand. In [4] we have shown that for any projective variety $\mathcal{V}$ there is a natural number $n$ (depending on $\mathcal{V}$) such that $\mathcal{V}$ can be realized as the quiver Grassmannian $\mathbb{G}_{(1,1)}(N)$ of a reduced representation $N$ of $K(n)$ (see also [1]). Our present investigation relies on this special case.

Note that the elements of $\mathbb{G}_{(1,1)}(N)$ are certain submodules of $N$ of length 2, and all the indecomposable submodules of length 2 belong to $\mathbb{G}_{(1,1)}(N)$. We call indecomposable modules of length 2 *bristles.* For any representation $N$ of $K(n)$, the set $\beta(N)$ of bristle submodules of $N$ is an open subset of $\mathbb{G}_{(1,1)}(N)$ which we call the *bristle variety* of $N$. In general, $\beta(N)$ is a proper subset of $\mathbb{G}_{(1,1)}(N)$, but for a reduced representation $N$, we have $\beta(N) = \mathbb{G}_{(1,1)}(N)$.

The procedure of the present paper is as follows: Given any wild acyclic quiver $Q$, and a natural number $m$, we will construct for some $n \geq m$ an orthogonal pair $X, Y$ of bricks with $\dim \operatorname{Ext}^1(Y, X) = n$ (a *brick* is a module with endomorphism ring $k$ and $X, Y$ are said to be *orthogonal* provided $\operatorname{Hom}(X, Y) = 0 = \operatorname{Hom}(Y, X)$).

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Always, $\mathbf{x}$ and $\mathbf{y}$ will denote the dimension vectors of $X$ and $Y$, respectively. Let $\mathcal{E} = \mathcal{E}(Y, X)$ be the full subcategory of all $kQ$-modules $M$ with an exact sequence of the form

$$0 \longrightarrow X^a \longrightarrow M \longrightarrow Y^b \longrightarrow 0,$$

where $a, b$ are natural numbers. Note that $\mathcal{E}$ is equivalent to $\operatorname{mod} kK(n)$ with an equivalence being given by an exact fully faithful functor

$$\eta : \operatorname{mod} kK(n) \to \operatorname{mod} kQ$$

with image $\mathcal{E}$. We say that a module $M$ in $\mathcal{E}$ is *$\mathcal{E}$-reduced* provided it has no direct summand isomorphic to $Y$, thus provided it is the image of a reduced $kK(n)$-module under $\eta$.

An indecomposable $kQ$-module $U$ will be called an *$\mathcal{E}$-bristle* provided there is an exact sequence of the form $0 \to X \to U \to Y \to 0$, thus provided $U$ is the image of a bristle in $\operatorname{mod} kK(n)$ under $\eta$. For any $kK(n)$-module $N$ with $M = \eta N$, the functor $\eta$ identifies the bristle variety $\beta(N)$ of $N$ with the set $\beta_{\mathcal{E}}(M)$ of submodules of $M$ which are $\mathcal{E}$-bristles. Since $\mathcal{E}$-bristles have dimension vector $\mathbf{x}+\mathbf{y}$, we have $\beta_{\mathcal{E}}(M) \subseteq \mathbb{G}_{\mathbf{x}+\mathbf{y}}(M)$. It remains to find conditions such that any submodule $U$ of $M$ with dimension vector $\mathbf{x}+\mathbf{y}$ is indeed an $\mathcal{E}$-bristle.

To be precise, we are looking for $kQ$-modules $X, Y$ so that the following closure condition (C) is satisfied:

(C) *If $M$ is an $\mathcal{E}$-reduced module in $\mathcal{E}(Y, X)$ and $U$ is a submodule of $M$ with* $\mathbf{dim}\, U = \mathbf{x}+\mathbf{y}$, *then $U$ is an $\mathcal{E}$-bristle.*

If the condition (C) is satisfied, then for any reduced representation $N$ of $K(n)$, there is a canonical bijection between $\mathbb{G}_{(1,1)}(N)$ and $\mathbb{G}_{\mathbf{x}+\mathbf{y}}(M)$, where $M = \eta N$. Namely, if $B$ is a submodule of the $kK(n)$-module $N$ with $\mathbf{dim}\, B = (1,1)$, then $\eta B$ is a submodule of $M$ with dimension vector $\mathbf{x}+\mathbf{y}$. Conversely, if $U$ is a submodule of $M$ with $\mathbf{dim}\, U = \mathbf{x}+\mathbf{y}$, then, by condition (C), $U$ belongs to $\mathcal{E}(Y, X)$, say $U = \eta B$ for some $K(n)$-submodule $B$ and the dimension vector of $B$ is $(1,1)$.

