EXCEPTIONAL COLLECTIONS ON SOME FAKE QUADRICS

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ABSTRACT. We construct exceptional collections of maximal length on four families of surfaces of general type with $p_g = q = 0$ which are isogenous to a product of curves. From these constructions we obtain new examples of quasiphantom categories as their orthogonal complements.

1. INTRODUCTION

Derived categories of coherent sheaves are one of the most attractive and mysterious invariants of algebraic varieties, and the notion of semiorthogonal decomposition plays a key role in the study of derived categories of algebraic varieties. Semiorthogonal decompositions tell us the structure of derived categories, and many interesting semiorthogonal decompositions of Fano and rational varieties were constructed. However, in contrast to the many studies of derived categories of Fano or rational varieties, we do not know much about the structure of derived categories of varieties of general type.

One of the easiest ways to construct a semiorthogonal decomposition is to find an exceptional collection. When a triangulated category has an exceptional collection we can divide it into the category generated by the exceptional collection and its orthogonal complement. For a surface with $p_g = q = 0$, every line bundle is an exceptional object, and we can construct semiorthogonal decompositions using line bundles. Then we can hope that for some surfaces with $p_g = q = 0$ there are exceptional collections of maximal length and we can study derived categories of these surfaces using semiorthogonal decompositions induced from them. Böhning, Graf von Bothmer and Sosna proved in [4] that there exists an exceptional collection of maximal length on the classical Godeaux surface. They constructed the first example of a quasiphantom category as the orthogonal complement of this exceptional collection. Motivated by their results there are now lots of studies on derived categories of surfaces of general type with $p_g = q = 0$. See the papers of Böhning, Graf von Bothmer, and Sosna [4], Alexeev and Orlov [1], Galkin and Shinder [12], Böhning, Graf von Bothmer, Katzarkov and Sosna [3], Fakhruddin [10], Galkin, Katzarkov, Mellit and Shinder [11], Coughlan [8], Keum [14] and the first author

[15, 16] for more details. They constructed categories with vanishing Hochschild homology groups as orthogonal complements of exceptional collections of line bundles of maximal length. Some of them are known to have finite Grothendieck groups and they provide examples of quasiphantom categories. Supported by these examples, it seems that the following question is now considered by many experts.

Question. Let S be a smooth projective surface of general type with $p_g = q = 0$. Is there an exceptional collection whose length is equal to the rank of Grothendieck group of S or the total dimension of $H^*(S, \mathbb{C})$? Especially can we construct such an exceptional collection using line bundles on S?

We want to answer the above question for some special surfaces of general type with $p_g = q = 0$. Bauer, Catanese and Grunewald have classified surfaces of general type with $p_g = q = 0$ which are quotients of a product of curves by the free diagonal action of a finite group in [6]. There are 12 families of such surfaces, and these are called the surfaces isogenous to a higher product of unmixed type. The rank of Grothendieck group of every such surface is 4, and the total dimension of cohomology group of every such surface is also 4 [12]. Therefore the maximal possible length of the exceptional collection on every such surface is 4. For the 4 families of such surfaces with abelian group quotients, exceptional collections of maximal length were constructed in [12, 15, 16]. In this paper we construct such collections in four more cases where G is $D_4 \times \mathbb{Z}/2$, S_4, $S_4 \times \mathbb{Z}/2$ and $(\mathbb{Z}/4 \times \mathbb{Z}/2) \rtimes \mathbb{Z}/2$ ($G(16)$ in the notation of [6]).

Theorem 1.1. *Let $S = (C \times D)/G$ be a surface isogenous to a higher product with $p_g = q = 0$, where G is one of $D_4 \times \mathbb{Z}/2$, S_4, $S_4 \times \mathbb{Z}/2$ and $(\mathbb{Z}/4 \times \mathbb{Z}/2) \rtimes \mathbb{Z}/2$. Then there are exceptional collections of line bundles of maximal length 4 on S, and the orthogonal complements of these exceptional collections are quasiphantom categories.*

We think that we can generalize this result to any surface isogenous to a higher product with $p_g = q = 0$. The following conjecture has also appeared in [12].

Conjecture 1.2. *Let S be a surface isogenous to a higher product with $p_g = q = 0$. Then there are exceptional collections of line bundles of maximal length 4 on S.*

We recall some basic facts about these surfaces (see [5, 21] for more details) and sketch the idea of the construction. Let $S = (C_1 \times C_2)/G$ be one of them. We have $C_i/G \cong \mathbb{P}^1$, $|G| = (g_1 - 1)(g_2 - 1)$ where g_i is the genus of C_i. Since $p_g = q = 0$, the Chern class map $Pic(S) \to H^2(S, \mathbb{Z})$ is an isomorphism. It follows from Noether's formula that $H^2(S, \mathbb{Z})$ has rank 2. Thus up to a finite torsion subgroup $Pic(S)$ is an unimodular indefinite lattice of rank 2, that is, a hyperbolic plane. Let $p_i : C_1 \times C_2 \to C_i$ be the projections and denote by $\mathcal{F} \boxtimes \mathcal{G} = p_1^*(\mathcal{F}) \otimes p_2^*(\mathcal{G})$ the external tensor product of coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on C_1 and C_2. Let us denote by $\mathcal{O}(2, 0)$ and $\mathcal{O}(0, 2)$ the classes of $p_*^G p_1^*(\Omega_{C_1}^1)$ and $p_*^G p_2^*(\Omega_{C_2}^1)$ in the lattice $H^2(S, \mathbb{Z})/Tors$. Then we have $\mathcal{O}(2, 0)^2 = 0$, $\mathcal{O}(0, 2)^2 = 0$, $\mathcal{O}(2, 0) \cdot \mathcal{O}(0, 2) = 4$. We see that the lattice $H^2(S, \mathbb{Z})/Tors$ must be generated by some numerical halves $\mathcal{O}(1, 0)$ and $\mathcal{O}(0, 1)$ of canonical classes of curves C_1, C_2. The Euler characteristic of a line bundle on S of numerical type $\mathcal{O}(i, j)$ is $(i - 1)(j - 1)$.

The category $\mathrm{coh}(S)$ of coherent sheaves on S is equivalent to the category of G-equivariant coherent sheaves on $C_1 \times C_2$, and we denote the functor

$$\mathrm{coh}^G(C_1 \times C_2) \to \mathrm{coh}(S)$$

by p_*^G. Therefore we are going to construct exceptional collections of line bundles in $D^b(S)$ by constructing exceptional collections of G-equivariant line bundles in $D_G^b(C_1 \times C_2)$. Recall the definition of exceptional sequence.

Definition 1.3.

