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ON EMBEDDINGS OF FINITE SUBSETS OF ℓ_p

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Abstract. We study finite subsets of ℓ_p and show that, up to a nowhere dense and Haar null complement, all of them embed isometrically into any Banach space that uniformly contains ℓ_p^n.

1. Introduction

One of the basic directions of research in Banach space theory is to find some structure in every infinite dimensional Banach space or in an infinite-dimensional Banach space belonging to some class. One of the most important results of this type is the following.

Theorem 1.1 (Dvoretzky's theorem [6]). *For each infinite-dimensional Banach space X, each $n \in \mathbb{N}$, and each $\varepsilon > 0$, there is an n-dimensional subspace $X_n \subset X$ and an isomorphism $T_n : X_n \to \ell_2^n$ such that $\|T\|\|T^{-1}\| \leq 1 + \varepsilon$.*

This theorem was conjectured by Grothendieck in [7].

It is easy to see that we cannot replace $(1+\varepsilon)$ by 1 in this theorem. This follows, for example, from the fact that the unit ball of any finite-dimensional subspace of c_0 is a polytope. The fact that ℓ_p does not contain ℓ_2^n isometrically, unless p is an even integer, was proven in [5].

Another result we would like to mention here is an analogue of Dvoretzky's theorem in the case of ℓ_p. For this, we restrict ourselves to the class of Banach spaces that are isomorphic to ℓ_p and note the following result.

Theorem 1.2 (Krivine's theorem [9]). *For each infinite-dimensional Banach space X isomorphic to ℓ_p, each $n \in \mathbb{N}$, and each $\varepsilon > 0$, there is an n-dimensional subspace $X_n \subset X$ and an isomorphism $T_n : X_n \to \ell_p^n$ such that $\|T\|\|T^{-1}\| \leq 1 + \varepsilon$.*

One of the important directions of current research is the study of embeddability of finite metric spaces into Banach spaces; see Chapter 15 in [10].

Recently Ostrovskii [11, 12] suggested the possibility of proving finite isometric versions of the Dvoretzky and Krivine theorems, respectively. Note that the above mentioned counterexamples to the "Dvoretzky theorem with $\varepsilon = 0$" do not work for finite sets. For c_0 this follows from the well-known observation of Fréchet that each n-element metric space admits an isometric embedding into ℓ_∞^n, and for ℓ_p $(1 \leq p < \infty)$ this follows from results of Ball [2], which we shall discuss later.

This paper is devoted to the following question.

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Question 1. Suppose $1 < p < \infty$, and that X is a Banach space that contains an isomorphic copy of ℓ_p. Then does any finite subset of ℓ_p embed isometrically into X?

The following partial result for Question 1 in the case $p = 2$ was proved by Shkarin in [13].

Theorem 1.3 (Lemma 3 of [13])**.** *Suppose X is any infinite-dimensional Banach space and that Z is any affinely independent subset of ℓ_2. Then Z embeds isometrically into X.*

This result was a strengthening of results obtained in [4]. A different proof was given in [8], and the methods of both [8] and [13] inspired the proofs in this article. We note that in the case $p = 2$, Theorem 1.3 provides a partial positive answer to the following variant of Question 1.

Question 2. Suppose that $1 < p < \infty$ and that X is a Banach space uniformly containing the spaces ℓ_p^n, $n \in \mathbb{N}$. Then does any finite subset of ℓ_p embed isometrically into X?

As before, the weaker conclusion that finite subsets of ℓ_p embed almost isometrically into such a space X follows from a finite quantitative version of Krivine's theorem (see Theorem 2.1 below).

There are natural analogues of Questions 2 and 1 for $p = \infty$. Since any n-point metric space embeds isometrically into ℓ_∞^n, the conclusion in any such analogue is that X contains isometrically all finite metric spaces. The assumption on X is one of the following (in decreasing order of strength): X contains an isomorphic copy of ℓ_∞; X contains an isomorphic copy of c_0; X contains the spaces ℓ_∞^n, $n \in \mathbb{N}$, uniformly. The answer for each of these questions is, however, negative. Indeed, let X be a strictly convex renorming of ℓ_∞. Then subsets of X have the unique metric midpoint property, i.e., there is no collection of 4 distinct points $x, y, z, w \in X$ such that $d(x, z) = d(z, y) = d(x, w) = d(w, y) = \frac{1}{2} d(x, y)$. However, there are finite metric spaces with this property, and thus such a metric space does not embed isometrically into X. In [8] we showed a positive result similar to Theorem 1.3. Let us call a metric space *concave* if it contains no three distinct points x, y, z such that $d(x, z) = d(x, y) + d(y, z)$. Then we have the following.

Theorem 1.4 (Theorem 4.3 of [8])**.** *Suppose that X is some infinite-dimensional Banach space such that the spaces ℓ_∞^n, $n \in \mathbb{N}$, uniformly embed into X. Then if Z is any finite concave metric space, Z embeds isometrically into X.*

In this paper we obtain a partial positive answer to Question 2 similar to Theorems 1.3 and 1.4. As in the case of $p = 2$ there remains a class of subsets of ℓ_p that our proof does not handle. This collection is certainly small in a strong sense. Our main theorem is as follows.

Theorem 1.5. *Suppose $1 < p < \infty$ and that Z is a Banach space that uniformly contains the spaces ℓ_p^n, $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, the set of n-point subsets of ℓ_p that do not embed isometrically into Z is nowhere dense and Haar null as a subset of $\ell_p \times \cdots \times \ell_p$ endowed with one of its standard norms.*

We now describe how our paper is organized. We shall also explain why the case of $p \in (1, \infty)$ is more difficult than the special cases of $p = 2$ and $p = \infty$ and how we handle the additional difficulty.