**The minimal wild acyclic quivers.** As we have mentioned, our aim is to exhibit for any wild acyclic quiver $Q$ and any natural number $m$ an orthogonal pair $X, Y$ of $kQ$-modules which are bricks such that $\dim_k \operatorname{Ext}^1(Y, X) = n \geq m$ and such that the condition (C) is satisfied. Of course, it is sufficient to deal with minimal wild acyclic quivers. (We recall that a quiver $Q$ is *wild* provided it is not the disjoint union of Dynkin and Euclidean quivers, and $Q$ is said to be *minimal wild* provided it is wild, and no quiver obtained from $Q$ by deleting a vertex or an arrow is wild.)

The following well-known proposition suggests to deal with two different cases.

**Proposition.** *A minimal wild acyclic quiver $Q$ different from $K(3)$ is obtained from a Euclidean quiver $Q'$ by adding a vertex $\omega$ and a single arrow which connects $\omega$ with some vertex of $Q'$ (in particular, $\omega$ is a sink or a source).*

*Sketch of proof.* If $Q$ has cycles, then there is a subquiver $Q'$ of type $\widetilde{\mathbb{A}}_n$ for some $n$ such that $Q'$ is obtained from $Q$ by deleting one vertex and one arrow.

Now assume that $Q$ is a tree. If there is a vertex with at least four neighbors, then $Q'$ is obtained from a quiver of type $\widetilde{\mathbb{D}}_4$ by deleting one vertex and one arrow. If $Q$ has two vertices which have three neighbors each, then $Q'$ is obtained from a quiver of type $\widetilde{\mathbb{D}}_n$ with $n \geq 5$ by deleting one vertex and one arrow. If $Q$ has a star with three arms, then $Q'$ is obtained from a quiver of type $\widetilde{\mathbb{E}}_m$ with $m = 6, 7, 8$ by deleting one vertex and one arrow. □

*Case* 1 (One-point extensions of representation-infinite quivers). We assume now that $Q$ is a connected quiver with a vertex $\omega$ which is a sink or a source such that the quiver $Q'$ obtained from $Q$ by deleting $\omega$ and the arrows which start or end in $\omega$ is connected and representation-infinite. Up to duality, we can assume that $\omega$ is a source, thus there is an arrow $\omega \to p$ with $p \in Q'_0$.

Let $Y = S(\omega)$, the simple $kQ$-module corresponding to the vertex $\omega$. Since $Q'$ is connected and representation-infinite, there is an exceptional $kQ'$-module $X$ with $\dim_k X_p \geq m$. The arrow $\omega \to p$ shows that $\dim_k \operatorname{Ext}^1(Y, X) \geq \dim_k X_p$. This pair $X, Y$ is the orthogonal pair of bricks which we use in order to look at $\mathcal{E}(Y, X)$.

**Lemma 1.** *Let $a$ be a natural number. Any submodule $W$ of $X^a$ with $\mathbf{dim}\, W = \mathbf{x}$ is isomorphic to $X$.*

*Proof.* We denote by $\langle -, - \rangle$ the bilinear form on the Grothendieck group $K_0(kQ)$ with $\langle \mathbf{dim}\, M, \mathbf{dim}\, M' \rangle = \dim_k \operatorname{Hom}(M, M') - \dim_k \operatorname{Ext}^1(M, M')$. Since $X$ is exceptional, we have $\langle X, W \rangle = \langle X, X \rangle > 0$, Therefore, there is a non-zero homomorphism $f : X \to W$. Let $\iota : W \to X^a$ be the inclusion map. The composition $\iota f : X \to X^a$ is non-zero. Since $X$ is a brick, we see that $f : X \to W$ is a split monomorphism, in particular injective. Now $\mathbf{dim}\, X = \mathbf{dim}\, W$ implies that $f$ is an isomorphism. □

*Proof of condition (C).* Let $M$ be an $\mathcal{E}$-reduced $kQ$-module in $\mathcal{E}(Y, X)$, say with an exact sequence

$$0 \longrightarrow X^a \xrightarrow{\ \mu\ } M \xrightarrow{\ \pi\ } Y^b \longrightarrow 0.$$

Let $U$ be a submodule of $M$ with dimension vector $\mathbf{x}+\mathbf{y}$ and inclusion map $\iota : U \to M$. The composition $\pi\iota$ is non-zero, since otherwise $U$ would be a submodule of $X^a$, but $\dim_k U_\omega = 1$ whereas $X_\omega = 0$. If follows that the image of $\pi\iota$ is isomorphic to $Y$. If we denote the kernel of $\pi\iota$ by $W$, we obtain the following commutative diagram with exact rows and vertical monomorphisms:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \longrightarrow & U & \longrightarrow & Y & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \iota} & & \downarrow & & \\ 0 & \longrightarrow & X^a & \xrightarrow{\ \mu\ } & M & \xrightarrow{\ \pi\ } & Y^b & \longrightarrow & 0. \end{array}$$

Of course, $\mathbf{dim}\, W = \mathbf{x}$, thus Lemma 1 shows that $W$ is isomorphic to $X$. In particular, $U$ belongs to $\mathcal{E}$.

