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DISCRETE MONOTONICITY OF MEANS AND ITS APPLICATIONS

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ABSTRACT. In this paper, we obtain new inequalities for the logarithmic mean and the complete elliptic integral of the first kind. In order to prove the inequalities, we use the monotonicity property of sequences defined by these functions. Additionally, we apply our approach to previous studies. As a result, we get refinements of known inequalities.

1. INTRODUCTION

The complete elliptic integral of the first kind [4, 12] is defined for $0 < k < 1$ by

$$K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 t}} dt \tag{1}$$

and $K'(k) = K(k')$, where $k' = \sqrt{1 - k^2}$. The complete elliptic integral of the first kind has many important applications in mathematics and physics and the approximation by inequalities has been intensively studied by many researchers in the past decades [1–3, 7, 11]. Very recently, Alzer and Ricards obtained the following beautiful inequality [2]:

$$\frac{K'(k)}{K'(k^{1/2})} < \frac{2}{1 + \sqrt{k}}. \tag{2}$$

They showed this inequality by using the monotonicity of the complete elliptic integral of the first kind. As an application of (2), they refined the classical arithmetic geometric mean. Meanwhile, we should note that this inequality was first proved by M. K. Vamanamurthy and M. Vuorinen in [11, p. 162].

Here, we give another interesting application of the inequality (2). Using the inequality repeatedly, we obtain

$$K'(k) < \frac{2}{1 + \sqrt{k}} K'(k^{1/2}) < \cdots < K'(1) \prod_{n=1}^{\infty} \frac{2}{1 + k^{1/2^n}}. \tag{3}$$

Let us note that $K'(1) = \pi/2$ and the infinite product [6, 10]

$$\frac{\log k}{k - 1} = \prod_{n=1}^{\infty} \frac{2}{1 + k^{1/2^n}}. \tag{4}$$

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Therefore, we get

$$K'(k) < \frac{\pi}{2}\frac{\log k}{k-1}. \tag{5}$$

Recall the following notation of the arithmetic mean, the geometric mean, and the logarithmic mean [1, 4–6, 11] of 1 and k:

$$A(1,k) = \frac{1+k}{2}, G(1,k) = \sqrt{k}, L(1,k) = \frac{1-k}{\log 1 - \log k}. \tag{6}$$

Then, we find that the inequality (5) is equivalent to the second inequality of the following well-known inequalities [1, 6, 7]:

$$\frac{1}{A(1,k)} < \frac{2}{\pi}K'(k) < \frac{1}{L(1,k)} < \frac{1}{G(1,k)}. \tag{7}$$

The first inequality can be shown by the basic property of the arithmetic-geometric mean, the second is due to Carlson et al. [7], and the third is a well-known inequality [6].

As shown in the above, we can show that these means are closely related to iterative procedures and infinite products such as (3) and (4). Actually, by using the idea, the author established a new method for the inequalities for trigonometric functions, the lemniscate function, and the arithmetic geometric mean in [8]. The main purpose of this paper is applying the method to the complete elliptic integral of the first kind and the logarithmic mean. In the following sections, we will see that the key ingredient of the inequalities (7) is showing the monotonicity of corresponding sequences. To distinguish the monotonicity from a standard meaning, in this paper, we call it "discrete monotonicity".

This paper is organized as follows. In the beginning of Section 2, we give a fundamental lemma for discrete monotonicity. Then, we show the discrete monotonicity of means in (7) and prove the inequalities as the typical examples of the lemma. In Section 3, we prove and refine several known inequalities for the complete elliptic integral by using the results in Section 2. For example, we get the following refinement of the inequality (5):

$$\frac{2}{\pi}K'(k) < \frac{1}{L(k^{1/2}, (k+1)/2)} < \frac{1}{L(1,k)}. \tag{8}$$

Note that the first inequality in (8) was conjectured in [1] without proof and it was suggested that the inequality gives a better bound for $K(k)$ than several results in the paper. Also, we discuss the approximation of $K(k)$ by hyperbolic arctangent by Anderson et al. [3] and Alzer et al. [1]. This attempt yields refinements of their results, together with a generalization and a lower bound for the ratio of the elliptic integral in (2).

