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THURSTON'S BOUNDARY FOR TEICHMÜLLER SPACES OF INFINITE SURFACES: THE LENGTH SPECTRUM

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Abstract. Let X_0 be an infinite area geodesically complete hyperbolic surface which can be decomposed into geodesic pairs of pants. We introduce Thurston's boundary to the Teichmüller space $T(X_0)$ of the surface X_0 using the length spectrum analogous to Thurston's construction for finite surfaces. Thurston's boundary using the length spectrum is a "closure" of projective bounded measured laminations $PML_{bdd}(X_0)$, and it coincides with $PML_{bdd}(X_0)$ when X_0 can be decomposed into a countable union of geodesic pairs of pants whose boundary geodesics $\{\alpha_n\}_{n\in\mathbb{N}}$ have lengths pinched between two positive constants. When a subsequence of the lengths of the boundary curves of the geodesic pairs of pants $\{\alpha_n\}_n$ converges to zero, Thurston's boundary using the length spectrum is strictly larger than $PML_{bdd}(X_0)$.

1. Introduction

A *geodesic pair of pants* is a bordered hyperbolic surface homeomorphic to a sphere minus 3 disks such that the boundary consists of 3 closed geodesics (called *cuffs*) with possibly 1 or 2 geodesics degenerated to have length 0, i.e. a cusp. Let X_0 be a fixed, geodesically complete, borderless hyperbolic surface which is decomposed into a union of infinitely many geodesic pairs of pants called the *geodesic pants decomposition*. Each two geodesic pairs of pants are either disjoint or share a cuff. No end of X_0 is a hyperbolic funnel, while an end can be a cusp. The fundamental group of X_0 is infinitely generated and X_0 has an infinite area.

The space of all quasiconformal deformations of X_0 modulo post-compositions by conformal maps and homotopies is an infinite-dimensional manifold called the *Teichmüller space* $T(X_0)$ of X_0. Denote by $[f]\in T(X_0)$ the equivalence class of a quasiconformal map $f: X_0\to X$. We study the limiting behaviour of the quasiconformal deformations of X_0 when the dilatations of the quasiconformal maps increase without a bound using the marked length spectrum of the image surfaces. Thurston [27], [14] used the length spectrum to compactify the Teichmüller space of a closed surface by adding to it the space of projective measured laminations of the surface. Bonahon [9] used geodesic currents to give an alternative description of Thurston's boundary for the Teichmüller space of a closed surface. In [10], geodesic currents were used to introduce a boundary to $T(X_0)$, and one of our goals is to compare how this geodesic currents boundary differs from Thurston's boundary defined using the length spectrum in the case of the above surface X_0.

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Alvarez and Rodriguez [4] proved that any geodesically complete hyperbolic surface is obtained by gluing geodesic pairs of pants along their cuffs and by attaching at most countably many funnels with closed geodesic boundary and half-planes with boundary infinite geodesics (see also [5]). Alessandrini, Liu, Papadopoulos, Su and Sun [3, Theorem 4.5] proved that if a complete hyperbolic surface has a geodesic pants decomposition, then any topological pants decomposition can be straightened to a geodesic pants decomposition (see [7, Proposition 3.1] for a related statement).

We restrict our attention to geodesically complete, infinite area hyperbolic surfaces that have geodesic pants decomposition into infinitely many geodesic pairs of pants (cf. [26], [6], [1]) since, in this case, the Teichmüller space is completely determined by the marked length spectrum. Shiga [26] initiated the study of Teichmüller spaces of such surfaces using the length spectrum, and this was continued by various authors (e.g. [1], [2], [6], [18], [17], [21]).

Let S be a closed hyperbolic surface and let $\mathcal{S}$ be the set of all simple closed geodesics on S. The homotopy class of a quasiconformal map $f : S \to S_1$ induces a function from $\mathcal{S}$ to $\mathbb{R}$ which assigns to each $\alpha \in \mathcal{S}$ the length of a geodesic in S_1 that is homotopic to $f(\alpha)$. Thus we have an injective map

$$\mathcal{X} : T(S) \to \mathbb{R}^{\mathcal{S}}_{\geq 0}.$$

The above map is a homeomorphism onto its image if $\mathbb{R}^{\mathcal{S}}_{\geq 0}$ is equipped with the weak* topology (cf. [14]).