(1) An object E of a triangulated category D is called exceptional if

$$Hom^k(E, E) = \begin{cases} \mathbb{C}, & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

(2) A collection of exceptional objects $E_1, \ldots, E_n$ is called exceptional if

$$Hom^k(E_i, E_j) = 0$$

for all $i > j$ and all k.

From the definition it is clear that when $\mathcal{L}, \mathcal{O}$ is an exceptional collection then $\chi(\mathcal{L})$ should be 0. Therefore we need some numerical halves $\mathcal{O}(1, 0)$ and $\mathcal{O}(0, 1)$ of canonical classes of curves C_1, C_2 to construct an exceptional collection of line bundles. However there are some cases when we cannot give a G-equivariant structure to the numerical halves of canonical bundles. In these cases we construct equivariant bundles on $C_1 \times C_2$ by finding two divisors D_1 and D_2 on C_1 and C_2 such that each of them is not equivariant on the curve C_i, but they have mutually inverse obstructions and therefore $p_1^*\mathcal{O}(D_1) \otimes p_2^*\mathcal{O}(D_2)$ is equivariant on the product. From now on we will omit p_*^G, p_1^* and p_2^* from our notation.

Let us explain how this is possible. For a smooth projective variety X with an action of a finite group G there is an exact sequence

$$(1) \qquad 0 \to \widehat{G} \to Pic^G(X) \to Pic(X)^G \to H^2(G, \mathbb{C}^*),$$

where $\widehat{G} = Hom(G, \mathbb{C}^*)$ is the group of characters of G, $Pic^G(X)$ is the group of G-equivariant line bundles on X and $Pic(X)^G$ is the group of line bundles whose classes in the Picard group are invariant under the action of G. The last map in (1) providing the obstruction to the existence of an equivariant structure on a line bundle $\mathcal{L}$ may be described as follows. Fix $\mathcal{L}$ in $Pic(X)^G$. For each $g \in G$ pick some isomorphisms $\phi_g : g_*\mathcal{L} \to \mathcal{L}$. Then

$$(2) \qquad \eta_{\mathcal{L}}(g, h) = \phi_g \cdot g_*\phi_h \cdot \phi_{gh}^{-1}$$

is an automorphism of $\mathcal{L}$, that is, an element of $\mathbb{C}^*$. Therefore $\eta_{\mathcal{L}}$ is a 2-cocycle in $Z^2(G, \mathbb{C}^*)$ and the map $Pic(X)^G \to H^2(G, \mathbb{C}^*)$ is given by $\mathcal{L} \mapsto \eta_{\mathcal{L}}$.

Let $S = (C_1 \times C_2)/G$ be surface isogenous to a higher product of unmixed type with $p_g = q = 0$ and $g_1 \leq g_2$. For $G = D_4 \times \mathbb{Z}/2, S_4, S_4 \times \mathbb{Z}/2$ cases we cannot give G-equivariant structure to any half of canonical line bundle on C_1. What we can do is to construct G-invariant acyclic line bundle $\mathcal{L}$ on C_1 which is a numerical half of the canonical line bundle but not G-equivariant. Then we need to find a G-invariant line bundle $\mathcal{M}$ such that $\eta_{\mathcal{L}} \cdot \eta_{\mathcal{M}} = 0$ in $H^2(G, \mathbb{C}^*)$ to make $\mathcal{L} \boxtimes \mathcal{M}$ a G-equivariant acyclic line bundle on $C_1 \times C_2$. The following proposition of Dolgachev [9] tells us that we can always find such a line bundle.

Proposition 1.4 ([9, Proposition 2.2]). *If X is a curve, then the map $Pic(X)^G \to H^2(G, \mathbb{C}^*)$ is surjective.*

In fact there are infinitely many such line bundles. Then we show that there are G-equivariant acyclic line bundles on C_2. From Serre duality, the Künneth formula

and the Riemann-Roch theorem on the curves C_1, C_2 one obtains the following lemma.

Lemma 1.5. *Suppose that $\mathcal{L}$ is a G-invariant acyclic line bundle on C_1, $\mathcal{M}$ and $\mathcal{N}$ are line bundles on C_2 such that $\mathcal{M}$ is G-invariant and $\eta_{\mathcal{L}} \cdot \eta_{\mathcal{M}} = 0$ in $H^2(G, \mathbb{C}^*)$ and $\mathcal{N}$ is acyclic and admits G-equivariant structure on C_2. Then the sequence*

$$\mathcal{L} \boxtimes (\mathcal{M} \otimes \mathcal{N})(\chi_1), \ \mathcal{L} \boxtimes \mathcal{M}(\chi_2), \ \mathcal{O} \boxtimes \mathcal{N}(\chi_3), \ \mathcal{O}$$

is an exceptional collection on S. Here χ_i denotes arbitrary characters of G by which we can twist the equivariant structure.

We will construct exceptional collections of maximal length on $S = (C_1 \times C_2)/G$ by this method when $G = D_4 \times \mathbb{Z}/2, S_4, S_4 \times \mathbb{Z}/2$. When $G = G(16)$ we can find acyclic G-equivariant line bundles on C_1, C_2, and the construction becomes much easier.

2. INVARIANT LINE BUNDLES

In this section we recall some results of Beauville [2] about curves with G-action and invariant line bundles on them which will be extremely useful for our construction. Let C be a smooth projective curve and G be a finite group acting on C. Let B be the quotient curve, $\pi : C \to B$ the quotient map. Denote by R_C the field of rational functions on the curve C. From the short exact sequence of G-modules

$$0 \to \mathbb{C}^* \to R_C^* \to R_C^*/\mathbb{C}^* \to 0,$$

we get

$$0 \to \mathbb{C}^* \to R_B^* \to (R_C^*/\mathbb{C}^*)^G \to H^1(G, \mathbb{C}^*) \to 0,$$

since $H^1(G, R_C^*) = 0$ by Hilbert's Theorem 90. From the short exact sequence of G-modules

$$0 \to R_C^*/\mathbb{C}^* \to Div(C) \to Pic(C) \to 0,$$

we get the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & R_B^*/\mathbb{C}^* & \longrightarrow & Div(B) & \longrightarrow & Pic(B) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (R_C^*/\mathbb{C}^*)^G & \longrightarrow & Div(C)^G & \longrightarrow & Pic(C)^G & \longrightarrow & H^1(G, R_C^*/\mathbb{C}^*).
\end{array}
$$

If we change the lower exact sequence by

$$0 \to (R_C^*/\mathbb{C}^*)^G \to Div(C)^G \to Im \to 0,$$

we still have a commutative diagram and we can apply the snake lemma as follows:

(3)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R_B^*/\mathbb{C}^* & \longrightarrow & Div(B) & \longrightarrow & Pic(B) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (R_C^*/\mathbb{C}^*)^G & \longrightarrow & Div(C)^G & \longrightarrow & Im & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& \longrightarrow & H^1(G, \mathbb{C}^*) & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0.
\end{array}
$$

Sometimes we can compute X, Y, Z explicitly and then the above diagram becomes very useful. For example when B is isomorphic to $\mathbb{P}^1$ we get $X = 0$.