In what follows, we often write $K'(k)$, $1/A(1,k)$, $1/G(1,k)$, and $1/L(1,k)$ as K', $1/A$, $1/G$, and $1/L$. Additionally, we suppose that $0 < k < 1$.

2. Discrete monotonicity

First, we shall introduce an important tool for discrete monotonicity.

Definition 2.1. Suppose that the sequence a_n $(n \geq 0)$ is defined by $x < a_0 < y$, $a_{n+1} = p(a_n)$, where p is a function from (x, y) onto (x, y) and $\lim_{n\to\infty} a_n = \alpha$.

Let f, g be positive functions on (x, y) and continuous at α. We define the function $D[a_n; f, g]$ as

$$D[a_n; f, g] := \frac{f(a_0)}{f(p(a_0))} - \frac{g(a_0)}{g(p(a_0))}, \tag{9}$$

where the right-hand side is defined.

The following lemmas are important in the proof of inequalities for means.

Lemma 2.2. *If $D[a_n; f, g] > 0$ for an arbitrary a_0, the sequence*

$$\frac{f(a_n)}{g(a_n)} g(a_0) \tag{10}$$

is strictly decreasing as n increases where the sequence is defined. Moreover, we have

$$\frac{f(a_0)}{f(\alpha)} > \frac{g(a_0)}{g(\alpha)} \tag{11}$$

where both sides are defined. If $D[a_n; f, g] < 0$, the sequence (10) *is strictly increasing as n increases and the inequality* (11) *is reversed where the sequence is defined.*

Proof. Assume that $D[a_n; f, g] > 0$. Then we obtain

$$f(a_0) > \frac{g(a_0)}{g(p(a_0))} f(p(a_0)). \tag{12}$$

Applying this inequality repeatedly, we obtain

$$\begin{aligned} f(a_0) > \frac{f(a_1)}{g(a_1)} g(a_0) > \frac{f(a_2)}{g(a_2)} g(a_0) > \cdots \\ > \frac{f(a_n)}{g(a_n)} g(a_0) > \cdots > \frac{f(\alpha)}{g(\alpha)} g(a_0). \end{aligned} \tag{13}$$

If $D[a_n; f, g] < 0$, we obtain the assertion in the same way. This completes the proof. □

Lemma 2.3. *Suppose that h is a positive function on (x, y) and continuous at α. Then,*

$$D[a_n; f, f] = 0, D[a_n; f, g] + D[a_n; g, f] = 0, \tag{14}$$

$$D[a_n; f, g] + D[a_n; g, h] + D[a_n; h, f] = 0. \tag{15}$$

Since Lemma 2.3 immediately follows from the definition of $D[a_n; f, g]$, we omit the proof. The inequalities in (13) show the discrete monotonicity between the functions f and g. Using Lemma 2.2, we obtain the following theorems.

Theorem 2.4. *Let f be a continuous positive function on $(0, 1]$ and define a sequence k_n by*

$$k_{n+1} = \frac{2{k_n}^{1/2}}{1 + k_n}. \tag{16}$$

Then,

$$D[k_n; f, (K')^p] = \frac{f(k)}{f\left(\frac{2k^{1/2}}{1+k}\right)} - \left(\frac{2}{1+k}\right)^p. \tag{17}$$

Moreover, if the inequality $D[k_n; f, K'] < 0$ *holds true for* $0 < k < 1$*, the sequence*

$$\left(\frac{K'(k)}{K'(k_n)}\right)^p f(k_n) \tag{18}$$

is strictly increasing as n *increases. Additionally, we have*

$$f(k) < f(1)\left(\frac{2}{\pi}K'(k)\right)^p. \tag{19}$$

If $D[k_n; f, K'] > 0$ *holds true for* $0 < k < 1$*, the sequence* (18) *is strictly decreasing and the inequality* (19) *is reversed.*

Proof. We first observe that

$$D[k_n; f, (K')^p] = \frac{f(k)}{f\left(\frac{2k^{1/2}}{1+k}\right)} - \left(\frac{K'(k)}{K'(k_1)}\right)^p \tag{20}$$

$$= \frac{f(k)}{f\left(\frac{2k^{1/2}}{1+k}\right)} - \left(\frac{2}{1+k}\right)^p, \tag{21}$$

where we used the identity

$$K'(k) = \frac{2}{1+k}K'(k_1) \tag{22}$$

in the last equation. It is well known [4] that $\lim_{n\to\infty} k_n$=1. Therefore, by using Lemma 2.2 we obtain the assertion. □

