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A FIXED POINT THEOREM FOR MONOTONE ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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ABSTRACT. Let C be a nonempty, bounded, closed, and convex subset of a Banach space X and $T : C \to C$ be a monotone asymptotically nonexpansive mapping. In this paper, we investigate the existence of fixed points of T. In particular, we establish an analogue to the original Goebel and Kirk's fixed point theorem for asymptotically nonexpansive mappings.

1. INTRODUCTION

Recently a new direction has been discovered dealing with the extension of the Banach Contraction Principle [2] to partially ordered metric spaces. Ran and Reurings [14] successfully carried out such an attempt while investigating the solution(s) to the matrix equation:

$$X = Q \pm \sum_{i=1}^{i=m} A_i^* F(X) A_i,$$

where $X \in H(n)$, the set of $n \times n$ Hermitian matrices, $F : H(n) \to H(n)$ is a monotone function, i.e., $F(X_1) \leq F(X_2)$ if $X_1 \leq X_2$, which maps the set of all $n \times n$ positive definite matrices $P(n)$ into itself, $A_1, \ldots, A_m$ are arbitrary $n \times n$ matrices and $Q \in P(n)$, a result known before to Turinici [15]. Another similar approach was carried out in [12] with applications to some differential equations. Jachymski [9] was the first to give a more general unified version of these extensions by considering graphs instead of a partial order. In all these works, the mappings considered are monotone contractions. The case of monotone nonexpansive mappings was first considered in [1]. Then the race was on to find out whether the classical fixed point theorems for nonexpansive mappings still hold for monotone nonexpansive mappings. In particular, an analogue to Browder [4] and Göhde [8] fixed point theorems for monotone mappings does hold [3]. But it is still unknown whether an analogue to the classical Kirk's fixed point theorem [10] holds for monotone nonexpansive mappings. The difficulty in doing this resides in the fact that the monotone Lipschitzian mappings enjoy nice properties only on comparable elements. In fact, they may not even be continuous, a property obviously shared by Lipschitzian mappings. In this paper, we extend Goebel and Kirk's fixed point theorem [7] for asymptotically nonexpansive mappings to the case of monotone mappings.

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An interesting reference with many applications of the fixed point theory of monotone mappings is the excellent book by Carl and Heikkilä [5].

2. Preliminaries

Let (M, d) be a metric space endowed with a partial order $\preceq$. We will say that $x, y \in M$ are comparable whenever $x \preceq y$ or $y \preceq x$. Next we give the definition of monotone mappings.

Definition 2.1. Let $(M, d, \preceq)$ be a metric space endowed with a partial order. Let $T : M \to M$ be a map. T is said to be monotone or order-preserving if

$$x \preceq y \Longrightarrow T(x) \preceq T(y),$$

for every $x, y \in M$.

Next we give the definition of monotone Lipschitzian mappings.

Definition 2.2. Let $(M, d, \preceq)$ be a metric space endowed with a partial order. Let $T : M \to M$ be a map. T is said to be monotone Lipschitzian mapping if T is monotone and there exists $k \geq 0$ such that

$$d(T(x), T(y)) \leq k\ d(x, y),$$

for every $x, y \in M$ such that x and y are comparable. We will say that T is a monotone asymptotically nonexpansive mapping if there exists $\{k_n\}$ a sequence of positive numbers such that $\lim\limits_{n\to+\infty} k_n = 1$ and

$$d(T^n(x), T^n(y)) \leq k_n\ d(x, y),$$

for every comparable element $x, y \in M$. A point $x \in M$ is said to be a fixed point of T whenever $T(x) = x$. The set of fixed points of T will be denoted by $Fix(T)$.

Note that monotone Lipschitzian mappings are not necessarily continuous. They usually have a good topological behavior on comparable elements but not on the entire set on which they are defined.

Before we close this section, recall that a sequence $\{x_n\}_{n\in\mathbb{N}}$ in a partially ordered set $(M, \preceq)$ is said to be

(i) monotone increasing if $x_n \preceq x_{n+1}$, for every $n \in \mathbb{N}$;
(ii) monotone decreasing if $x_{n+1} \preceq x_n$, for every $n \in \mathbb{N}$;
(iii) a monotone sequence if it is either monotone increasing or decreasing.