Lemma 2.1 ([2])**.** *Let C be a curve with involution σ and B be the quotient curve $C/\langle\sigma\rangle$. If the covering $\pi : C \to B$ is unramified, then $Y = Z = 0$ and $X \cong H^1(G, \mathbb{C}^*) \cong \mathbb{Z}/2$. If the set R of ramification points of π is nonempty, then $X = 0$ and the last row is isomorphic to*

$$0 \to \mathbb{Z}/2 \to (\mathbb{Z}/2)^R \to Pic(C)^\sigma/\pi^* Pic(B) \to 0,$$

where the kernel is generated by $(1, \ldots, 1)$.

The next result of Beauville will be very important for our computations. Recall that a theta characteristic is a nontrivial line bundle whose square is the canonical line bundle of C.

Lemma 2.2 ([2])**.** *In the situation of the previous lemma, let $\mathcal{L}$ be a σ-invariant theta characteristic on C. There are some $\mathcal{L}' \in Pic(B)$ and $E \subset R$ such that $\mathcal{L} = \pi^*(\mathcal{L}')(E)$. Then we also have $\mathcal{L} = \pi^*(K_B \otimes \mathcal{L}'^{-1})(R - E)$. The pushforward of $\mathcal{L}$ splits: $\pi_*(\mathcal{L}) = \mathcal{L}' \oplus (K_B \otimes \mathcal{L}'^{-1})$ and we have*

$$H^0(C, \mathcal{L}) = H^0(B, \mathcal{L}') \oplus H^1(B, \mathcal{L}')^*.$$

3. CASE $G = D_4 \times \mathbb{Z}/2$

The group G has a presentation

$$\langle x, y, z \mid x^4 = y^2 = z^2 = [x, z] = [y, z] = 1, x^y = x^{-1} \rangle,$$

where $x^y = y^{-1}xy$, $[x, y] = xyx^{-1}y^{-1}$.

A covering $\pi : C \to \mathbb{P}^1$ with Galois group G can be specified by its ramification type $(m_1^{k_1}, \ldots, m_l^{k_l})$, which means that π has k_i ramification points of multiplicity m_i and by the tuple of generators $(g_1, \ldots, g_n)$, $g_i \in G$, $n = k_1 + \cdots + k_l$ such that a simple geometric loop around the j-th ramification point on $\mathbb{P}^1$ lifts to the action of g_j on C. We must have $g_1 \ldots g_n = 1$ and $g_1, \ldots, g_n$ must generate G. Of course these data do not specify the covering completely because one can move the ramification points on $\mathbb{P}^1$. If $p_1, \ldots, p_n \in \mathbb{P}^1$ are the ramification points, then we will denote by E_i the reduced fiber of π over p_i.

The covering $C_1 \to \mathbb{P}^1$ has ramification type $(2^3, 4)$ and the corresponding elements of G are (z, yz, xy, x). The covering $C_2 \to \mathbb{P}^1$ has ramification type (2^6) and the corresponding tuple is $(y, x^3yz, x^2y, x^3yz, x^2z, x^2z)$. The curve C_1 has genus 3 $(2g - 2 = 4)$, C_2 has genus 9 $(2g - 2 = 16)$. Divisors E_1, E_2, E_3 on C_1 have degree 8 and E_4 has degree 4. All divisors E_i on C_2 consist of 8 points.

Lemma 3.1. *There is a G-invariant theta characteristic $\mathcal{L}$ on C_1 which has no sections.*

Proof. Consider the mapping $\pi : C_1 \to C_1/\langle z \rangle$. The quotient has genus 0, so π is a hyperelliptic structure on C_1. It is ramified in the 8 points of E_1. The quotient of C_1 by subgroup $\langle x^2, xy \rangle$ also has genus 0. The divisor E_1 consists of two $\langle x^2, xy \rangle$-orbits. Let B_1 be one of them. Let also B_2 be any full fiber of π. Since the subgroups $\langle x^2, xy \rangle$ and $\langle z \rangle$ are normal in G, divisors B_1 and B_2 have G-invariant classes in the Picard group. Let $L = B_1 - B_2$ and $\mathcal{L} = \mathcal{O}(L)$. The canonical class of C_1 is equivalent to $E_1 - 2B_2 \sim 2B_1 - 2B_2$, so L is a theta characteristic. By Lemma 2.2 applied to π we have $h^0(C_1, L) = h^0(\mathbb{P}^1, \mathcal{O}(-1)) + h^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0$. $\square$

Proposition 1.4 implies the next lemma.

Lemma 3.2. *There is a G-invariant line bundle $\mathcal{M}$ on C_2 such that $\eta_{\mathcal{L}} \cdot \eta_{\mathcal{M}} = 0$.*

Now we construct an explicit G-equivariant acyclic theta characteristic on C_2.

Lemma 3.3. *There is a G-equivariant theta characteristic $\mathcal{N}$ on C_2 which has no sections.*

Proof. Let $N = E_1 - E_2 + E_5$ and $\mathcal{N} = \mathcal{O}(N)$. Quotients of C_2 by subgroups $\langle xyz, x^2, z \rangle, \langle y, x^2, z \rangle, \langle y, xyz, x^2 \rangle$ all have genus 0. From these three quotients we see that $E_1 \sim E_3$, $E_2 \sim E_4$, $E_5 \sim E_6$. It follows that $\mathcal{N}$ is a theta characteristic. Consider the quotient $\pi : C_2 \to C_2/\langle x^2 z \rangle$. We have $N = \pi^* N' + E_5$, where N' is a divisor of degree 0. The curve $C_2/\langle x^2 z \rangle$ has genus 1. We have $2N' \sim 0$ which is the canonical class of $C_2/\langle x^2 z \rangle$. There is an induced action of y on $C_2/\langle x^2 z \rangle$. Applying Lemma 2.2 first to π, then to the quotient of $C_2/\langle x^2 z \rangle$ by $\langle y \rangle$ we find $h^0(C_2, N) = 2h^0(C_2/\langle x^2 z \rangle, N') = 4h^0(\mathbb{P}^1, \mathcal{O}(-1)) = 0$. $\qquad\square$

From the above lemmas we get the following theorem.