It remains to show that $U$ is indecomposable. Otherwise, $U$ would be isomorphic to $W \oplus Y$. Thus $M$ would have a submodule isomorphic to $Y$. But $Y$ is relative injective inside $\mathcal{E}$, thus $M$ would have a direct summand isomorphic to $Y$, in contrast to our assumption that $M$ is $\mathcal{E}$-reduced. This shows that $U$ is indecomposable, thus an $\mathcal{E}$-bristle. □

*Case* 2 (The Kronecker quiver $K(3)$). Here we consider the Kronecker quiver $Q = K(3)$, with the three arrows $\alpha, \beta, \gamma : 2 \to 1$. Let $\lambda_1, \ldots, \lambda_n$ be pairwise different non-zero elements of $k$ with $n \geq 2$. Let $X = X(\lambda_1, \ldots, \lambda_n) = (k^n, k^n; \alpha, \beta, \gamma)$ be defined by

$$\alpha(e(i)) = e(i), \quad \beta(e(i)) = \lambda_i e(i), \quad \gamma(e(i)) = e(i+1),$$

for $1 \le i \le n$, where $e(1), \dots, e(n)$ is the canonical basis of $k^n$ and $e(n+1) = e(1)$. Let $Y = (k, k; 1, 0, 0)$. We denote by $Q'$ the subquiver of $Q$ with arrows $\alpha, \beta$, this is the 2-Kronecker quiver $K(2)$. For the structure of the category $\operatorname{mod} K(2)$, see for example [2]. The restriction of $X, Y$ to $Q'$ shows that $\operatorname{Hom}(X, Y) = \operatorname{Hom}(Y, X) = 0$. The endomorphism ring of $X|Q'$ is $k \times \cdots \times k$; and the only endomorphisms of $X|Q'$ which commute with $\gamma$ are the scalar multiplications. This shows that $X$ is a brick. Also, it is easy to see that $\dim_k \operatorname{Ext}^1(Y, X) = n$.

**Lemma 2.** *Let $a$ be a natural number. Any submodule $W$ of $X^a$ with* $\mathbf{dim}\, W$ *of the form $(w, w)$ is isomorphic to $X^s$ for some $s$.*

*Proof.* Let $M = X^a$ and decompose $M|Q' = \bigoplus_{i=1}^n M(i)$, where $\beta(x) = \lambda_i x$ for $x \in M(i)_1$. Here, we use $\alpha$ in order to identify $M_1$ and $M_2$. Now we consider the submodule $W$ of $M$. Note that $W|Q'$ has to be regular, since it cannot have any non-zero preinjective direct summand. As a regular submodule of a semisimple regular Kronecker module it has to be a direct summand of $M|Q'$, thus we have a similar direct decomposition $W = \bigoplus W(i)$, where $W(i) = W \cap M(i)$.

The linear map $\gamma$ restricted to $W(i)_1$ is a monomorphism $W(i)_1 \to W(i+1)_2 = W(i+1)_1$ for $1 \le i \le n$; we obtain in this way a monomorphism $W(1)_1 \to W(1)_2 = W(1)_1$. This shows that all the monomorphisms $W(i)_1 \to W(i+1)_2 = W(i+1)_1$ are actually bijections. Let $\dim_k W(1)_1 = s$. It follows that $W$ is isomorphic to $X^s$. □

*Proof of condition (C).* Let $M$ be an $\mathcal{E}$-reduced $kQ$-module in $\mathcal{E}$ and let $U$ be a submodule of $M$ with dimension vector $\mathbf{x} + \mathbf{y} = (n+1, n+1)$ and with inclusion map $\iota : U \to M$.