In Section 2 we recall various definitions and results that will be used throughout the paper (and have already been used in this introduction). The proof of Theorem 1.5 begins in Section 3. Here we prove a result (see Theorem 3.1) that may be of independent interest: almost all n-point subsets of ℓ_p^n have the property that small perturbations of that subset remain subsets of ℓ_p^n. In Section 4 we introduce *Property* K of finite subsets of ℓ_p. Our aim will be to show that every finite subset of ℓ_p with Property K embeds isometrically into a Banach space X that satisfies the assumption of Theorem 1.5.

For general $p \in (1, \infty)$, Property K plays the rôle of affine independence in the case $p = 2$ or concavity in the case $p = \infty$. For $p = 2$, any n-point subset of ℓ_2 embeds isometrically into ℓ_2^n via an orthogonal transformation which preserves affine independence. For $p = \infty$, any n-point metric space embeds into ℓ_∞^n via an isometry, which preserves concavity. For general $p \in (1, \infty)$ it is not even clear if a finite subset of ℓ_p embeds isometrically into ℓ_p^N for any N. In fact, this is true: the result of Ball mentioned earlier states that any n-point subset of ℓ_p embeds isometrically into ℓ_p^N with $N = \binom{n}{2}$. The difficulty is that Ball's proof is not constructive, and Property K is somewhat technical. In Section 4 we prove a version of Ball's result (Lemma 4.1) which is much weaker, in the sense that N will depend on the subset. However, Lemma 4.1 will show that our embedding will preserve Property K.

Remark 1.6. In this article, we do not pay much attention to the case $p = 1$. Indeed, as stated, Question 1 is false. As for ℓ_∞, there is a strictly convex renorming X of ℓ_1, and no finite subset of ℓ_1 that fails the unique metric midpoint property embeds isometrically into such an X. However, one might expect a result similar to Theorem 1.4 to hold when there's a restriction on the type of subset we consider. The methods of this paper rely heavily on the differentiability of the norm of ℓ_p for $1 < p < \infty$ which fails for $p = 1$. Thus our techniques only produce weak conclusions in the case $p = 1$.

2. Classical results and notation

2.1. **Banach space definitions and classical results.** Throughout this paper, for simplicity, we will only be interested in *real* Banach spaces.

Suppose that X and Y are Banach spaces. The *Banach–Mazur distance between X and Y* is defined by $d(X, Y) = \inf\{\|T\|\|T^{-1}\| : T$ is an isomorphism from X to $Y\}$. We say that a Banach space X is *C-isomorphic* to a Banach space Y if there is a linear isomorphism $T : X \to Y$ such that $\|T\|\|T^{-1}\| \leq C$. We say that a Banach space X *almost isometrically contains* a Banach space Y, or that Y *almost isometrically embeds into* X, if for each $\varepsilon > 0$ there is a subspace Z of X such that Z is $(1 + \varepsilon)-$isomorphic to Y. We say that a Banach space X *uniformly contains* spaces X_n, $n \in \mathbb{N}$, if there exist a constant C and subspaces Y_n of X such that Y_n is C-isomorphic to X_n for all n.

We will need the following quantitative version of Krivine's theorem:

Theorem 2.1. *Let $1 \leq p \leq \infty$, let $C \geq 1$, let $\varepsilon > 0$ and let $k \in \mathbb{N}$. Then there is some n (dependent on $p, C, k,$ and ε) such that if a Banach space X is C-isomorphic to ℓ_p^n, then there is a subspace of X that is $(1 + \varepsilon)$-isomorphic to ℓ_p^k.*

For a proof of this theorem, including estimates of the constants involved, we refer the reader to [1].

We introduce the notion of a null set in infinite-dimensional Banach spaces. A well-known fact is that if X is infinite dimensional and separable, and μ is a translation-invariant Borel measure on X, then μ either assigns 0 or ∞ to every open subset of X. However, there are several useful notions of a null set in Banach spaces under which the null sets form a translation-invariant σ-ideal. One such notion that we shall use is that of a *Haar null* set. A Borel set $A \subset X$ is called *Haar null* if there is a Borel probability measure μ on X such that $\mu(x + A) = 0$ for every $x \in X$. It is easy to see that if for some $n \in \mathbb{N}$ there is an n-dimensional subspace Y of X such that the measure $\lambda_n(Y \cap (A + x)) = 0$ for all $x \in X$, where λ_n is n-dimensional Lebesgue measure, then A is Haar null. More on sets of this type, and on other notions of nullity, can be found in [3, Chapter 6].

2.2. **Submersions.** We will need a fact from differential geometry related to submersions. Suppose we have a C^1-map $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ where $n \geq m$. We say that Φ is a *submersion* at a point x if the derivative $D\Phi|_x$ of Φ at x has rank m. The following result is known as the Submersion Theorem and can be found in any introductory text on differential geometry.

Theorem 2.2. *Suppose $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ is a C^1-map, where $n \geq m$. If Φ is a submersion at a point x, then there are open sets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ with $x \in A$, $\Phi(x) \in B$, and $\Phi(A) = B$. Moreover, there is a C^1-map $\Psi : B \to A$ such that $\Phi \circ \Psi$ is the identity on B and $\Psi(\Phi(x)) = x$.*

3. A theorem about finite subsets of ℓ_p^n

In this section we establish a preliminary result that may be of independent interest. Suppose that Z is a metric space on a sequence of points $(z_i)_{i=1}^n$ and Y is a metric space on a sequence of points $(y_i)_{i=1}^n$. We say that Y is an *ε-perturbation* of Z if for each pair i, j we have that $|d_Z(z_i, z_j) - d_Y(y_i, y_j)| < \varepsilon$.