Theorem 3.4. *Let $S = (C_1 \times C_2)/G$ be a surface isogenous to a higher product with $p_g = q = 0$ and $G = D_4 \times \mathbb{Z}/2$. Then there are exceptional collections of line bundles of maximal length 4 on S.*

4. CASE $G = S_4 \times \mathbb{Z}/2$

The covering $C_1 \to \mathbb{P}^1$ has ramification type $(2, 4, 6)$. The corresponding tuple is $((12), 0), ((1234), 1), ((432), 1)$. The covering $C_2 \to \mathbb{P}^1$ has ramification type (2^6), and the tuple is $((12)(34), 1), ((12), 1), ((34), 1), ((14)(23), 1), ((23), 1), ((14), 1)$.

Lemma 4.1. *There is a G-invariant acyclic theta characteristic $\mathcal{L}$ on C_1.*

Proof. Note that the curve C_1 is hyperelliptic and the quotient of C_1 by the action of the element $(1, 1)$ of order two in G is isomorphic to $\mathbb{P}^1$. We denote by $\pi : C_1 \to \mathbb{P}^1$ the quotient morphism. Consider the Klein subgroup $V_4 = V_4 \times \{0\} = \{1, ((12)(34), 0), ((13)(24), 0), ((14)(23), 0)\} \le S_4 \times \mathbb{Z}_2$. The quotient C_1/V_4 has genus 0. The group V_4 acts freely on E_3, so E_3 consists of two free V_4-orbits, which are equivalent to each other. Let W be one of them. Then $\mathcal{O}(W)$ is a G-invariant line bundle. Let $\mathcal{L} = \pi^* \mathcal{O}(-1) \otimes \mathcal{O}(W)$. Note that the ramification points of π are precisely E_3. Looking at the morphism π we see that $\pi^* \mathcal{O}(-2)(E_3)$ is a canonical bundle on C_1, but $E_3 \sim 2W$, so $\mathcal{L}$ is a theta characteristic on C_1. It is G-invariant since G must preserve the hyperelliptic structure. From Lemma 2.2 applied to the hyperelliptic involution we get $h^0(C_1, \mathcal{L}) = h^0(\mathbb{P}^1, \mathcal{O}(-1)) + h^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0$. $\quad\square$

The next lemma follows from Proposition 1.4.

Lemma 4.2. *There is a G-invariant line bundle $\mathcal{M}$ on C_2 such that $\eta_{\mathcal{L}} \cdot \eta_{\mathcal{M}} = 0$.*

Then we construct an explicit G-equivariant acyclic theta characteristic on C_2.

Lemma 4.3. *There is a G-equivariant acyclic theta characteristic $\mathcal{N}$ on C_2.*

Proof. Note that the elements $((12)(34), 1), ((13)(24), 1), ((14)(23), 1)$ generate the subgroup $V_4 \times \mathbb{Z}_2$ isomorphic to $\mathbb{Z}_2^3$. Consider the system of quotients $C_2 \to C_2' \to C_2'' \to C_2'''$, where C_2' is the quotient by the action of $((12)(34), 1)$, C_2'' by the induced action of $((13)(24), 1)$ and C_2''' by $((14)(23), 1)$. We divide E_1 into three parts

F_1, F_2, F_3 according to their stabilizers $((12)(34), 1)$, $((13)(24), 1)$ or $((14)(23), 1)$ and analogously E_4 into F_4, F_5, F_6. The ramification points of the morphism $C_2 \to C_2'$ are $F_1 \cup F_4$, the ramification of $C_2' \to C_2''$ is $F_2' \cup F_5'$, where F_i' is the image of F_i on C_2' (without multiplicities) and the ramification of $C_2'' \to C_2'''$ is $F_3'' \cup F_6''$. The curve C_2''' is elliptic, so its canonical divisor $K_{C_2'''}$ is zero. Thus we see that $K_{C_2''} = F_3'' + F_6''$, $K_{C_2'} = F_2' + F_3' + F_5' + F_6'$ and $K_{C_2} = F_1 + \cdots + F_6 = E_1 + E_4$.

The element $((14), 1)$ stabilizes the subgroup generated by $((12)(34), 1)$, $((13)(24), 1)$. Therefore it acts on the curve C_2''. The quotient by this action is $\mathbb{P}^1$ and F_3'', F_6'' are its free orbits. Therefore $F_3'' \sim F_6''$ and $F_3' \sim F_6'$, $F_3 \sim F_6$. Analogously $F_2' \sim F_5'$ and so on.

Let $N = E_1 + E_2 - E_3$, $N' = F_2' + F_3' + E_2' - E_3'$, $N'' = F_3'' + E_2'' - E_3''$. From the natural map $C_2''' \to C_2/G$ we get $2E_2''' \sim 2E_3'''$. Therefore N, N', N'' are theta characteristics on the corresponding curves. From Beauville's lemma applied repeatedly we get $h^0(C_2, N) = 2h^0(C_2', N') = 4h^0(C_2'', N'')$.

Now consider again the quotient $C_2'' \to \mathbb{P}^1$ by the action of $((14), 1)$. Applying Beauville's lemma to it we find $h^0(C_2'', N'') = 2h^0(\mathbb{P}^1, \mathcal{O}(-1)) = 0$. $\qquad \square$

From the above lemmas we get the following theorem.

Theorem 4.4. *Let $S = (C_1 \times C_2)/G$ be a surface isogenous to a higher product with $p_g = q = 0$ and $G = S_4 \times \mathbb{Z}/2$. Then there are exceptional collections of line bundles of maximal length 4 on S.*

5. Case $G = S_4$

The covering $C_1 \to \mathbb{P}^1$ has ramification type $(3, 4^2)$. The corresponding tuple is $((123), (1234), (1243))$. The covering $C_2 \to \mathbb{P}^1$ has ramification type (2^6) and the tuple is $((12), (12), (23), (23), (34), (34))$. The following lemma was stated in [21]. We give a proof as follows.