In the case of a hyperbolic surface X_0 equipped with a geodesic pants decomposition containing infinitely many geodesic pairs of pants, the *length spectrum distance* between $[f] \in T(X_0)$ and $[g] \in T(X_0)$ is defined by (cf. [26], [2])

$$d_{ls}([f],[g]) = \sup_{\alpha \in \mathcal{S}} \frac{1}{2}\Big|\log \frac{l_{f(X_0)}(f(\alpha))}{l_{g(X_0)}(g(\alpha))}\Big|,$$

where $\mathcal{S}$ is the set of all closed geodesics on X_0 and $l_{f(X_0)}(f(\alpha))$ is the length of the closed geodesic on $f(X_0)$ homotopic to the closed curve $f(\alpha)$. Shiga [26] proved that the topology induced by the length spectrum distance on $T(X_0)$ is equal to the Teichmüller topology when the surface X_0 has a geodesic pants decomposition with lengths of cuffs pinched between two positive constants. Alessandrini, Liu, Papadopoulos and Su [1] proved that the length spectrum distance on $T(X_0)$ is not complete when X_0 contains a sequence of simple closed geodesics whose lengths go to zero. In fact, they [1] introduced a new space called the length spectrum Teichmüller space $T_{ls}(X_0)$ which contains $T(X_0)$ on which d_{ls} is complete. We do not pursue the study of this space since we are interested in comparing the geodesic currents boundary of $T(X_0)$ to that of the length spectrum Thurston's boundary.

Define $l^{\infty}_{X_0}$ to be the set of all $h \in \mathbb{R}^{\mathcal{S}}_{\geq 0}$ that satisfy $\sup_{\alpha\in\mathcal{S}} \Big|\frac{h(\alpha)}{l_{X_0}(\alpha)}\Big| < \infty$. We introduce a *normalized supremum norm* on $l^{\infty}_{X_0}$ by

$$\|h\|^{norm}_{\infty} = \sup_{\alpha\in\mathcal{S}} \Big|\frac{h(\alpha)}{l_{X_0}(\alpha)}\Big|$$

for $h \in l^{\infty}_{X_0}$. Then $\mathcal{X}(T(X_0)) \subset l^{\infty}_{X_0}$ and the normalized supremum norm on $l^{\infty}_{X_0}$ makes the map $\mathcal{X} : T(X_0) \to l^{\infty}_{X_0}$ a homeomorphism onto its image (cf. Proposition 3.1).

Analogous to the closed surface case, we projectivize $\mathcal{X}$ and obtain an injective map

$$P\mathcal{X} : T(X_0) \to Pl^{\infty}_{X_0}.$$

By definition, *(the length spectrum) Thurston's boundary* of $T(X_0)$ consists of the boundary points of the image $P\mathcal{X}(T(X_0))$ of $T(X_0)$, where $Pl^{\infty}_{X_0}$ is given the quotient topology with respect to the normalized supremum norm on $l^{\infty}_{X_0}$.

Let μ be a bounded measured lamination on X_0 (see section 2 for the definition). For $\alpha \in \mathcal{S}$, let $i(\mu,\alpha)$ denote the geometric intersection number of the measured lamination μ and the closed geodesic α. Then $i(\mu,\cdot) : \mathcal{S} \to \mathbb{R}$ is an element of $l^{\infty}_{X_0}$ (see section 4); $ML_{bdd}(X_0)$ is identified with its image in $l^{\infty}_{X_0}$; $PML_{bdd}(X_0)$ is identified with its image in $Pl^{\infty}_{X_0}$.

Theorem 1. *Let X_0 be a borderless infinite area geodesically complete hyperbolic surface that has a geodesic pants decomposition with infinitely many geodesic pairs of pants. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be the family of cuffs of the pants decomposition that are closed geodesics; i.e. cusps are excluded. The (length spectrum) Thurston's boundary of $T(X_0)$ is the closure of the space of projective bounded measured laminations $PML_{bdd}(X_0)$ in $Pl^{\infty}_{X_0}$, where $Pl^{\infty}_{X_0}$ has the quotient topology induced by the topology on $l^{\infty}_{X_0}$ coming from the normalized supremum norm.*

If the lengths of $\{\alpha_n\}_{n\in\mathbb{N}}$ are pinched between two positive constants, then the length spectrum Thurston's boundary is equal to $PML_{bdd}(X_0)$ as a set.