In finite dimensions, the phrase *almost all* will only be used with respect to Lebesgue measure. Throughout this section, we fix some $n \in \mathbb{N}$ and $p \in \mathbb{R}$ with $1 < p < \infty$. We denote by $\|.\|$ the p-norm on ℓ_p^n.

Theorem 3.1. *For almost all n-point subsets X of ℓ_p^n, there is an $\varepsilon > 0$ such that if Y is an ε-perturbation of X, then Y isometrically embeds into ℓ_p^n.*

For our purposes, we will need a slightly stronger property of an n-point subset X of ℓ_p^n. We will need that an ε-perturbation of X isometrically embeds into ℓ_p^n in a way that depends continuously on the perturbation (in a way we will make precise below). This is the content of Theorem 3.2 below, from which Theorem 3.1 will easily follow. To state Theorem 3.2 we will first develop some notation.

Let $M = M_n = \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n \text{ times}}$ and let $U = U_n$ denote the $n \times n$ upper triangular matrices with 0 on the diagonal. We let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$ and let e_i^j be the element of M with e_j in the ith coordinate and zero everywhere else. Note that e_i^j, $1 \leq i, j \leq n$, form a basis of M. Given $x = (x_1, \ldots, x_n) \in M$ we denote the jth coordinate (with respect to the standard basis) of the vector x_i as x_i^j so that $x = \sum_{i,j} x_i^j e_i^j$. Let E_{ij} be the $n \times n$ matrix with 1 in the (i, j)-entry and 0 elsewhere. Note that E_{ij}, $1 \leq i < j \leq n$, forms a basis for U, so the dimension of U is $\binom{n}{2}$.

We define the map $F = F_n : M \to U$ by
$$F(x_1, \dots, x_n) = (\|x_i - x_j\|^p)_{1 \le i < j \le n}.$$
We observe that F is a C^1-map. Indeed, by computing the partial derivatives in the direction e_l^k we get

$$\frac{\partial F}{\partial e_l^k}(z_1, \dots, z_n) = \big(p|z_i^k - z_j^k|^{p-1} \operatorname{sgn}(z_i^k - z_j^k)(\delta_{il} - \delta_{jl})\big)_{1 \le i < j \le n}, \tag{1}$$

and these are evidently continuous. Theorem 3.1 says that F is locally open at almost all n-tuples $(x_1, \dots, x_n)$. This is contained in the following theorem.

Theorem 3.2. *Let $F : M \to U$ be defined as above. Set $G = G_n = \{x \in M : DF|_x$ has rank $\binom{n}{2}\}$. Then G is an open subset of M whose complement has measure zero (and is thus nowhere dense). Moreover, given $x \in G$, there is an open subset A of M containing x, an open subset B of U containing $F(x)$, and a C^1-map $\Phi : B \to A$ such that $F \circ \Phi = Id_B$ and $\Phi(F(x)) = x$.*

Let us briefly spell out how Theorem 3.1 follows from Theorem 3.2. Suppose that $\{x_1, \dots, x_n\}$ is an n-point subset of ℓ_p^n and that $x = (x_1, \dots, x_n) \in G$. Define $X_{ij} = \|x_i - x_j\|^p$ and $X = (X_{ij})_{1 \le i < j \le n}$. Then, since $x \in G$, by Theorem 3.2 there are open subsets A of M and B of U such that $x \in A$, $F(x) = X \in B$ and $F(A) = B$. Thus there is some $\varepsilon > 0$ such that if $|Y_{ij} - X_{ij}| < \varepsilon$ for all i, j, then $(Y_{ij})_{1 \le i < j \le n}$ is an element of B and thus is the image under F of some $y = (y_1, \dots, y_n) \in A$. Hence Y defines a metric on an n-point set and the resulting metric space embeds isometrically into ℓ_p^n. This is slightly more than the statement that ε-perturbations of the metric space $\{x_1, \dots, x_n\}$ with the inherited metric embed isometrically into ℓ_p^n.

Proof of Theorem 3.2. We first show that G is open. Indeed, if $x \in G$, then there is a linear map $B : U \to M$ such that $DF|_x \circ B = \mathrm{Id}_U$. Since DF is continuous, there is some $\varepsilon > 0$ such that whenever y is such that $\|x - y\| < \varepsilon$, $\|DF|_y \circ B - \mathrm{Id}_U\| < 1$. Thus, $DF|_y \circ B$ is invertible and $DF|_y$ has full rank.

We now show that $M \setminus G$ has measure zero. Once we do this, the proof of the theorem is then complete. Indeed, the rest of the statement of Theorem 3.2 follows immediately from the Submersion Theorem.

The proof that $M \setminus G$ has measure zero is done in several steps. We first identify a certain subset of G.