Lemma 5.1. *Let us denote by C_1' the curve C_1 from the previous section with its $S_4 \times \mathbb{Z}/2$-action. Consider the subgroup S_4 embedded into $S_4 \times \mathbb{Z}/2$ by the mapping $(id, sign)$. We claim that the curve C_1' with the action of this subgroup is isomorphic to C_1 with its S_4-action.*

Proof. Consider the quotient $C_1' \to C_1'/S_4$ and its ramification. There is a map $C_1'/S_4 \to C_1'/(S_4 \times \mathbb{Z}/2)$ which is a twofold covering of $\mathbb{P}^1$ by $\mathbb{P}^1$, and we want to choose coordinates in such a way that the mapping will be given by $z \mapsto z^2$. Suppose that the covering $C_1' \to C_1'/(S_4 \times \mathbb{Z}/2) \cong \mathbb{P}^1$ is ramified over points $0, 1, \infty$ and the loop around 0 corresponds to the action of $((12), 0)$ on the covering and the loops around 1 and ∞ correspond to $((1234), 1)$, $((432), 1)$. Choose -1 as a base point on the quotient. The covering C_1' is specified by the map

$$\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, -1) = \langle t_0, t_1, t_\infty \mid t_0 t_1 t_\infty = 1 \rangle \to S_4 \times \mathbb{Z}/2,$$

given by

$$t_0 \mapsto ((12), 0), \ t_1 \mapsto ((1234), 1), \ t_\infty \mapsto ((432), 1).$$

There are four points $0, 1, -1, \infty$ on C_1'/S_4 above $0, 1, \infty$ on $C_1'/(S_4 \times \mathbb{Z}/2)$. We want to compute the composite map
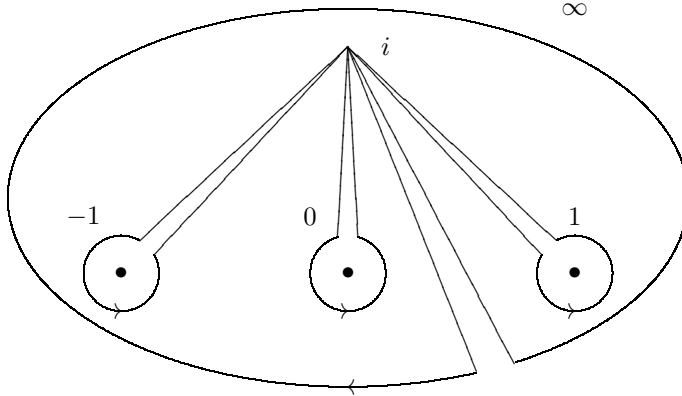
$$\pi_1(\mathbb{P}^1 \setminus \{0, \pm 1, \infty\}, i) \to \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, -1) \to S_4 \times \mathbb{Z}/2,$$

check that its image lies in $S_4 \leq S_4 \times \mathbb{Z}/2$ that loop around one of the points maps to the trivial element, so the ramification is actually in the three points, and finally check that the map gives our covering C_1.
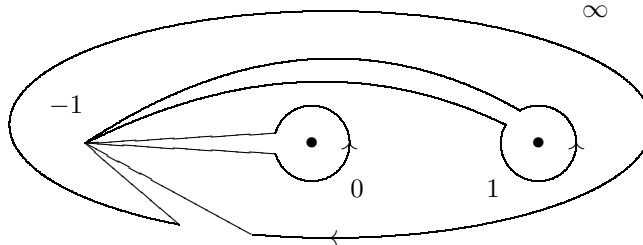
If we choose the generators of

$$\pi_1(\mathbb{P}^1 \setminus \{0, \pm 1, \infty\}, i) = \langle s_1, s_{-1}, s_0, s_\infty \mid s_1 s_{-1} s_0 s_\infty = 1 \rangle$$

as follows:

(these are loops around $1, -1, 0, \infty$) and generators t_0, t_1, t_∞ as follows:

then we can just draw the images of $s_1, \ldots, s_\infty$ under the map $z \mapsto z^2$ and then write them as combinations of t_0, t_1. We obtain that the homomorphism

$$\pi_1(\mathbb{P}^1 \setminus \{0, \pm 1, \infty\}, i) \to \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, -1)$$

is given by

$$s_1 \mapsto t_1, \ s_{-1} \mapsto t_0 t_1 t_0^{-1}, \ s_0 \mapsto t_0^2, \ s_\infty \mapsto t_0^{-1} t_1^{-1} t_0^{-1} t_1^{-1}.$$

So the composition

$$\pi_1(\mathbb{P}^1 \setminus \{0, \pm 1, \infty\}, i) \to \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, -1) \to S_4 \times \mathbb{Z}/2$$

is given by

$$s_1 \mapsto ((1234), 1), \ s_{-1} \mapsto ((1342), 1), \ s_0 \mapsto (1, 0), \ s_\infty \mapsto ((134), 0).$$

We see that it factors through $S_4 \leq S_4 \times \mathbb{Z}/2$ and that s_0 is mapped to a trivial element, so $C_1' \to C_1'/S_4$ is ramified only over $\pm 1, \infty$. The group $\pi_1(\mathbb{P}^1 \setminus \{\pm 1, \infty\})$ can be represented as

$$\langle s_\infty, s_1, s_{-1} \mid s_\infty s_1 s_{-1} = 1 \rangle$$

and these three generators map to $(134), (1234), (1342)$. If we conjugate this sequence by $(13)(24)$, we get $(123), (1234), (1243)$. Therefore, the covering $C_1' \to C_1'/S_4$ is isomorphic to $C_1 \to C_1/S_4$ with S_4-action. $\qquad \square$

Lemma 5.2. *There is a G-invariant acyclic theta characteristic $\mathcal{L}$ on C_1.*

Proof. Indeed, we can use the same $\mathcal{L}$ as in Lemma 4.1. $\qquad \square$

Proposition 1.4 implies the next lemma.

Lemma 5.3. *There is a G-invariant line bundle $\mathcal{M}$ on C_2 such that $\eta_{\mathcal{L}} \cdot \eta_{\mathcal{M}} = 0$.*

Then we prove that there is a G-equivariant acyclic theta characteristic on C_2.

Lemma 5.4. *There is a G-equivariant acyclic theta characteristic $\mathcal{N}$ on C_2.*

Proof. We let $N = E_i + E_j - E_k$ for i, j, k all different and $\mathcal{N} = \mathcal{O}(N)$. We will prove that for some choices of i, j, k such $\mathcal{N}$ is acyclic, but we can't say for which ones precisely.

Note that N has degree 12, so we only have to check that N has no regular sections. The subgroup A_4 acts freely on C_2, the quotient C_2/A_4 has genus 2, and the morphism $C_2/A_4 \to C_2/S_4$ has 6 ramification points $E_1', \ldots, E_6'$, where E_i' is the image of E_i on the quotient C_2/A_4. We have $E_1 + E_2 + E_3 \sim E_4 + E_5 + E_6$, and the S_4 acts by a sign character on the function with divisor $E_1 + E_2 + E_3 - E_4 - E_5 - E_6$. All relations between divisors E_i follow from this one and $2E_i \sim 2E_j$ since G must act via character on a function giving such relation. Now it is not hard to see that possible choices of i, j, k give 10 different classes in $Pic(C_2)$. It follows from these relations that $\mathcal{N}$ is a theta characteristic.