Theorem 2.5. *Let* f *be a continuous positive function on* $(0, 1]$*. Then,*

$$D[k^{1/2^n}; f, 1/L^p] = \frac{f(k)}{f(k^{1/2})} - \left(\frac{2}{1+k^{1/2}}\right)^p. \tag{23}$$

Moreover, if the inequality $D[k^{1/2^n}; f, 1/L^p] < 0$ *holds true for* $0 < k < 1$*, the sequence*

$$\left(\frac{L(1, k^{1/2^n})}{L(1, k)}\right)^p f(k^{1/2^n}) \tag{24}$$

is strictly decreasing as n *increases; and we then have*

$$f(k) < f(1)\left(\frac{1}{L(1, k)}\right)^p. \tag{25}$$

If $D[k^{1/2^n}; f, 1/L^p] > 0$ *holds true for* $0 < k < 1$*, the sequence* (24) *is strictly decreasing and the inequality* (25) *is reversed.*

Proof. It is easy to see that

$$D[k^{1/2^n}; f, 1/L^p] = \frac{f(k)}{f(k^{1/2})} - \left(\frac{L(1, k^{1/2})}{L(1, k)}\right)^p \tag{26}$$

$$= \frac{f(k)}{f(k^{1/2})} - \left(\frac{2}{1+k^{1/2}}\right)^p. \tag{27}$$

Using Lemma 2.2, we obtain the desired result. □

Note that the computation of (17) or (23) is algebraic if f is an algebraic function. Moreover, even if the function f is not algebraic and the sequence k_n or $k^{1/2^n}$ is replaced by each other, we can compare $1/L$ or K' with other means by differentiation and the discrete monotonicity as in the following corollaries.

Corollary 2.6. *We have*

$$D[k^{1/2^n}; 1/A, K'] < 0, \tag{28}$$

$$D[k^{1/2^n}; K', 1/L] < 0, \tag{29}$$

$$D[k^{1/2^n}; 1/L, 1/G] < 0, \tag{30}$$

and then the sequences

$$\frac{1}{A(1, k^{1/2^n})} \frac{K(k)}{K(k^{1/2^n})}, \tag{31}$$

$$K'(k^{1/2^n}) \frac{L(1, k^{1/2^n})}{L(1, k)}, \tag{32}$$

$$\frac{1}{L(1, k^{1/2^n})} \frac{G(1, k^{1/2^n})}{G(1, k)} \tag{33}$$

are strictly increasing as n increases.

Proof. First, we have

$$D[k^{1/2^n}; 1/A, K'] = \frac{1 + k^{1/2}}{1 + k} - \frac{K'(k)}{K'(k^{1/2})} \tag{34}$$

$$= \frac{1 + k^{1/2}}{1 + k} - \frac{1 + k^{1/2}}{1 + k} \frac{K'(\frac{2k^{1/2}}{1+k})}{K'(\frac{2k^{1/4}}{1+k^{1/2}})} \tag{35}$$

$$= \frac{1 + k^{1/2}}{1 + k} \left(1 - \frac{K'(\frac{2k^{1/2}}{1+k})}{K'(\frac{2k^{1/4}}{1+k^{1/2}})} \right) < 0, \tag{36}$$

where we used the identity (22) in (34) and where the last inequality follows from the monotonicity property of $K'(k)$ since $2k^{1/4}/(1 + k^{1/2}) > 2k^{1/2}/(1 + k)$. Next, we have

$$D[k^{1/2^n}; K', 1/L] = \frac{K'(k)}{K'(k^{1/2})} - \frac{2}{1 + k^{1/2}} < 0, \tag{37}$$

which follows from (2). Finally,

$$D[k^{1/2^n}; 1/L, 1/G] = \frac{2}{1 + k^{1/2}} - \frac{1}{k^{1/4}} = -\frac{(k^{1/4} - 1)^2}{k^{1/4}(k^{1/2} + 1)} < 0. \tag{38}$$

Using Lemma 2.2, we obtain the monotonicity of the corresponding sequences. This completes the proof. ☐

Corollary 2.7. *We have*

$$D[k_n; 1/A, K'] < 0, \tag{39}$$

$$D[k_n; K', 1/L] < 0, \tag{40}$$

$$D[k_n; 1/L, 1/G] < 0, \tag{41}$$

and then the sequences

$$\frac{1}{A(1,k_n)}\frac{K'(k)}{K'(k_n)}, \tag{42}$$

$$K'(k_n)\frac{L(1,k_n)}{L(1,k)}, \tag{43}$$

$$\frac{1}{L(1,k_n)}\frac{G(1,k_n)}{G(1,k)} \tag{44}$$

are strictly increasing as n increases.