If the lengths of $\{\alpha_n\}_{n\in\mathbb{N}}$ are bounded from above and there exists a subsequence $\{\alpha_{n_k}\}$ whose lengths converge to 0, then the length spectrum Thurston's boundary contains $PML_{bdd}(X_0)$ as a proper subset.

In addition, Thurston's boundary of $T(X_0)$ when the hyperbolic surface X_0 whose every geodesic pants decomposition does not have an upper bound on the lengths of cuffs but can be decomposed into bounded polygons with at most n sides (introduced by Kinjo [17]) equals $PML_{bdd}(X_0)$. On the other hand, if X_0 is the surface constructed by Shiga [26] such that the length spectrum distance is incomplete, then the length spectrum Thurston's boundary is strictly larger than $PML_{bdd}(X_0)$ (cf. section 6).

Recall that the quasiconformal Mapping Class Group $MCG_{qc}(X_0)$ consists of all quasiconformal maps $g : X_0 \to X_0$ up to homotopy (cf. [15]). The action of $MCG_{qc}(X_0)$ on the Teichmüller space $T(X_0)$ is given by $[f] \mapsto [f \circ g^{-1}]$ and it is continuous in the Teichmüller metric d_T (for the definition of d_T see, for example, [15] or section 2). Since $d_{ls}([f],[g]) \leq d_T([f],[g])$ for all $[f],[g] \in T(X_0)$, the action is also continuous for the length spectrum distance. We prove

Theorem 2. *The action of the quasiconformal Mapping Class Group $MCG_{qc}(X_0)$ on the Teichmüller space $T(X_0)$ extends to a continuous action on the length spectrum Thurston's closure of $T(X_0)$ for the topology induced by the normalised supremum norm.*

The following is a natural question regarding the convergence towards a boundary point.

Open problem. Assume that a sequence in $T(X_0)$ converges to a bounded projective measured lamination in the length spectrum Thurston's boundary of $T(X_0)$. Is it true that the sequence converges to the same point in the closure introduced using geodesic currents?

2. Teichmüller space, measured geodesic laminations and earthquakes for geometrically infinite hyperbolic surfaces

A geodesic pair of pants is a bordered hyperbolic surface homeomorphic to a sphere minus 3 disks such that the boundary components are closed geodesics with possibly one or two of them degenerated to a cusp. The boundary components are called *cuffs*. Let X_0 be a fixed, borderless, geodesically complete hyperbolic surface equipped with a geodesic pants decomposition with infinitely many geodesic pairs of pants. Let $\{\alpha_n\}_{n=1}^{\infty}$ be the family of cuffs which are closed geodesics (i.e. non-cusps) of the fixed pants decomposition of X_0.

The *Teichmüller space* $T(X_0)$ of the surface X_0 is the space of equivalence classes of all quasiconformal maps $f : X_0 \to X$ where X is an arbitrary complete hyperbolic surface. Two quasiconformal maps $f_1 : X_0 \to X_1$ and $f_2 : X_0 \to X_2$ are *equivalent* if there exists an isometry $I : X_1 \to X_2$ such that $f_2^{-1} \circ I \circ f_1$ is homotopic to the identity. Denote by $[f]$ the equivalence class of a quasiconformal map $f : X_0 \to X$.

The *Teichmüller distance* on $T(X_0)$ is defined by

$$d_T([f_1],[f_2]) = \frac{1}{2} \log \inf_{g \in [f_2 \circ f_1^{-1}]} K(g)$$

where the infimum is taken over all quasiconformal maps g equivalent to $f_2 \circ f_1^{-1}$ and $K(g)$ is the quasiconformal constant of g. The *Teichmüller topology* on $T(X_0)$ is the topology induced by the Teichmüller distance.