Step 1. There are no 1-dimensional subrepresentations in $H^0(C_2, N)$.

The subgroup A_4 must act trivially on such a subrepresentation, so if it exists, then we must have $E_i + E_j \sim E_k + E_l$. This never holds when i, j, k are different.

Step 2. The dimension of $H^0(C_2, N)$ is 0, 2 or 4.

We apply Beauville's lemma to the quotient $C_2 \to C_2/(12)$. We have $h^0(C_2, N) = h^0(A) + h^1(A) = 2h^0(A)$ where A is a line bundle of degree 3 on the quotient $C_2/(12)$. By Clifford's theorem $h^0(A) \leq 2$.

Step 3. There are i, j, k such that N has no regular sections.

From previous steps we know that $H^0(C_2, N)$ is a sum of 2-dimensional irreducible representations of S_4. Let W be one of them. Then W comes from an irreducible representation of S_3 via the map $S_4 \to S_4/V_4 \cong S_3$. There is a basis ϕ_1, ϕ_2 of W such that $A_3 \leq S_3$ acts on each ϕ_i by a character, and elements not in A_3 exchange ϕ_1 and ϕ_2, also multiplying them by some constant. We have $E_1 + E_2 - E_3 + (\phi_1) = F_1$ and $E_1 + E_2 - E_3 + (\phi_2) = F_2$ where F_1 and F_2 are A_4-invariant divisors of degree 12. Elements not in A_4 exchange F_1 and F_2. We denote by E_i' and F_i' images of E_i and F_i on the quotient C_2/A_4 (without multiplicities). Then E_i' and F_i' are single points on C_2/A_4. Note that F_1', F_2' are not equal to any E_i'. We have $\sigma(F_1') = F_2'$, where σ is the nontrivial automorphism of the covering $C_2/A_4 \to C_2/S_4$.

We want to prove that there are no more than 2 different N's of the form $E_i + E_j - E_k$ which have sections. Since the morphism $C_2 \to C_2/A_4$ is unramified,

from the diagram (3) we see that the kernel X of the map $Pic(C_2/A_4) \to Pic(C_2)$ is isomorphic to $H^1(A_4, \mathbb{C}^\times) \cong \mathbb{Z}/3$. Suppose that $N(i,j,k) = E_i + E_j - E_k$ and different $N(l,m,n)$ have sections, and we have $N(i,j,k) \sim F_1 \sim F_2$, $N(l,m,n) \sim F_3 \sim F_4$, where $\sigma(F_1) = F_2$, $\sigma(F_3) = F_4$, and each F_i is a pullback of a point on C_2/A_4. Note that F_1' is not equal to F_3' or F_4', as otherwise we would have $N(i,j,k) \sim N(l,m,n)$. Divisors $F_1' - F_2'$ and $F_3' - F_4'$ are nontrivial elements of X. Since $X \cong \mathbb{Z}/3$, we have either $F_1' - F_2' \sim F_3' - F_4'$ or $2(F_1' - F_2') \sim F_3' - F_4'$. The first variant is impossible, because then the divisor $F_1' + F_4'$ has at least 2-dimensional sections, but such a divisor is unique, so it must be equivalent to $F_1' + F_2'$, which is not true. There can't be a third pair F_5', F_6' because then $F_5' - F_6'$ must be equivalent to one of $F_1' - F_2', F_3' - F_4'$. Therefore, there are not more than 2 N's which have sections, so acyclic N exists. $\qquad\square$

From the above lemmas we get the following theorem.

Theorem 5.5. *Let $S = (C_1 \times C_2)/G$ be a surface isogenous to a higher product with $p_g = q = 0$ and $G = S_4$. Then there are exceptional collections of line bundles of maximal length 4 on S.*

6. Case $G = (\mathbb{Z}/4 \times \mathbb{Z}/2) \rtimes \mathbb{Z}/2$

The group G has a presentation

$$\langle x, y, z \mid x^4 = y^2 = z^2 = [x,y] = [y,z] = 1, x^z = xy \rangle.$$

Both coverings $C_1, C_2 \to \mathbb{P}^1$ have ramification type $(2^2, 4^2)$. The corresponding tuples are (z, z, x, x^{-1}) for C_1 and $(x^2yz, x^2yz, xyz, x^3z)$ for C_2. Both curves C_1 and C_2 have genus 5 $(2g - 2 = 8)$. The reduced fibers E_1, E_2 consist of 8 points each and E_3, E_4 consist of 4 points on both curves. Now we construct explicit G-equivariant acyclic theta characteristics on C_1 and C_2.

Lemma 6.1. *There is an acyclic G-equivariant theta characteristic $\mathcal{L}$ on C_1.*

Proof. Let $L = E_1 - E_3$ and $\mathcal{L} = \mathcal{O}(L)$. Looking at the quotients $C_1/\langle x \rangle, C_1/\langle y, z \rangle$ we find $E_1 \sim E_2, E_3 \sim E_4$. It follows that $\mathcal{L}$ is a theta characteristic. Consider the quotient $C_1 \to C_1/\langle z \rangle$. From Lemma 2.2 we have $h^0(L) = h^0(L') + h^1(L')$ where L' is a divisor of degree 0 on the curve $C_1/\langle z \rangle$ of genus 1. One checks that L' is again a theta characteristic. From the quotient $C_1/\langle z \rangle \to C_1/\langle x^2, z \rangle$ we find $h^0(L) = 2h^0(L') = 4h^0(\mathbb{P}^1, \mathcal{O}(-1)) = 0$. $\qquad\square$

Lemma 6.2. *There is an acyclic G-equivariant line bundle $\mathcal{N}$ on C_2.*

Proof. The curve C_2 is abstractly the same as C_1 (lies in the same family), but the action of G on it is twisted by an automorphism. Namely, consider the automorphism $\phi : G \to G$ given by

$$x \mapsto xyz,$$
$$y \mapsto y,$$
$$z \mapsto x^2yz.$$

Then the curve C_2 is one of the possible curves C_1 with the G-action given by $g \cdot x = \phi^{-1}(g)x$ where on the right hand side we consider the action on C_1. Thus if we let $\mathcal{N} = \mathcal{O}(E_1 - E_3)$ on C_2, then $\mathcal{N}$ has no regular sections and is equivariant. $\qquad\square$

From the above lemmas we get the following theorem.