Proof. The inequality (39) is shown by

$$D[k_n;1/A,K'] = \frac{1+\frac{2k^{1/2}}{1+k}}{1+k} - \frac{2}{1+k} = -\frac{(k^{1/2}-1)^2}{(k+1)^2} < 0. \tag{45}$$

Next, we get

$$D[k_n;K',1/L] = \frac{k_1-1}{\log k_1}\left(\frac{2}{1+k}\frac{\log k_1}{k_1-1} - \frac{\log k}{k-1}\right) \tag{46}$$

$$= -\frac{k_1-1}{\log k_1}\left(\frac{\log k}{k-1} + \frac{2\log\frac{2k^{1/2}}{k+1}}{(k^{1/2}-1)^2}\right) \tag{47}$$

$$= \frac{-1}{(k^{1/2}-1)^2}\frac{k_1-1}{\log k_1}\left(\frac{k^{1/2}-1}{k^{1/2}+1}\log k + 2\log\left(\frac{2k^{1/2}}{k+1}\right)\right). \tag{48}$$

Differentiating the functions in the brackets, we obtain

$$\frac{d}{dk}\left(\frac{k^{1/2}-1}{k^{1/2}+1}\log k + 2\log\left(\frac{2k^{1/2}}{k+1}\right)\right) \tag{49}$$

$$= \frac{(k-1)}{k^{1/2}(1+k^{1/2})^2}\left(\frac{\log k}{k-1} - \frac{2}{1+k}\right) < 0, \tag{50}$$

where the last inequality is shown by putting $p = 1$ and $f(t) = 2/(1+t)$ in Theorem 2.5. Since

$$\lim_{k\to 1}\left(\frac{k^{1/2}-1}{k^{1/2}+1}\log k + 2\log\left(\frac{2k^{1/2}}{k+1}\right)\right) = 0, \tag{51}$$

we find that $D[k_n;K',1/L] < 0$. Next, we have

$$D[k_n;1/L,1/G] = \frac{k_1-1}{\log k_1}\left(\frac{\log k}{k-1} - \frac{\log k_1}{k_1-1}\left(\frac{2}{k^{1/2}(1+k)}\right)^{1/2}\right) \tag{52}$$

$$< \frac{k_1-1}{\log k_1}\left(\frac{\log k}{k-1} - \frac{\log\frac{2k^{1/2}}{1+k}}{\frac{2k^{1/2}}{1+k}-1}\frac{4}{(1+k^{1/2})^2}\right) \tag{53}$$

$$= \frac{k_1-1}{\log k_1}\frac{k+1}{(k-1)^2}\left(\frac{k-1}{k+1}\log k + 4\log\left(\frac{2k^{1/2}}{1+k}\right)\right). \tag{54}$$

Here, we note that

$$\frac{d}{dk}\left(\frac{k-1}{k+1}\log k + 4\log\left(\frac{2k^{1/2}}{1+k}\right)\right) \tag{55}$$

$$= \frac{2k(1-k)}{k(k+1)^2}\left(\frac{1+k}{2k} - \frac{\log k}{k-1}\right) > 0, \tag{56}$$

where the last inequality follows by setting $p = 1$ and $f(t) = (1+t)/2t$ in Theorem 2.5. Since

$$\lim_{k\to 1}\left(\frac{k-1}{k+1}\log k + 4\log\left(\frac{2k^{1/2}}{1+k}\right)\right) = 0, \tag{57}$$

we find that $D[k_n; 1/L, 1/G] < 0$. We get the monotonicity of the corresponding sequences by Lemma 2.2. □

Corollaries 2.6 and 2.7 show the discrete monotonicity of these means, and we can easily find that the inequalities in (7) follow from the inequality (19) or (25).