Lemma 3.3. *Let $H = \{(x_1, \dots, x_n) \in M : x_i = e_i + \sum_{j=i+1}^n x_i^j e_j$ for each $i = 1, \dots, n\}$. Then if $x \in H$, the partial derivatives $\frac{\partial F}{\partial e_l^k}(x)$, $1 \le k < l \le n$, are linearly independent. In particular, $H \subset G$.*

Proof. Fix $x = (x_1, \dots, x_n) \in H$. By (1) we see that the (i, j)-entry of $\frac{\partial F}{\partial e_l^k}(x)$ is zero unless $j = l$ and $i \le k$. We can hence expand $\frac{\partial F}{\partial e_l^k}(x)$ in terms of the matrices E_{kl} as
$$\frac{\partial F}{\partial e_l^k}(x) = -pE_{kl} + \sum_{i=1}^{k-1} \alpha_i^k E_{il},$$
where α_i^k are constants depending on x. It follows by induction on k that E_{kl} is in the span of $\frac{\partial F}{\partial e_i^j}(x)$ for all $1 \le k < l \le n$. This completes the proof of the lemma. □

Let us now define $V = \{x = (x_1, \ldots, x_n) \in M :$ there are $i, j, k \in \{1, \ldots, n\}$ such that $i \neq j$ and $x_i^k = x_j^k\}$. Note that $M \setminus V$ has finitely many connected components which are open and convex. Since $\mu(V) = 0$, in order to show that $\mu(M \setminus G) = 0$, it suffices to show that $\mu(C \setminus G) = 0$ for every connected component C of $M \setminus V$. The following lemma will be vital to this aim.

Lemma 3.4. *Suppose that $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are two points in the same connected component of $M \setminus V$, and suppose that $\frac{\partial F}{\partial e_i^j}(x)$, $1 \leq i < j \leq n$, are linearly independent. Then, for all but finitely many values of $t \in [0, 1]$, the partial derivatives $\frac{\partial F}{\partial e_i^j}((1-t)x + ty)$, $1 \leq i < j \leq n$, are linearly independent. In particular, for all but finitely many values of $t \in [0, 1]$, we have that $(1-t)x + ty \in G$.*

Proof. Define J to be the set $\{(k, l) : 1 \leq k < l \leq n\}$. For $\sigma = (i, j) \in J$ we will write $e_\sigma = e_i^j$, and for $X \in U$ we will write X_σ for the (i, j)-entry of X. By assumption, the $J \times J$ matrix given by $((\frac{\partial F}{\partial e_\sigma}(x))_\rho)$ has nonzero determinant. We now define a function $g : [0, 1] \to \mathbb{R}$ by setting

$$g(t) = \det\left(\left(\frac{\partial F}{\partial e_\sigma}((1-t)x + ty)\right)_\rho\right) = \det(X(t)).$$

Using (1) and the fact that x and y are from the same component of $M \setminus V$, for each $\sigma, \rho \in J$, the matrix $X(t)$ has (σ, ρ)-entry $p(a_{\sigma,\rho}t + b_{\sigma,\rho})^{p-1}\varepsilon_{\sigma,\rho}$, where $a_{\sigma,\rho}$ and $b_{\sigma,\rho}$ are nonzero constants with $a_{\sigma,\rho}t + b_{\sigma,\rho} > 0$ for all $t \in [0, 1]$ and $\varepsilon_{\sigma,\rho} \in \{-1, 0, 1\}$.

By compactness there is an open connected subset U of $\mathbb{C}$ containing $[0, 1]$ such that the real part of $a_{\sigma,\rho}t + b_{\sigma,\rho}$ is positive for each $t \in U$. It follows that the function g extends analytically to all of U, and therefore by the identity principle (and the fact that $g(0)$ is nonzero), g has at most finitely many zeroes in $[0, 1]$. $\square$

Consider the subset R of M defined by

$$R = \{(x_1, \ldots, x_n) \in M : x_i^i > x_j^i \text{ for each } 1 \leq i < j \leq n\}.$$

Note that for each component C of $M \setminus V$ either $C \subset R$ or $C \cap R = \varnothing$. We next show that in order to prove that $\mu(C \setminus G) = 0$ for every component C of $M \setminus V$, it is sufficient to consider components C such that $C \subset R$.

Fix $(x_1, \ldots, x_n) \in M \setminus V$. Define a permutation $\pi \in S_n$ recursively as follows: for $j = 1, \ldots, n$, let $\pi(j)$ be the unique $i \in \{1, \ldots, n\} \setminus \{\pi(1), \ldots, \pi(j-1)\}$ such that

$$x_i^j > x_k^j \text{ for all } k \in \{1, \ldots, n\} \setminus \{\pi(1), \ldots, \pi(j-1), i\}.$$

It then follows that $x_{\pi(j)}^j > x_{\pi(k)}^j$ for all $1 \leq j < k \leq n$, and hence $(x_{\pi(1)}, \ldots, x_{\pi(n)}) \in R$.

Define a map $A_\pi : M \to M$ by $A_\pi(y_1, \ldots, y_n) = (y_{\pi(1)}, \ldots, y_{\pi(n)})$ and a map $B_\pi : U \to U$ by $B_\pi((X_{ij})_{1 \leq i < j \leq n}) = (Y_{ij})_{1 \leq i < j \leq n}$, where

$$Y_{ij} = \begin{cases} X_{\pi(i),\pi(j)} & \text{if } \pi(i) < \pi(j), \\ X_{\pi(j),\pi(i)} & \text{if } \pi(j) < \pi(i). \end{cases}$$

We note that $B_\pi^{-1} F A_\pi = F$, and thus $B_\pi^{-1} DF|_{A_\pi(x)} A_\pi = DF|_x$, so to verify that F has full rank at x, it is sufficient to verify that F has full rank at $A_\pi(x)$, which lies in R. This completes the proof that it is sufficient to show that $\mu(C \setminus G) = 0$ whenever C is a component of $M \setminus V$ with $C \subset R$.