If Y is a hyperbolic surface and α a closed curve on Y not homotopic to a point or a cusp of Y, we denote by $l_Y(\alpha)$ the length of the unique closed geodesic homotopic to α. The *length spectrum distance* on $T(X_0)$ is given by

$$d_{ls}([f_1],[f_2]) = \sup_{\delta \in \mathcal{S}} \left\{ \left| \log \frac{l_{f_2(X_0)}(f_2(\delta))}{l_{f_1(X_0)}(f_1(\delta))} \right| \right\}.$$

Shiga [26] proved that if the cuff family $\{\alpha_n\}_{n=1}^{\infty}$ has positive lower and upper bounds on their lengths, then the Teichmüller distance induces the same topology as the length spectrum distance on $T(X_0)$. However, there are examples of hyperbolic surfaces X_0 for which the two distances do not induce the same topology on $T(X_0)$ (cf. [26]). We use the topology obtained from the length spectrum distance since our construction uses the length spectrum of surfaces.

A *geodesic lamination* on a hyperbolic surface X_0 is a closed subset of X_0 that is foliated by non-intersecting complete geodesics called *leaves* of the lamination. A *stratum* of a geodesic lamination is either a leaf of the lamination or a connected component of the complement. A *measured lamination* μ on X_0 is an assignment of a positive Radon measure on each arc transverse to a geodesic lamination $|\mu|$ that is invariant under homotopies setwise preserving each leaf of $|\mu|$, and the measure of a subarc is the restriction of the measure of the arc containing it (cf. [28]). The geodesic lamination $|\mu|$ is called the *support* of μ. A measured lamination on X lifts to a measured lamination on the hyperbolic plane $\mathbb{H}$ that is invariant under the covering group of X.

A *(left) earthquake* $E : X_0 \to X$ with support geodesic lamination λ is a surjective map that is an isometry on each stratum of λ such that each stratum is moved to the left relative to any other stratum. An earthquake E induces a measured lamination on its support given by the amount of the relative movement to the left;

an earthquake is uniquely determined by its induced measured lamination up to the post-composition by an isometry (cf. [28]).

An earthquake $E : X_0 \to X$ lifts to an earthquake of $\tilde{E} : \mathbb{H} \to \mathbb{H}$ where the support of $\tilde{E}$ is the lift of the support on E (cf. Thurston [27]). The lifted earthquake $\tilde{E}$ extends by the continuity to a homeomorphism of the unit circle S^1. Thurston's earthquake theorem states that any homeomorphism of the unit circle S^1 can be obtained by continuous extension of a left earthquake (cf. Thurston [27]). Thus an earthquake induces a homeomorphism class of mappings from X_0 to X.

We define *Thurston's norm* of a measured lamination μ as

$$\|\mu\|_{Th} = \sup_J i(\mu, J)$$

where the supremum is over all hyperbolic arcs J of length 1 and $i(\cdot, \cdot)$ is the intersection number (cf. [28], [27]).

A quasiconformal map of X_0 onto another surface X lifts to a quasiconformal map of $\mathbb{H}$, and the latter extends to a quasisymmetric map of the unit circle S^1. Therefore we consider measured laminations whose earthquakes induce quasisymmetric maps of S^1. An earthquake $\tilde{E}^\mu$ extends by continuity to a quasisymmetric map of S^1 if and only if $\|\mu\|_{Th} < \infty$ (cf. [27], [16], [21], [22]).

Denote by $ML_{bdd}(X_0)$ the space of all measured laminations with finite Thurston's norm on X_0. When $ML_{bdd}(X_0)$ is equipped with an appropriate topology, the map

$$EM : T(X_0) \to ML_{bdd}(X_0)$$

is a homeomorphism (cf. [19]).

Note that $\|t\mu\|_{Th} = t\|\mu\|_{Th}$, for $t > 0$. Then, for $\|\mu\|_{Th} < \infty$, we have that $t \mapsto E^{t\mu}$, for $t > 0$, is a path in $T(X_0)$ called an *earthquake path.* An earthquake path in $T(X_0)$ leaves every compact subset as $t \to \infty$ and is a convenient tool for studying Thurston's boundary. On the other hand, we note that not every earthquake path that starts in the length spectrum Teichmüller space $T_{ls}(X_0)$ stays inside $T_{ls}(X_0)$ (cf. [25]).