3. Applications

The study in Section 2 leads us to new proofs and refinements of known inequalities for the logarithmic means and the complete elliptic integral of the first kind. Unaware of the discrete monotonicity of these functions and its generality, many authors published results which are related to this property. The first example is the following assertion due to Carlson [6].

Corollary 3.1.

(58)
$$(xy)^{1/2^{n+1}}\prod_{m=1}^{n}\frac{x^{1/2^m}+y^{1/2^m}}{2} < L(x,y) < \frac{x^{1/2^n}+y^{1/2^n}}{2}\prod_{m=1}^{n}\frac{x^{1/2^m}+y^{1/2^m}}{2},$$

where $0 < x < y$ and the first and third members are increasing and decreasing as n increases, respectively.

Proof. It follows from Corollary 2.6 and Lemma 2.3 that

$$D[k^{1/2^n}; 1/G, 1/L] = -D[k^{1/2^n}; 1/L, 1/G] > 0, \tag{59}$$

$$D[k^{1/2^n}; 1/A, 1/L] = D[k^{1/2^n}; 1/A, K'] + D[k^{1/2^n}; K', 1/L] < 0. \tag{60}$$

Thus,

$$G(1,k^{1/2^n})\frac{L(1,k)}{L(1,k^{1/2^n})} < L(1,k) < A(1,k^{1/2^n})\frac{L(1,k)}{L(1,k^{1/2^n})}, \tag{61}$$

where the left-hand (or right-hand) side is strictly increasing (or decreasing) as n increases. Applying the infinite product representation

$$\frac{1}{L(1,k)} = \prod_{n=1}^{\infty}\frac{2}{1+k^{1/2^n}}, \tag{62}$$

and by setting $k = x/y$ where $0 < x < y$, we obtain the assertion. This completes the proof. □

Now, it is easy to see that the following inequalities are valid.

Corollary 3.2.

$$\frac{1}{A(1,k)} < \frac{1}{A(1,k^{1/2})^2} < \frac{2}{\pi}K'(k) < \frac{1}{L(k^{1/2},(k+1)/2)} < \frac{1}{L(1,k)}, \tag{63}$$

where the second and third inequalities are sharp as $k \to 1$.

Proof. Since $D[k_n; 1/A, K'] < 0$ and $D[k_n; K', 1/L] < 0$, by using the discrete monotonicity we have

$$\frac{1}{A(1,k)} < \frac{2}{1+k}\frac{1}{A(1,k_1)} < \frac{2}{\pi}K'(k) < \frac{2}{1+k}\frac{1}{L(1,k_1)} < \frac{1}{L(1,k)}. \tag{64}$$

Simple computations show that these inequalities are equivalent to (63). The sharpness immediately follows from $A(1,1) = 1$, $L(1,1) = 1$, and $2K'(0)/\pi = 1$. ☐

As we mentioned before, the third inequality in (63) was conjectured in [1] without proof and the inequality gives a better bound for the elliptic integral than Carlson's inequality (5). In [3], Anderson et al. gave the following beautiful bounds for the complete elliptic integral by hyperbolic arctangent:

$$\left(\frac{\operatorname{arth} k'}{k'}\right)^{1/2} < \frac{2}{\pi}K'(k) < \frac{\operatorname{arth} k'}{k'}, \tag{65}$$

where the first and second inequalities are sharp as $k \to 1$. They obtained these inequalities by analyzing the monotonicity properties of the corresponding functions. Now, we can prove and refine these inequalities by making use of the right-hand side of Corollary 3.2 and the discrete monotonicity.

Corollary 3.3.

$$\left(\frac{\operatorname{arth} k'}{k'}\right)^{1/2} < \frac{1}{L(1,k^2)^{1/2}} < \frac{2}{\pi}K'(k) < \frac{1}{L(1,k)} < \frac{\operatorname{arth} k'}{k'}, \tag{66}$$

where the second and third inequalities are sharp as $k \to 1$.