Fix a component C of $M \setminus V$ with $C \subset R$. If $\mu(C \setminus G) > 0$, then by Lebesgue's density theorem, there is a point $y \in C$ such that $\lim_{\varepsilon \to 0} \frac{\mu(B_\varepsilon)(y) \cap (C \setminus G))}{\mu(B_\varepsilon(y))} = 1$. For $i, j \in \{1, \ldots, n\}$, define

$$x_i^j = \begin{cases} 1 & \text{if } i = j, \\ 1 & \text{if } y_i^j > y_j^j, \\ 0 & \text{else.} \end{cases}$$

It is easy to verify that if $y_i^k < y_j^k$, then $x_i^k \leq x_j^k$, and thus $(1-t)x + ty \in C$ for all $t \in (0, 1]$. Moreover, since $y \in R$, we have $x \in H$. It follows by Lemma 3.3 that the partial derivatives $\frac{\partial F}{\partial e_i^j}(x)$, $1 \leq i < j \leq n$, are linearly independent. Hence there is an $\varepsilon > 0$ such that at each $z \in B_\varepsilon(x)$ the same holds, i.e., $\frac{\partial F}{\partial e_i^j}(z)$, $1 \leq i < j \leq n$, are linearly independent. Choose $t \in (0, 1)$ such that $z = (1-t)x + ty \in B_\varepsilon(x)$. Then $z \in B_\varepsilon(x) \cap C$, so there is some $\delta > 0$ such that $B_\delta(z) \subset B_\varepsilon \cap C$.

The Lebesgue density at y is equal to 1, so by making δ smaller, we may assume that $B_\delta(y) \subset C$ and $\mu(B_\delta(y) \setminus G) > 0$. By Lemma 3.4, each line in the direction $y - x$ through a point in $B_\delta(z)$ intersects $B_\delta(y) \setminus G$ in at most finitely many points. The lines in the direction $y - x$ through $B_\delta(z)$ can be parametrised whereby they intersect the hyperplane through z whose normal is $y - x$. This is a $(\binom{n}{2} - 1)$-dimensional hyperplane. The measure of $B_\delta(y) \setminus G$ can be given, by Fubini's theorem, as

$$\mu(B_\delta(y) \setminus G) = \int_{\mathbb{R}^{\binom{n}{2}-1}} \int_{[a_s, b_s]} 1_{L(s) \cap B_\delta(y) \setminus G} \, d\mu' ds,$$

where $L(s)$ is the line through the point s in the previously mentioned hyperplane going through s, $[a_s, b_s]$ is the interval for which the line $L(s)$ intersects the sphere $B_\delta(y)$, and μ' is 1-dimensional Lebesgue measure. This integral is equal to zero, as $L(s) \cap B_\delta(y) \setminus G$ is finite. This is a contradiction on y being a point of Lebesgue density, and thus of $C \setminus G$ having nonzero measure. Thus $\mu(C \setminus G) = 0$ and the proof of Theorem 3.2 is complete. □

4. The proof of Theorem 1.5

Given a subset $M = \{m_1, \ldots, m_n\}$ of $\mathbb{N}$ with $m_1 < m_2 < \cdots < m_n$, if $x = (x_i)_{i=1}^\infty \in \ell_p$ or $x = (x_i)_{i=1}^N \in \ell_p^N$ with $N \geq m_n$, we define $P_M(x) = (x_{m_1}, \ldots, x_{m_n})$. If $N \in \mathbb{N}$, we write P_N instead of $P_{\{1,\ldots,N\}}$.

We say that an n-tuple $(x_1, \ldots, x_n)$ in ℓ_p (or ℓ_p^N) has *Property K* if there is an $M \subset \mathbb{N}$ (or $M \subset \{1, \ldots, N\}$, respectively) of size n such that $(P_M x_1, \ldots, P_M x_n) \in G_n$, where G_n is the set defined in Theorem 3.2. Note that the set of n-tuples with property K is open since the set G_n is open.

We prove Theorem 1.5 by showing that the closed set of n-tuples without Property K is Haar null (and thus nowhere dense) and that an n-tuple with Property K embeds isometrically into a Banach space that satisfies the assumption of Theorem 1.5. We will need three lemmas.

Lemma 4.1. *Suppose that $x = (x_1, \ldots, x_n)$ is an n-tuple in ℓ_p with Property K. Then there is some $N \in \mathbb{N}$, and vectors $y_1, \ldots, y_n \in \ell_p^N$ such that $\|y_i - y_j\| = \|x_i - x_j\|$, and the n-tuple $(y_1, \ldots, y_n)$ has Property K.*

Remark 4.2. This is the variant of Ball's result mentioned in the Introduction. Here $\|.\|$ denotes the ℓ_p norm.

Proof of Lemma 4.1. Let $M \subset \mathbb{N}$ be such that $|M| = n$ and $(P_M x_1, \dots, P_M x_n) \in G_n$. After an isometry (permuting the indices), we may assume without loss of generality that $M = \{1, \dots, n\}$. Then, since $(P_n x_1, \dots, P_n x_n) \in G_n$ and G_n is open, there is some $\varepsilon > 0$ such that if $z_i \in \ell_p^n$ and $\|z_i - P_n x_i\| < \varepsilon$, then $(z_1, \dots, z_n) \in G_n$.

Since $(P_n x_1, \dots, P_n x_n) \in G_n$, by Theorem 3.2, there are open sets $A \ni (P_n x_1, \dots, P_n x_n)$, $B \ni F(P_n x_1, \dots, P_n x_n)$, and a C^1-map $\Phi : B \to A$ such that $F \circ \Phi = \mathrm{Id}_B$ and $\Phi(F(x)) = x$.