Starting with the exact sequence $0 \longrightarrow X^a \xrightarrow{\mu} M \xrightarrow{\pi} Y^b \longrightarrow 0$ and the inclusion map $\iota : U \to M$, let $W$ be the kernel and let $\overline{U}$ be the image of $\pi\iota : U \to Y^b$. We obtain the following commutative diagram with exact rows and injective vertical maps:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \longrightarrow & U & \longrightarrow & \overline{U} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \iota} & & \downarrow & & \\ 0 & \longrightarrow & X^a & \xrightarrow{\mu} & M & \xrightarrow{\pi} & Y^b & \longrightarrow & 0. \end{array}$$

Let us consider the restriction of these modules to $Q'$. Since $M|Q'$ is regular, it has no non-zero preinjective direct summand. Thus any submodule of $M|Q'$ with dimension vector $(n+1, n+1)$ has to be regular. This shows that $U|Q'$ is regular. Actually, $M|Q'$ is semisimple regular, thus also its regular submodule $U|Q'$ is semisimple regular (and a direct summand of $M|Q'$). Next, $\pi\iota$ is a map between regular $kQ'$-modules. It follows that the kernel $W|Q'$ and the image $\overline{U}|Q'$ are regular $kQ'$-modules. In particular, the dimension vector of $W$ is of the form $\mathbf{dim}\, W = (w, w)$ for some $0 \le w \le n+1$.

Now $\overline{U}|Q'$ is a regular submodule of the semisimple regular $kQ'$-module $Y^b|Q'$, thus $\overline{U}|Q'$ is a direct sum of copies of $Y|Q'$. By construction, $Y$ is annihilated by $\gamma$. Since $\overline{U}$ is a submodule of $Y^b$, it follows that $\overline{U}$ is annihilated by $\gamma$. Altogether, we see that $\overline{U}$ is the direct sum of copies of $Y$.

We claim that $W \neq 0$. Otherwise $U = \overline{U} = Y^{n+1}$, thus $Y$ is a submodule of $M$. But $Y$ is relative injective in $\mathcal{E}$, thus $Y$ would be a direct summand of $M$. However, by assumption, $M$ is $\mathcal{E}$-reduced. This contradiction shows that $W \neq 0$.

Now $W$ is a submodule of $X^a$ with dimension vector $(w, w)$, thus, according to Lemma 2, $W$ is a direct summand of say $s$ copies of $X$ and $s \geq 1$. The equality $(w, w) = (sn, sn)$ implies that $s = 1$, since $w \leq n+1$ and $n \geq 2$. In this way, we see that $W$ is isomorphic to $X$. It follows that $\mathbf{dim}\,\overline{U} = (1, 1)$ and therefore $\overline{U} = Y$.

Finally, as in Case 1, we see that $U$ is indecomposable, using again the assumption that $M$ is $\mathcal{E}$-reduced. This shows that $U$ is an $\mathcal{E}$-bristle. □

*Remark.* We should stress that given orthogonal bricks $X, Y$ in $\operatorname{mod} kQ$, the condition (C) is usually not satisfied. Here is a typical example for $Q = K(3)$. As above, let $Y = (k, k; 1, 0, 0)$, but for $X$ we now take $X = X'(\lambda_1, \lambda_2) = (k^2, k^2; \alpha, \beta, \gamma)$, defined by

$$\alpha(e(i)) = e(i), \quad \beta(e(i)) = \lambda_i e(i), \quad \gamma(e(1)) = e(2), \quad \gamma(e(2)) = 0$$

for $1 \leq i \leq 2$. Again, $e(1), e(2)$ is the canonical basis of $k^2$ and $\lambda_1 \neq \lambda_2$ are assumed to be non-zero elements of $k$. Since $\dim_k \operatorname{Ext}^1(Y, X) = 2$, there is an equivalence $\eta : \operatorname{mod} kK(2) \to \mathcal{E}(Y, X)$. Let $N$ be an indecomposable $kK(2)$-module with dimension vector $(2, b)$ (note that $b$ has to be equal to $1, 2$ or $3$) and $M = \eta N$. Thus there is an exact sequence

$$0 \longrightarrow X^2 \longrightarrow M \longrightarrow Y^b \longrightarrow 0.$$

Since we assume that $N$ is indecomposable, it is reduced, thus $M$ is $\mathcal{E}$-reduced. Note that $X$ has a (unique) $kQ$-submodule $V$ with dimension vector $(1, 1)$: the vector spaces $V_1$ and $V_2$ both are generated by $e(2)$. The submodule $U = X \oplus V$ of $X^2$ is a submodule of $M$ with dimension vector $(3, 3) = \mathbf{x} + \mathbf{y}$, and it is not an $\mathcal{E}$-bristle. Thus, condition (C) is not satisfied. Here, $\eta$ defines a proper embedding of $\beta(N) = \mathbb{G}_{(1,1)}(N)$ into $\mathbb{G}_{\mathbf{x}+\mathbf{y}}(M)$.

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