Theorem 6.3. *Let $S = (C_1 \times C_2)/G$ be a surface isogenous to a higher product with $p_g = q = 0$ and $G = (\mathbb{Z}/4 \times \mathbb{Z}/2) \rtimes \mathbb{Z}/2$. Then there are exceptional collections of line bundles of maximal length 4 on S.*

7. APPENDIX: EXPLICIT CONSTRUCTION IN THE CASE $G = D_4 \times \mathbb{Z}/2$

In this section we give explicit construction of the line bundle $\mathcal{M}$ on the curve C_2 in the case $G = D_4 \times \mathbb{Z}/2$ (see section 3). We compute obstructions to the existence of the equivariant structure on line bundles $\mathcal{L}$ and $\mathcal{M}$ and prove that they are inverse to each other, so the bundle $\mathcal{L} \boxtimes \mathcal{M}$ on the product $C_1 \times C_2$ is equivariant. The next lemma is elementary.

Lemma 7.1. *Let C be a hyperelliptic curve of genus g. The twofold covering $\pi : C \to \mathbb{P}^1$ has $2g+2$ ramification points. Let us label them arbitrarily as $x_1, \ldots, x_{g+1}$, $y_1, \ldots, y_{g+1}$. Then there is a unique rational function (up to constant) f on C which has a simple zero in each x_i, simple pole in each y_i and for such f we have $\sigma_* f = -f$ where σ is the nontrivial automorphism of the covering $C \to \mathbb{P}^1$.*

Proof. Let $D = -x_1 - \cdots - x_{g+1} + y_1 + \cdots + y_{g+1}$ be a divisor on C. By the Serre duality, we have $h^1(C, \mathcal{O}(D)) = h^0(C, \mathcal{O}(K - D)) = h^0(C, \pi^*(\mathcal{O}_{\mathbb{P}^1}(-2)) \otimes \mathcal{O}(2x_1 + \cdots + 2x_{g+1})) = h^0(C, \pi^* \mathcal{O}_{\mathbb{P}^1}(g - 1)) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(g - 1)) + h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = g$. From the Riemann-Roch formula, we see that $h^0(C, \mathcal{O}(D)) - h^1(C, \mathcal{O}(D)) = 1 - g$ and $h^0(C, \mathcal{O}(D)) = 1, h^1(C, \mathcal{O}(D)) = g$. Therefore there is a unique (up to constant) rational function f on C which has a simple zero in each x_i and simple pole in each y_i. Moreover we can write f explicitly and we see that f^2 is a rational function on $\mathbb{P}^1$, but f is not a rational function on $\mathbb{P}^1$. Therefore we have $\sigma_* f = -f$ where σ is the nontrivial automorphism of the covering $C \to \mathbb{P}^1$. □

The group G has a presentation

$$\langle x, y, z \mid x^4 = y^2 = z^2 = [x, z] = [y, z] = 1, x^y = x^{-1} \rangle,$$

where $x^y = y^{-1} x y$, $[x, y] = x y x^{-1} y^{-1}$. We say that an element $g \in G$ is written in standard form if $g = x^k y^l z^m$, where $0 \le k \le 3$, $0 \le l, m \le 1$. Each element of G has the unique standard form. One can show that $H^2(G, \mathbb{C}^*) \cong (\mathbb{Z}/2)^2$ (see [17] for more details).

Now we compute the obstruction of $\mathcal{L}$. Recall that when we fix an isomorphism $\phi_g : g_* \mathcal{L} \to \mathcal{L}$ for each $g \in G$, the obstruction can be computed via

$$(4) \qquad \eta_{\mathcal{L}}(g, h) = \phi_g \cdot g_* \phi_h \cdot \phi_{gh}^{-1}$$

where g, h are elements of G.

Lemma 7.2. *The line bundle $\mathcal{L}$ on C_1 has obstruction*

$$\eta_{\mathcal{L}}(x^k y^l z^m, x^{k'} y^{l'} z^{m'}) = (-1)^{m(k'+l')} \cdot i^{-lk'}$$

for elements of G in the standard form (here $i = \sqrt{-1}$).

Proof. Recall from section 3 that $\mathcal{L} = \mathcal{O}(B_1 - B_2)$, where B_1 is one of $\langle x^2, xy \rangle$-orbits in E_1 and B_2 is any free $\langle z \rangle$-orbit. We will compute obstructions for B_1 and B_2.

Consider the covering $q : C_1 \to C'_1 = C_1/\langle x^2, xy \rangle$. The curve C'_1 has genus 0. The image of E_1 on C'_1 is two points P, xP. Let $B_1 = q^* P$. The divisor B_1 has degree 4. The G-orbit of divisor B_1 consists of 2 points: B_1 and $xB_1 = q^*(xP)$. Let f' be a rational function on C'_1 with divisor $xP - P$ and $f = q^* f'$ be its pullback

to C_1. The quotient map $C_1' \to C_1'/\langle z \rangle$ is ramified in 2 points P, xP. By Lemma 7.1 we have $z_* f' = -f'$. The function f is $\langle x^2, xy \rangle$-invariant since it comes from C_1'. The function f has divisor $xB_1 - B_1$, and $x_* f$ has inverse divisor $B_1 - xB_1$. Then for the right choice of a constant we have $x_* f = \frac{1}{f}$, $y_* f = \frac{1}{f}$, $z_* f = -f$. The divisor B_1 is invariant under the action of $\langle x^2, xy, z \rangle$, and we can put $\phi_g = 1$ if $g \in \langle x^2, xy, z \rangle$ and $\phi_g = f$ if $g \notin \langle x^2, xy, z \rangle$. Then we have

$$\eta_{B_1}(x^k y^l z^m, x^{k'} y^{l'} z^{m'}) = (-1)^{m(k'+l')}.$$

Consider the covering $q : C_1 \to C_1' = C_1/\langle z \rangle$. The curve C_1' has genus 0. The image of E_2 on C_1' consists of 4 points: two of them have stabilizer $\langle y \rangle$, the other two have $\langle x^2 y \rangle$. Let P be one of the points which is stabilized by y and let $B_2 = q^* P$. It is a divisor of degree 2. There is a function f' on C_1' with divisor $xP - P$, and its pullback $f = q^* f'$ has divisor $xB_2 - B_2$. The functions $x_*^k f$ have divisors $x^{k+1} B_2 - x^k B_2$. The function $f \cdot x_* f \cdot \ldots \cdot x_*^{k-1} f$ has divisor $x^k B_2 - B_2$. The function $f \cdot x_* f \cdot x_*^2 f \cdot x_*^3 f$ has trivial divisor since $x^4 = 1$. Thus multiplying f by a constant we can assume that