Fix $N \geq n$, and define $\rho_{ij} = \rho_{ij}(N)$ by $\|x_i - x_j\|^p = \|P_N x_i - P_N x_j\|^p + \rho_{ij}$. Since $\rho_{ij} \to 0$ as $N \to \infty$, there is an $N > n$ such that the element $Z = Z(N) = (\|P_n x_i - P_n x_j\|^p + \rho_{ij})_{1 \leq i < j \leq n}$ of U is in the set B. Set $z = z(N) = (z_1, \dots, z_n) = \Phi(Z)$. By the continuity of Φ at the point $F(P_n x_1, \dots, P_n x_n)$, if N is sufficiently large, then $\|z_i - P_n x_i\| < \varepsilon$, and hence $(z_1, \dots, z_n) \in G_n$.

We now define the points $y_1, \dots, y_n \in \ell_p^N$ by

- $P_n y_i = z_i$,
- $(P_N - P_n) y_i = (P_N - P_n) x_i$.

We now verify that $(y_1, \dots, y_n)$ has Property K and that $\|y_i - y_j\| = \|x_i - x_j\|$. The first of these is clear, $(P_n y_1, \dots, P_n y_n)$ is in G_n by construction.

To verify that $\|y_i - y_j\| = \|x_i - x_j\|$, note that

$$\|y_i - y_j\|^p = \|P_n y_i - P_n y_j\|^p + \|(P_N - P_n) y_i - (P_N - P_n) y_j\|^p,$$

which is equal to

$$\|z_i - z_j\|^p + \|(P_N - P_n) x_i - (P_N - P_n) x_j\|^p.$$

By the definition of $(z_1, \dots, z_n)$, we see that $\|z_i - z_j\|^p = \|P_n x_i - P_n x_j\|^p + \rho_{ij}$. By the definition of ρ_{ij}, we thus get that $\|y_i - y_j\|^p = \|x_i - x_j\|^p$. ☐

We have now shown that if a subset of ℓ_p has Property K, then it is isometric to a subset of ℓ_p^N with Property K. We next show a slight variant of Theorem 3.2.

Lemma 4.3. *Suppose $x = (x_1, \dots, x_n)$ is an n-tuple in ℓ_p^N, $N \geq n$, with Property K. Then there is some $\varepsilon > 0$ such that any ε-perturbation of X can be embedded into ℓ_p^N with the embedding depending continuously on the perturbation.*

At the beginning of the proof of Lemma 4.3 we will make it clear what continuous dependence on the perturbation means in a way similar to the precise statement of Theorem 3.2.

Proof. Define $\tilde{F} : \underbrace{\mathbb{R}^N \times \cdots \times \mathbb{R}^N}_{n \text{ times}} \to U_n$ by

$$\tilde{F}(y_1, \dots, y_n) = (\|y_i - y_j\|)_{1 \leq i < j \leq n},$$

where we note that there is no pth power of the norm. Our goal is to show that there is an open subset $\tilde{B}$ of U_n and a continuous map $\Psi : \tilde{B} \to \underbrace{\mathbb{R}^N \times \cdots \times \mathbb{R}^N}_{n \text{ times}}$ such that:

- $\tilde{F}(x) \in \tilde{B}$,
- $\Psi(\tilde{F}(x)) = x$,
- $\tilde{F} \circ \Psi = \mathrm{Id}_{\tilde{B}}$.

Let $M \subset \{1, \dots, N\}$ be such that $|M| = n$ and $(P_M x_1, \dots, P_M x_n) \in G_n$. Again, without loss of generality, we may assume that $M = \{1, \dots, n\}$.

By Theorem 3.2, there exist open sets $A \ni (P_n x_1, \dots, P_n x_n)$, $B \ni F(P_n x_1, \dots, P_n x_n)$ and a C^1-map $\Phi : B \to A$ such that $\Phi(F(P_n x_1, \dots, P_n x_n)) = (P_n x_1, \dots, P_n x_n)$ and $F \circ \Phi = \mathrm{Id}_B$. Fix $\varepsilon > 0$ such that if $Y = (Y_{ij})_{1 \le i < j \le n}$ is such that $|Y_{ij} - \|x_i - x_j\|^p| < \varepsilon$, then $Y \in B$.

Choose $\delta = \delta(\varepsilon) > 0$ to be specified later. We set $\tilde{B} = \{Y \in U_n : |Y_{ij} - \|x_i - x_j\|| < \delta$ for all pairs $i, j\}$.

Fix $Y = (Y_{ij})_{1 \le i < j \le n} \in \tilde{B}$. We define $\Psi(Y)$ similarly to the definition of the points $(y_1, \dots, y_n)$ in the proof of Lemma 4.1. Define $\rho_{ij} = Y_{ij} - \|x_i - x_j\|$ and $\varepsilon_{ij} = \varepsilon_{ij}(Y)$ by $(\|x_i - x_j\| + \rho_{ij})^p = \|x_i - x_j\|^p + \varepsilon_{ij}$. If $|\rho_{ij}|$ is sufficiently small (i.e., our choice of δ is sufficiently small), then $(\|P_n x_i - P_n x_j\|^p + \varepsilon_{ij})_{1 \le i < j \le n}$ is in B. Define $z_i = \Phi((\|P_n x_i - P_n x_j\|^p + \varepsilon_{ij})_{1 \le i < j \le n})$. We then set $\Psi(Y)$ to be the n-tuple $(y_1, \dots, y_n)$ where:

- $P_n y_i = z_i$,
- $(P_N - P_n) y_i = x_i$.