3. Monotone asymptotic nonexpansive mappings

The fixed point theory for asymptotically nonexpansive mappings finds its root in the work of Goebel and Kirk [7]. Following some successful results on monotone mappings in recent years, the original fixed point theorem of Goebel and Kirk for these mappings was elusive till now. The setting will be uniformly convex Banach spaces partially ordered.

Definition 3.1. Let $(X, \|.\|)$ be a Banach space. We say that X is uniformly convex (in short, UC) if for every $\varepsilon > 0$

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\|; \|x\| \leq 1,\ \|y\| \leq 1,\ \|x - y\| \geq \varepsilon\right\} > 0.$$

The function δ is known as the modulus of uniform convexity of X.

The following technical lemma will be useful in the proof of our main result. This lemma is stated in terms of type functions but its origin may be found in the work of Edelstein [6] on an asymptotic center of a sequence.

Lemma 3.2. *Let C be a nonempty closed convex subset of uniformly convex Banach space $(X, \|.\|)$. Let $\tau : C \to [0, +\infty)$ be a type function, i.e., there exists a bounded sequence $\{x_n\} \in X$ such that*

$$\tau(x) = \limsup_{n\to+\infty} \|x_n - x\|,$$

for every $x \in C$. Then τ has a unique minimum point $z \in C$ such that

$$\tau(z) = \inf\{\tau(x);\ x \in C\} = \tau_0.$$

Moreover, if $\{z_n\}$ is a minimizing sequence in C, i.e., $\lim\limits_{n\to+\infty} \tau(z_n) = \tau_0$, then $\{z_n\}$ converges strongly to z.

Proof. Since τ is continuous and convex, the existence and uniqueness of the minimum point of τ is well known and its proof will be omitted. We will focus on the proof of the second part of the lemma. Let $\{z_n\}$ be a minimizing sequence of τ. Let us prove that $\{z_n\}$ converges to the minimum point z. This conclusion is obvious if $\tau_0 = 0$. Assume $\tau_0 > 0$ and $\{z_n\}$ does not converge to z. Since $\{x_n\}$ is bounded, then $\{z_n\}$ is also bounded. Therefore, there exists $R > 0$ such that

$$\max\Big(\|x_n - z_m\|, \|x_n - z\|\Big) \le R,$$

for every $n, m \in \mathbb{N}$. Since $\{z_n\}$ does not converge to z, we may assume that

$$\varepsilon = \inf\left\{\frac{\|z_m - z\|}{R};\ m \in \mathbb{N}\right\} > 0.$$

Using the definition of the modulus of uniform convexity of X, we get

$$\left\|x_n - \frac{z_m + z}{2}\right\| = \left\|\frac{(x_n - z_m) + (x_n - z)}{2}\right\| \le \max\Big(\|x_n - z_m\|, \|x_n - z\|\Big)\left(1 - \delta(\varepsilon)\right),$$

for every $n, m \in \mathbb{N}$. If we let $n \to +\infty$, taking the limit-sup, we get

$$\tau\left(\frac{z_m + z}{2}\right) \le \max\Big(\tau(z_m), \tau(z)\Big)\left(1 - \delta(\varepsilon)\right),$$

for every $m \in \mathbb{N}$, which implies

$$\tau_0 \le \tau(z_m)\ \left(1 - \delta(\varepsilon)\right).$$

If we let $m \to +\infty$, we get $\tau_0 \le \tau_0\ (1 - \delta(\varepsilon)) < \tau_0$. This contradiction implies that $\{z_n\}$ does converge to z. The proof of Lemma 3.2 is complete. $\square$

Since the main result of this work is set in a partially ordered Banach space, we assume that $(X, \|.\|)$ is endowed with a partial order $\preceq$. Throughout, we assume that order intervals are convex and closed. Recall that an order interval is any of the subsets

$$[a, \to) = \{x \in X; a \preceq x\} \quad \text{and} \quad (\leftarrow, b] = \{x \in X; x \preceq b\},$$

for every $a, b \in X$.

Now we are ready to state the main result of this work.