$$(5) \qquad\qquad f \cdot x_* f \cdot x_*^2 f \cdot x_*^3 f = 1.$$

The divisor B_2 has an orbit of order 4 under the action of G; it consists of $B_2, \ldots, x^3 B_2$. We have $yB_2 = zB_2 = B_2$. Then for the divisor B_2 we can put $\phi_{x^k y^l z^m} = \phi_{x^k}$, where $\phi_{x^k} = f \cdot x_* f \cdot \ldots \cdot x_*^{k-1} f$ (by (5) it depends only on k mod 4). The divisor of the function $f' \cdot x_* f'$ is $x^2 P - P$, but P and $x^2 P$ are the only ramification points of the morphism $C_1' \to C_1'/\langle y \rangle$. Therefore by Lemma 7.1 we have $y_*(f' \cdot x_* f') = -f' \cdot x_* f'$. The divisor of $y_* f$ is $y(xB_2 - B_2) = x^3 B_2 - B_2$ and $x_*^{-1} f$ has divisor $B_2 - x^3 B_2$, so we have

$$y_* f = \frac{C}{x_*^{-1} f}$$

for some constant C and

$$y_*(f \cdot x_* f) = y_* f \cdot x_*^{-1} y_* f = \frac{C}{x_*^2 f} \cdot \frac{C}{x_*^3 f} = C^2 f \cdot x_* f.$$

Thus $C^2 = -1$. Changing f to if if needed we can assume that $C = i$. Note that such a change preserves (5). Now a simple calculation gives the cocycle for B_2:

$$\eta_{B_2}(x^k y^l z^m, x^{k'} y^{l'} z^{m'}) = i^{lk'}.$$

It remains to add two obstructions to finish the proof of the lemma. $\qquad\square$

Then we construct $\mathcal{M}$, which gives us an explicit construction of exceptional collections.

Lemma 7.3. *There is a G-invariant line bundle $\mathcal{M}$ of degree zero on C_2 with obstruction inverse to $\eta_{\mathcal{L}}$.*

Proof. We will construct divisors A_1, A_2 on C_2 with G-invariant classes in the Picard group with obstructions $\eta_{A_1} = (-1)^{mk'}$ and $\eta_{A_2} = i^{l(k'+2m')}$. Consider the covering $q : C_2 \to C_2' = C_2/H$, where $H = \langle y, x^2 y \rangle$. In each of the fibers E_1, E_3 there are 4 points with stabilizer $\langle y \rangle$ and 4 points with stabilizer $\langle x^2 y \rangle$. Thus the curve C_2' has genus 1. The subgroup H is normal in G, and there is an action of G/H on C_2'. Each of E_5, E_6 consists of two free H-orbits and is mapped to two points on C_2'. Let us denote by P a point in the image of E_5 and by Q a point in

the image of E_6. The image of E_5 on C_2' is $\{P, xP\}$ because P is stabilized by H and x^2z, so xP is a different point in the image of E_5. The image of E_6 is $\{Q, xQ\}$. Let the divisor A_1 on C_2 be equal to $q^*(P - Q)$. It has degree 0.

We claim that the line bundle $\mathcal{O}(A_1)$ lies in the $Pic(C_2)^G$. We need to find isomorphisms $g_*\mathcal{O}(A_1) \to \mathcal{O}(A_1)$ for each $g \in G$, or in other words, rational functions ϕ_g which have divisors $gA_1 - A_1$. The divisor A_1 is invariant under the action of the subgroup $\langle x^2, y, z \rangle = \langle H, x^2z \rangle$. There are only two elements in the G-orbit of A_1: A_1 and xA_1. So we only need to find a function on C_2 with divisor $xA_1 - A_1$. Consider the covering $C_2' \to C_2'' = C_2/\langle x, x^2y, x^2z \rangle$. Then C_2'' has genus 0, and $C_2' \to C_2''$ is the covering of degree two ramified in the points P, xP, Q, xQ.

By Lemma 7.1 there is a function f' on C_2' with divisor $P - xP - Q + xQ$. We denote its pullback to C_2 by f. The divisor of f on C_2 is $xA_1 - A_1$. With the right choice of the multiplicative constant G acts on f in the following way: $x_*f = \frac{1}{f}$, $y_*f = f$, $z_*f = -f$. Now we can put $\phi_g = 1$ if $g \in \langle x^2, y, z \rangle$ and $\phi_g = f$ if $g \notin \langle x^2, y, z \rangle$. We see that $\mathcal{O}(A_1)$ lies in the invariant part of the Picard group. Computing the obstruction by the formula (2) we see that it is equal to

$$\eta_{A_1}(x^k y^l z^m, x^{k'} y^{l'} z^{m'}) = (-1)^{mk'}.$$

Now we will construct another divisor with G-invariant class on C_2. Since we will not need any more details about the construction of A_1, we will reuse the notation P, Q, C, etc., for new objects. Consider the covering $q : C_2 \to C_2' = C_2/\langle x^2z \rangle$. The group $G/\langle x^2z \rangle$ acts on C_2'. Each of E_1, E_3 is mapped to 4 points on C_2'. In both cases two of the points are stabilized by y on C_2' and the other two by x^2y. Let P be one of the points in the image of E_1 stabilized by y, and Q be such a point in the image of E_3. Then the 4 points in the image of E_1 are P, xP, x^2P, x^3P, the points P and x^2P are stabilized by y and xP, x^3P are exchanged by y and stabilized by x^2y. The same is true about E_3 and points x^iQ. Let A_2' be the divisor $P - Q$ on C_2' and $A_2 = q^*A_2'$. Both divisors have degree 0. As in the case of B_2, the function $f' \cdot x_*f'$ on C_2' has simple zeros or poles in all the points stabilized by y and only in them; therefore $y_*(f \cdot x_*f) = -f \cdot x_*f$. Analogously we have

$$y_*f = \frac{C}{x_*^{-1}f}$$

and we can assume $C = i$. The only difference is that now $\phi_{x^k y^l z^m} = \phi_{x^{k\pm 2m}}$, because we took the quotient by $\langle x^2z \rangle$, not by $\langle z \rangle$. We get

$$\eta_{B_2}(x^k y^l z^m, x^{k'} y^{l'} z^{m'}) = i^{l(k' + 2m')}.$$

If we add the obstructions for line bundles $\mathcal{L}$ and $\mathcal{M}$ we will get

$$\eta(x^k y^l z^m, x^{k'} y^{l'} z^{m'}) = (-1)^{mk'} \cdot i^{l(k' + 2m')} \cdot (-1)^{mk' + ml'} \cdot i^{-lk'} = (-1)^{ml' + lm'},$$

and this cocycle is cohomologous to zero because it is the differential of the 1-cochain $\beta : G \to \mathbb{C}^*$ such that $\beta(x^k y^l z^m) = -1$ if both l and m are odd and $\beta = 1$ otherwise. $\square$

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