We verify that $\|y_i - y_j\| = \|x_i - x_j\| + \rho_{ij} = Y_{ij}$, i.e., that $\tilde{F}(\Psi(Y)) = Y$, as this is the only one of the three properties listed above that is nontrivial.

Indeed,

$$\|y_i - y_j\|^p = \|P_n y_i - P_n y_j\|^p + \|(P_N - P_n) y_i - (P_N - P_n) y_j\|^p,$$

which (by the definition of y_i) equals

$$\|z_i - z_j\|^p + \|(P_N - P_n) x_i - (P_N - P_n) x_j\|^p,$$

and this is equal (by the definition of z_i) to

$$\|x_i - x_j\|^p + \varepsilon_{ij}.$$

By the definition of ε_{ij}, this is equal to $(\|x_i - x_j\| + \rho_{ij})^p$, which is as required. $\square$

Our next lemma shows that if we have an n-point subset of ℓ_p^N with Property K, then it embeds isometrically into any Banach space satisfying the assumption of Theorem 1.5. This result is, in some sense, dual to Theorem 3.1. Where Theorem 3.1 says *small perturbations of the metric space embed into the Banach space*, this is saying that *the metric space embeds into small perturbations of the Banach space.*

Lemma 4.4. *Suppose $x = (x_1, \dots, x_n)$ is an n-tuple in ℓ_p^N, $N \ge n$, with Property K. Then there is some $\delta > 0$ such that if $d(E, \ell_p^N) < 1 + \delta$, then $\{x_1, \dots, x_n\}$ with the metric inherited from ℓ_p^N embeds isometrically into E.*

Proof. Let $\tilde{F}, \tilde{B}$, and Ψ be as in the proof of Lemma 4.3. Choose $\varepsilon > 0$ such that if $Y = (Y_{ij})_{1 \le i < j \le n} \in U$, $|Y_{ij} - \|x_i - x_j\|| < \varepsilon$, then $Y \in \tilde{B}$. Fix some $\delta > 0$ and let E be an N-dimensional Banach space such that $d(E, \ell_p^N) < 1 + \delta$. We will find the value of δ later, and it will be expressed in terms of x and ε only. We may assume that $E = (\mathbb{R}^N, \|.\|_E)$ and that the norm on E satisfies $\|y\|_E \le \|y\| \le (1 + \delta)\|y\|_E$, where $\|.\|$ denotes the ℓ_p norm.

Let $\rho = (\rho_{ij})_{1 \le i < j \le n}$ be an element of the space $[0, \varepsilon]^{\binom{n}{2}}$. We define a metric space $Z(\rho)$ as follows:

- $Z(\rho)$ is a metric space on n distinct points $z_1, \dots, z_n$.
- $d(z_i, z_j) = \|x_i - x_j\| + \rho_{ij}$.

By the choice of ε, and since $\tilde{F} \circ \Psi = \mathrm{Id}_{\tilde{B}}$, it follows that $Z(\rho)$ is a metric space isometric to a subset of ℓ_p^N. Through slight abuse of notation, in what follows we identify $Z(\rho)$ with its distance matrix, i.e., $Z(\rho) = (d(z_i, z_j))_{1 \leq i < j \leq n}$.

Now define $\varphi : [0, \varepsilon]^{\binom{n}{2}} \to [0, \varepsilon]^{\binom{n}{2}}$ by

$$\varphi(\rho) = (\|x_i - x_j\| + \rho_{ij} - \|\Psi(Z(\rho))_i - \Psi(Z(\rho))_j\|_E)_{1 \leq i < j \leq n}\,.$$

We claim that if δ is sufficiently small, then φ is well defined. To see that $\varphi(\rho)_{ij} > 0$, note that $\varphi(\rho)_{ij} \geq \|x_i - x_j\| + \rho_{ij} - \|\Psi(Z(\rho))_i - \Psi(Z(\rho))_j\| = 0$, where we have used that $\|y\| \geq \|y\|_E$ for all $y \in \mathbb{R}^N$.

On the other hand, $\varphi(\rho)_{ij} \leq \|x_i - x_j\| + \rho_{ij} - \frac{1}{1+\delta}\|\Psi(Z(\rho))_i - \Psi(Z(\rho))_j\| = \frac{\delta}{\delta+1}(\|x_i - x_j\| + \rho_{ij})$, where we have used that $\|y\| \leq (1+\delta)\|y\|_E$ for all $y \in \mathbb{R}^N$. So if δ is sufficiently small, then this is less than ε.

Since φ is a continuous map from a compact convex subset of $\mathbb{R}^{\binom{n}{2}}$ to itself, it follows from Brouwer's fixed point theorem that φ has a fixed point ρ. Letting $(y_1, \ldots, y_n) = \Psi(Z(\rho))$, the map sending x_i to y_i is an isometric embedding of $\{x_1, \ldots, x_n\}$ into E. $\square$

Remark 4.5. Suppose we had $x_1, \ldots, x_n \in \ell_p$ such that the map $\tilde{F} : \underbrace{\ell_p \times \ldots \times \ell_p}_{n \text{ times}} \to U$, $\tilde{F}(y_1, \ldots, y_n) = (\|y_i - y_j\|)_{1 \leq i < j \leq n}$, had a continuous right inverse at $\tilde{F}(x_1, \ldots, x_n)$. Then an identical argument to the proof of Lemma 4.4 would show that there is some $\delta > 0$ such that if $d(Y, \ell_p) < 1 + \delta$, then Y contains an isometric copy of $\{x_1, \ldots, x_n\}$. Since the assumption in Theorem 1.5 is *weaker* than the Banach space containing an isomorphic copy of ℓ_p, we had to choose a more technical version of Property K than simply "$\tilde{F}$ has a continuous right inverse at $(x_1, \ldots, x_n)$". This stronger assumption also motivated Lemma 4.1.