Proof. First, we note the following identities [4]:

$$K'(k) = \frac{2}{1+k}K\left(\frac{1-k}{1+k}\right), \tag{67}$$

$$\frac{1}{2k}\log\left(\frac{1+k}{1-k}\right) = \frac{\operatorname{arth} k}{k}. \tag{68}$$

Applying the identity (67) to the right-hand side of (63), we obtain

$$\frac{2}{\pi}K(\frac{1-k}{1+k}) < \log\frac{2k^{1/2}}{1+k}\Big/\left(\frac{2k^{1/2}}{1+k} - 1\right) < \frac{1+k}{2}\frac{\log k}{k-1}. \tag{69}$$

Replacing k in (69) with $(1-k')/(1+k')$ and by using the identity (68), we obtain the right-hand side of (66). Next, we show the discrete monotonicity of the function on the left-hand side, namely, we have

$$D[k_n; (K')^2, 1/L(1,k^2)] = \left(\frac{2}{1+k}\right)^2 - \frac{\log k^2}{k^2-1}\Big/\frac{\log {k_1}^2}{{k_1}^2-1} \tag{70}$$

$$= \frac{1}{(1+k)^2}\frac{1}{\log k_1}\left(\frac{k-1}{k+1}\log k + 4\log\left(\frac{2k^{1/2}}{1+k}\right)\right) > 0, \tag{71}$$

which follows from the observation in (55)–(57). Thus, we obtain

$$\frac{1}{L(1,k^2)} < \left(\frac{2}{1+k}\right)^2\frac{1}{L(1,{k_1}^2)} < \left(\frac{2}{\pi}K'(k)\right)^2. \tag{72}$$

Using identities (67)–(68) and replacing k by $(1-k')/(1+k')$, we get the left-hand side of (66). The sharpness immediately follows since $L(1,1)=1$ and $\pi K'(0)/2=1$. This completes the proof. □

Remark. The right-hand side of the equation in Corollary 3.3 indicates that Carlson's inequality (5) is sharper than the upper bound in (65).

As shown here, discrete monotonicity gives the reason why the logarithmic mean and the hyperbolic arctangent approximate the complete elliptic integral, namely, these functions are related by the discrete monotonicity.

Finally, we study the relation between discrete monotonicity and continuous monotonicity. If the function f/g and p are strictly increasing, we obtain

$$\begin{aligned} D[a_n; f, g] &= \frac{f(a_0)}{f(p(a_0))} - \frac{g(a_0)}{g(p(a_0))} \\ &= \frac{g(a_0)}{f(p(a_0))}\left(\frac{f(a_0)}{g(a_0)} - \frac{f(p(a_0))}{g(p(a_0))}\right) < 0 \end{aligned} \tag{73}$$

and if the function f/g is strictly decreasing and the function p is strictly increasing, the inequality is reversed.

Now, we apply this observation to the following results by Anderson et al. [3] and Alzer et al. [1].

Theorem 3.4. *The function*

$$\frac{kK(k)}{\operatorname{arth} k} \tag{74}$$

is strictly decreasing from $(0,1)$ *onto* $(\pi/2, \infty)$. *In particular, we have*

$$\frac{2}{\pi}K(k) < \frac{\operatorname{arth} k}{k}. \tag{75}$$

Theorem 3.5. *The function*

$$\frac{kK(k)^{4/3}}{\operatorname{arth} k} \tag{76}$$

is strictly increasing from $(0,1)$ *onto* $((\pi/2)^{(4/3)}, \infty)$. *In particular, we have*

$$\left(\frac{\operatorname{arth} k}{k}\right)^{3/4} < \frac{2}{\pi}K(k). \tag{77}$$

Using these theorems and the discrete monotonicity, we obtain the following results.

Theorem 3.6. *Let* $m \geq 2$. *Then*

$$\frac{K'(k)}{K'(k^{1/m})} < \frac{m(1-k^{1/m})}{1-k}, \tag{78}$$

$$\left(\frac{2}{1+k^{1/2}}\right)^{1/2}\left(\frac{2}{1+k}\right)^{1/4} < \frac{K'(k)}{K'(k^{1/2})}, \tag{79}$$

$$\left(\frac{\operatorname{arth} k'}{k'}\right)^{3/4} < \frac{1}{A(1,k)^{1/4}}\frac{1}{L(1,k)^{3/4}} < \frac{2}{\pi}K'(k), \tag{80}$$

where the inequalities (78) *and* (79) *and the second inequality in* (80) *are sharp as* $k \to 1$.