3. Thurston's boundary for Teichmüller spaces of general surfaces using the length spectrum

Recall that X_0 is a fixed, borderless, geodesically complete hyperbolic surface equipped with a geodesic pants decomposition that contains infinitely many geodesic pairs of pants. In other words, X_0 is a geodesically complete hyperbolic surface formed by gluing infinitely many geodesic pairs of pants along their boundaries. Not every surface obtained by gluing infinitely many geodesic pairs of pants is complete (cf. Basmajian [5]). However, each gluing can be adjusted by choosing an appropriate twist such that the surface is complete (cf. [7]). We are assuming that X_0 is such a surface.

Denote by $\mathcal{S}$ the set of all simple closed geodesics on X_0. Recall that $l^\infty_{X_0}$ is the space of functions $h : \mathcal{S} \to \mathbb{R}^+$ such that $\sup_{\alpha\in\mathcal{S}} \left|\frac{h(\alpha)}{l_{X_0}(\alpha)}\right| < \infty$, where $l_{X_0}(\alpha)$ is the length of the closed geodesic α. We define a map $\mathcal{X}$ from the Teichmüller space $T(X_0)$ into $\mathbb{R}^{\mathcal{S}}_{\geq 0}$, for $[f] \in T(X_0)$ and $\alpha \in \mathcal{S}$,

$$\mathcal{X}([f])(\alpha) = l_{f(X_0)}(f(\alpha)),$$

where $f(X_0)$ is the image hyperbolic surface under quasiconformal mapping f and $l_{f(X_0)}(f(\alpha))$ is the length of the simple closed geodesic on $f(X_0)$ homotopic to a simple closed curve $f(\alpha)$. The map $\mathcal{X} : T(X_0) \to \mathbb{R}^{\mathcal{S}}_{\geq 0}$ is injective because it uniquely determines the Fenchel-Nielsen coordinates which determine $f(X_0)$ up to an isometry (cf. [3]).

Wolpert [29] proved that if $f : X_0 \to X$ is a K-quasiconformal mapping and α a closed geodesic in X, then $\frac{1}{K} l_{X_0}(\alpha) \leq l_X(f(\alpha)) \leq K l_{X_0}(\alpha)$. This immediately gives $\mathcal{X}([f]) \in l^\infty_{X_0}$. Thus

$$\mathcal{X} : T(X_0) \to l^\infty_{X_0}.$$

We introduce the *normalized supremum norm* on $l^\infty_{X_0}$ by

$$\|h\|^{norm}_\infty = \sup_{\alpha\in\mathcal{S}} \frac{|h(\alpha)|}{l_{X_0}(\alpha)}$$

for all $h \in l^\infty_{X_0}$.

Proposition 3.1. *The length spectrum distance on $T(X_0)$ is locally bi-Lipschitz equivalent to the normalized supremum norm on $\mathcal{X}(T(X_0))$.*

Remark 3.2. This statement holds for $T_{ls}(X_0)$ in the place of $T(X_0)$ with the same proof.

Proof. Indeed, if

$$\sup_{\alpha\in\mathcal{S}} \left| \frac{l_{f_1(X_0)}(f_1(\alpha))}{l_{X_0}(\alpha)} - \frac{l_{f_2(X_0)}(f_2(\alpha))}{l_{X_0}(\alpha)} \right| < \epsilon,$$

then

$$\sup_{\alpha\in\mathcal{S}} \frac{l_{f_1(X_0)}(f_1(\alpha))}{l_{X_0}(\alpha)} \left| 1 - \frac{l_{f_2(X_0)}(f_2(\alpha))}{l_{f_1(X_0)}(f_1(\alpha))} \right| < \epsilon.$$

Since f_1 is a quasiconformal map, there exists $M > 1$ such that $1/M \leq \frac{l_{f_1(X_0)}(f_1(\alpha))}{l_{X_0}(\alpha)} \leq M$ (cf. Wolpert [29]). The above and symmetry imply that

$$\left| \frac{l_{f_2(X_0)}(f_2(\alpha))}{l_{f_1(X_0)}(f_1(\alpha))} - 1 \right|, \left| \frac{l_{f_1(X_0)}(f_1(\alpha))}{l_{f_2(X_0)}(f_2(\alpha))} - 1 \right| \leq M\epsilon$$

for all $\alpha \in \mathcal{S}$, and one direction is obtained since $|\log x|/|x-1|$ is between two positive constants for $1/2 < x < 2$. The other direction is obtained by reversing the order of the above inequalities, and the two distances are locally bi-Lipschitz. $\square$