We now give the proof of Theorem 1.5.

Proof of Theorem 1.5. By a combination of Lemmas 4.1, 4.3 and 4.4, we see that if an n-tuple $(x_1, \ldots, x_n)$ in ℓ_p has Property K, then there is some $N \in \mathbb{N}$ and $\delta > 0$ such that if Y is a Banach space with $d(Y, \ell_p^N) < 1 + \delta$, then $\{x_1, \ldots, x_n\}$ with the metric inherited from ℓ_p^N embeds isometrically into Y. By Krivine's theorem, Theorem 2.1, any Banach space X satisfying the assumption of the theorem (i.e., containing the spaces ℓ_p^n, $n \in \mathbb{N}$, uniformly) contains a subspace Y with $d(Y, \ell_p^N) < 1 + \delta$. Thus $\{x_1, \ldots, x_n\}$ with the metric inherited from ℓ_p^N embeds isometrically into X.

To conclude, we just need to show that the set A of all n-tuples that do not have Property K is Haar null. Indeed, the intersection of A with the finite-dimensional space $\ell_p^n \times \cdots \times \ell_p^n$ is contained in the complement of G_n, which by Theorem 3.2 has measure zero. Note also that A is translation-invariant. Thus, by the characterization of Haar null sets stated in Section 2.1, A is Haar null. Since A is closed, it follows that A is nowhere dense. $\square$

5. Further remarks and open problems

In this section we give some remarks on the special cases of ℓ_2, ℓ_∞, and ℓ_1, and pose some open problems.

In the case ℓ_2, we deduce Theorem 1.3 from our results.

Theorem 5.1. *Every finite affinely independent subset of ℓ_2 isometrically embeds into every infinite-dimensional Banach space X.*

Proof. First note that every affinely independent set has a linearly independent translate, so without loss of generality, we may reduce to the case of linearly independent sets. Let $e_1, e_2, \dots$ be an orthonormal basis of ℓ_2. If $\{x_1, \dots, x_n\}$ is a linearly independent subset of ℓ_2, then there is some isometry Θ such that $\Theta(x_1) \in \mathrm{span}\{e_1\}$, $\Theta(x_2) \in \mathrm{span}\{e_1, e_2\}$, etc. Such a Θ is constructed by induction and the Gram–Schmidt process applied to the vectors $\{x_1, \dots, x_n\}$. Then a minor variant of Lemma 3.3 (in which the coefficient of e_i in x_i is nonzero, but not necessarily one) shows that the n-tuple $(x_1, \dots, x_n)$ belongs to G_n. Thus $(\Theta x_1, \dots, \Theta x_n)$ (which is isometric to $(x_1, \dots, x_n)$) has Property K.

Applying Lemma 4.4 to $(\Theta x_1, \dots, \Theta x_n)$ we see that there exist some $\delta > 0$ such that whenever E is an n-dimensional Banach space with $d(E, \ell_2^n) < 1 + \delta$, then $(\Theta x_1, \dots, \Theta x_n)$ embeds isometrically into E. By Dvoretzky's theorem, if X is infinite dimensional, there is a subspace Z of X such that $d(Z, \ell_2^n) < 1 + \delta$, and thus Z contains an isometric copy of $(\Theta x_1, \dots, \Theta x_n)$ (which is isometric to $(x_1, \dots, x_n)$). $\square$

In the case of ℓ_∞, the proof of Theorem 1.4 (given as Theorem 4.3 in [8]) essentially proceeds by directly showing that if $(x_1, \dots, x_n)$ is a concave metric space, then the mapping $\tilde{F}$ is locally open at $(x_1, \dots, x_n)$. This argument does not use differentiation: the norm on ℓ_∞ is easy to compute.

In the case of ℓ_1, the majority of the proofs in this paper simply do not work. In the case $p = 1$ the computation of the derivative (Equation (1)) yields $\frac{\partial F}{\partial e_l^k} = (\mathrm{sgn}(x_i^k - x_j^k)_{1 \le i < j \le n})$. Thus the function is locally open if the collection forms linearly independent matrices. This is, however, not the case on a large set as it is for the case $1 < p < \infty$. However, if it is true at a point $x = (x_1, \dots, x_n)$, the rest of the proofs presented here work identically.

We now list some open problems. The case $p = 2$ was originally raised by Ostrovskii in [12], who asked:

Question 3. Let X be an infinite-dimensional Banach space and A a finite subset of ℓ_2. Then does A isometrically embed into X?

The general question of Ostrovskii, given in [11], still remains open.

Question 4. Let X be an infinite-dimensional Banach space containing ℓ_p isomorphically. Then does every finite subset of ℓ_p embed isometrically into X?

The way we approached this question leads to the following natural variant.

Question 5. Let X be an infinite-dimensional Banach space that uniformly contains ℓ_p^n, $n \in \mathbb{N}$. Then does every finite subset of ℓ_p embed isometrically into X?

As detailed in the introduction, there can be no positive results in the cases $p = 1$ and $p = \infty$. However, the known partial answers lead to the following open question.

Question 6. Let $p = 1$ or $p = \infty$. Which n-point subsets of ℓ_p embed isometrically into any Banach space X that uniformly contains the spaces ℓ_p^n, $n \in \mathbb{N}$?

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