Proof. By replacing k in (74) and (76) by $(1-k)/(1+k)$ and using (67), we find that the function

$$\frac{k-1}{\log k}K'(k) \tag{81}$$

is strictly increasing and the function

$$\left(\frac{1+k}{2}\right)^{1/3}\frac{k-1}{\log k}K'(k)^{4/3} \tag{82}$$

is strictly decreasing. Therefore, from the observation in (73) and since $k < k^{1/m}$, we have

$$D\left[k^{1/m^n};\frac{\log k}{k-1},K'(k)\right] = \frac{m(1-k^{1/m})}{1-k} - \frac{K'(k)}{K'(k^{1/m})} > 0 \tag{83}$$

and

$$\begin{aligned} &D\left[k^{1/2^n};\left(\frac{2}{1+k}\right)^{1/3}\frac{\log k}{k-1},K'(k)^{4/3}\right] \\ &\quad = \left(\frac{2}{1+k^{1/2}}\right)^{2/3}\left(\frac{2}{1+k}\right)^{1/3} - \left(\frac{K'(k)}{K'(k^{1/2})}\right)^{4/3} < 0, \end{aligned} \tag{84}$$

which are equivalent to (78) and (79), respectively.

Similarly, since $k < 2k^{1/2}/(1+k) = k_1$ we get

$$D\left[k_n;\left(\frac{2}{1+k}\right)^{1/3}\frac{\log k}{k-1},K'(k)^{4/3}\right] < 0, \tag{85}$$

which leads to

$$\left(\frac{2}{1+k}\right)^{1/3}\frac{\log k}{k-1} < \left(\frac{2}{1+k}\right)^{4/3}\frac{\log k_1}{k_1-1} < K'(k)^{4/3}. \tag{86}$$

By applying the identity (67)–(68) and replacing k by $(1-k')/(1+k')$ in (85), we obtain (80) in a similar way as in the proof of Corollary 3.3.

Finally, we need to show the sharpness of the above inequalities. The sharpness of inequality (78) follows from

$$\lim_{k\to 1}\frac{m(1-k^{1/m})}{1-k} = 1. \tag{87}$$

The sharpness of inequality (79) and the second inequality in (80) are obvious. This completes the proof. □

Remark. If we put $m = 2$ in (78), we obtain inequality (2).

It is worth asking whether the constants in the presented inequalities are the best ones possible. We close this section with the following conjectures for the sharpness of the inequalities in the above.

Conjecture 3.1. We have

$$\frac{1}{A(1,k^{\alpha_1})^{1/\alpha_1}} < \frac{2}{\pi}K'(k), \tag{88}$$

with the best possible constant $\alpha_1 = 1/2$.

Conjecture 3.2. We have

$$\frac{1}{L(1,k^{\alpha_2})^{1/\alpha_2}} < \frac{2}{\pi}K'(k) < \frac{1}{L(1,k^{\beta_2})^{1/\beta_2}}, \tag{89}$$

with the best possible constants $\alpha_2 = 2$, $\beta_2 = 1$.

Conjecture 3.3. We have

$$\frac{1}{A(1,k)^{1/\alpha_3}}\frac{1}{L(1,k)^{1/\beta_3}} < \frac{2}{\pi}K'(k), \tag{90}$$

with the best possible constants $\alpha_3 = 4$, $\beta_3 = 4/3$.

4. Concluding remarks

In this paper, we obtained several new inequalities for the logarithmic mean and the complete elliptic integral of the first kind by using discrete monotonicity. It was shown that a comparison of two means f and g is reduced to the computation of $D[a_n; f, g]$. Although we focused on these two functions in the present paper, we should note that there are other functions which have a similar infinite product representation such as that in (4). For example, the formula

$$\frac{\sqrt{1-t^2}}{\arccos(t)} = \sqrt{\frac{1+t}{2}} \times \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1+t}{2}}} \times \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1+t}{2}}}} \times \cdots \tag{91}$$

is known and the left-hand side is called the Schwab–Borchardt mean [5,8]. From this infinite product representation, we find that the discrete monotonicity of the Schwab–Borchardt mean can be utilized to obtain new proofs and refinements of some classical inequalities for trigonometric functions [8]. Remarkably, the lemniscate function and the arithmetic-geometric mean have similar properties [5,8,9].

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