Theorem 3.3. *Let $(X, \|.\|, \preceq)$ be a partially ordered Banach space for which order intervals are convex and closed. Assume $(X, \|.\|)$ is uniformly convex. Let C be a nonempty convex closed bounded subset of X not reduced to one point. Let $T : C \to C$ be a continuous monotone asymptotically nonexpansive mapping. Then T has a fixed point if and only if there exists $x_0 \in C$ such that x_0 and $T(x_0)$ are comparable.*

Proof. Obviously if x is a fixed point of T, then x and $T(x) = x$ are comparable. Let $x_0 \in C$ be such that x_0 and $T(x_0)$ are comparable. Without loss of any generality, assume that $x_0 \preceq T(x_0)$. Since T is monotone, then we have $T^n(x_0) \preceq T^{n+1}(x_0)$, for every $n \in \mathbb{N}$. In other words, the orbit $\{T^n(x_0)\}$ is monotone increasing. Since the order intervals are closed and convex and X is reflexive, we conclude that

$$C_\infty = \bigcap_{n\geq 0} [T^n(x_0), \to) \cap C = \bigcap_{n\geq 0} \{x \in C;\ T^n(x_0) \preceq x\} \neq \emptyset.$$

Let $x \in C_\infty$; then $T^n(x_0) \preceq x$ and since T is monotone, we get

$$T^n(x_0) \preceq T(T^n(x_0)) = T^{n+1}(x_0) \preceq T(x),$$

for every $n \geq 0$, i.e., $T(C_\infty) \subset C_\infty$. Consider the type function $\tau : C_\infty \to [0, +\infty)$ generated by $\{T^n(x_0)\}$, i.e., $\tau(x) = \limsup\limits_{n\to+\infty} \|T^n(x_0) - x\|$. Lemma 3.2 implies the existence of a unique $z \in C_\infty$ such that $\tau(z) = \inf\{\tau(x);\ x \in C_\infty\} = \tau_0$. Since $z \in C_\infty$, we have $T^p(z) \in C_\infty$, for every $p \in \mathbb{N}$, which implies

$$\tau(T^p(z)) = \limsup_{n\to+\infty} \|T^n(x_0) - T^p(z)\| \leq k_p\ \limsup_{n\to+\infty} \|T^n(x_0) - z\|,$$

where $\{k_p\}_{p\in\mathbb{N}}$ is given by Definition 2.2 such that $\lim\limits_{p\to+\infty} k_p = 1$ since T is asymptotically nonexpansive. Hence $\tau_0 \leq \tau(T^p(z)) \leq k_p\ \tau_0$, for every $p \in \mathbb{N}$. The main property of $\{k_p\}_{p\in\mathbb{N}}$ implies

$$\lim_{p\to+\infty} \tau(T^p(z)) = \tau_0,$$

which means that $\{T^p(z)\}_{p\in\mathbb{N}}$ is a minimizing sequence of τ. Using Lemma 3.2 again, we conclude that $\{T^p(z)\}_{p\in\mathbb{N}}$ converges to z. Since T is continuous, we have $\lim\limits_{p\to+\infty} T(T^p(z)) = \lim\limits_{p\to+\infty} T^{p+1}(z) = T(z) = z$, i.e., z is a fixed point of T. □

It is natural to ask whether the continuity assumption in Theorem 3.3 may be relaxed. This is the main motivation behind [12] where the authors relaxed the continuity assumption from the main result of [14]. Looking at the proof carefully, we see that the continuity assumption was used at the end to prove that the minimum point is a fixed point. The difficulty met here has to do with the fact that it is not clear whether the minimum point is comparable to its image under the map in question. While investigating this point, we came with a property satisfied by any Banach lattice, like the classical $L^p([0, 1])$-spaces (for $p \geq 1$), similar to the Opial condition [13]. It is well known that the classical ℓ^p spaces (for $p \geq 1$) enjoy the Opial condition for the weak topology while $L^p([0, 1])$-spaces (for $p > 1$) fail to enjoy such property despite the fact that these spaces are uniformly convex.

Definition 3.4. Let $(X, \|.\|, \preceq)$ be a partially ordered Banach space.

(i) [13] X is said to satisfy the weak-Opial condition if whenever any sequence $\{x_n\}$ in X which weakly converges to x, we have

$$\limsup_{n\to+\infty} \|x_n - x\| < \limsup_{n\to+\infty} \|x_n - y\|,$$

for every $y \in X$ such that $x \neq y$.

(ii) X is said to satisfy the monotone weak-Opial condition if whenever any monotone increasing (resp. decreasing) sequence $\{x_n\}$ in X which weakly converges to x, we have

$$\limsup_{n\to+\infty} \|x_n - x\| \leq \limsup_{n\to+\infty} \|x_n - y\|,$$

for every $y \in X$ such that $x \preceq y$ (resp. $y \preceq x$).

The following result is noteworthy and seems to be new.

Proposition 3.5. *Any Banach lattice satisfies the monotone weak-Opial condition.*

Proof. Let $(X, \|.\|, \preceq)$ be a Banach lattice. One of the properties of X states

$$0 \preceq u \preceq v \text{ implies } \|u\| \leq \|v\|,$$

for every $u, v \in X$, and the positive cone P of X is convex and closed [11]. In order to show that X satisfies the monotone weak-Opial condition, let $\{x_n\}_{n\in\mathbb{N}}$ be a monotone sequence in X which converges weakly to x. Without loss of generality, we assume that $\{x_n\}_{n\in\mathbb{N}}$ is monotone increasing. Let $y \in X$ be such that $x \preceq y$. Since order intervals of X are convex and closed, we conclude that $x_n \preceq x \preceq y$ which implies $0 \preceq x - x_n \preceq y - x_n$, for every $n \in \mathbb{N}$. Hence $\|x - x_n\| \leq \|y - x_n\|$, for every $n \in \mathbb{N}$, which implies

$$\limsup_{n\to+\infty} \|x_n - x\| \leq \limsup_{n\to+\infty} \|x_n - y\|.$$

□

Using the monotone Opial condition, we can relax the continuity assumption in Theorem 3.3.

Theorem 3.6. *Let $(X, \|.\|, \preceq)$ be a partially ordered Banach space for which order intervals are convex and closed. Assume $(X, \|.\|)$ is uniformly convex. Let C be a nonempty convex closed bounded subset of X not reduced to one point. Assume that X satisfies the monotone weak-Opial condition. Let $T : C \to C$ be a monotone asymptotically nonexpansive mapping. Then T has a fixed point if and only if there exists $x_0 \in C$ such that x_0 and $T(x_0)$ are comparable.*

Proof. As we did in the proof of Theorem 3.3, we first assumed that $x_0 \preceq T(x_0)$. Then the orbit $\{T^n(x_0)\}$ is a monotone increasing sequence. Since X is reflexive, it is easy to show that $\{T^n(x_0)\}$ is weakly convergent to some point $x \in C$ and $T^n(x_0) \preceq x$, for every $n \in \mathbb{N}$. Since X satisfies the monotone weak-Opial condition, we know that

$$\limsup_{n\to+\infty} \|T^n(x_0) - x\| \leq \limsup_{n\to+\infty} \|T^n(x_0) - y\|,$$

for every $y \in \widetilde{C} = C \cap [x, \to)$. Note that $\widetilde{C}$ is a nonempty closed convex subset of C. Therefore the type function τ generated by the orbit $\{T^n(x_0)\}$ has x as its unique minimum point in $\widetilde{C}$. As we did in the proof of Theorem 3.3, we see that $\{T^p(x)\}$ converges strongly to x. Let us show that $x \preceq T(x)$. We have $T^n(x_0) \in (\leftarrow, x] \cap C$, for every $n \in \mathbb{N}$. Since T is monotone we get $T^{n+1}(x_0) \in (\leftarrow, T(x)] \cap C$, for every $n \in \mathbb{N}$. Since $(\leftarrow, T(x)] \cap C$ is convex and closed, then the weak-limit of $\{T^{n+1}(x_0)\}$ also belongs to this set, i.e., $x \in (\leftarrow, T(x)] \cap C$. In other words, we have $x \preceq T(x)$. This will imply that $\{T^n(x)\}$ is a monotone increasing sequence which converges to x. Therefore we must have $T^n(x) \preceq x$, for every $n \in \mathbb{N}$. This will force $x = T(x)$, i.e., x is a fixed point of T. □

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