Denote by

$$\mathcal{PX} : T(X_0) \to Pl^\infty_{X_0}$$

the map from $T(X_0)$ into the projective space $Pl^\infty_{X_0} = (l^\infty_{X_0} - \{\bar{0}\})/\mathbb{R}_{>0}$. The map $\mathcal{PX}$ is injective on $T(X_0)$. The *length spectrum Thurston's boundary* of $T(X_0)$ is, by definition, the space of all limit points in $Pl^\infty_{X_0}$ of the set $\mathcal{PX}(T(X_0))$ for the topology induced by the normalized supremum norm (cf. [14] for Thurston's original discussion on closed surfaces).

Note that a measured lamination μ on X_0 induces a real valued function on $\mathcal{S}$ by the formula

$$\mu(\alpha) = i(\mu, \alpha)$$

for all $\alpha \in \mathcal{S}$, where $i(\mu, \alpha)$ is the intersection number. We have

Lemma 3.3. *If $\mu \in ML_{bdd}(X_0)$, then $i(\mu, \cdot) \in l^\infty_{X_0}$.*

Proof. Let α be a closed geodesic in X_0. If $l_{X_0}(\alpha) > 1$, then we define $N = [l_{X_0}(\alpha)] + 1$, where $[l_{X_0}(\alpha)]$ is the integer part of $l_{X_0}(\alpha)$. Since $\mu \in ML_{bdd}(X_0)$, we have

$$i(\mu,\alpha) \leq N\|\mu\|_{Th} \leq 2l_{X_0}(\alpha)\|\mu\|_{Th},$$

which gives

$$\frac{i(\mu,\alpha)}{l_{X_0}(\alpha)} \leq 2\|\mu\|_{Th}.$$

If $l_{X_0}(\alpha) \leq 1$, let $N = [1/l_{X_0}(\alpha)] + 1$. Note that $\frac{1}{l_{X_0}(\alpha)} \leq N \leq \frac{2}{l_{X_0}(\alpha)}$ and that $1 \leq Nl_{X_0}(\alpha) \leq 2$. Then we have

$$Ni(\mu,\alpha) = i(\mu, N\alpha) \leq 2\|\mu\|_{Th}$$

since the length of N consecutive copies of α is at most 2. The above gives

$$i(\mu,\alpha) \leq \frac{2\|\mu\|_{Th}}{N} \leq 2\|\mu\|_{Th} l_{X_0}(\alpha),$$

and the lemma follows. □

From now on, we identify μ with this element $i(\mu,\cdot) \in l^\infty_{X_0}$. The proof of the above lemma gives

$$\|i(\mu,\cdot)\|^{norm}_\infty \leq 2\|\mu\|_{Th}.$$

We prove our first result on Thurston's boundary for $T(X_0)$ using the length spectrum.

Proposition 3.4. *Let X_0 be a fixed borderless geodesically complete hyperbolic surface equipped with a geodesic pants decomposition with infinitely many geodesic pairs of pants. Then the length spectrum Thurston's boundary of $T(X_0)$ contains the space of projective bounded measured lamination $PML_{bdd}(X_0)$ and it equals the closure of $PML_{bdd}(X_0)$ for the topology on $Pl^\infty_{X_0}$ induced by the normalized supremum norm.*

Proof. Let $\mu \in ML_{bdd}(X_0)$ be a non-zero bounded measured lamination on X_0. Denote by $E^{t\mu}$, for $t > 0$, an earthquake path with the earthquake measure $t\mu$. Then $t \mapsto E^{t\mu}(X_0)$ is a path in $T(X_0)$ which leaves every bounded set in $T(X_0)$ because $\mu \in ML_{bdd}(X_0)$ (cf. [21]). Let f_t be a quasiconformal map from X_0 to X_t which belongs to the class represented by $E^{t\mu}$.

For $\alpha \in \mathcal{S}$, the inequality

$$l_{f_t(X_0)}(f_t(\alpha)) \leq ti(\mu,\alpha) + l_{X_0}(\alpha)$$

implies that

$$\frac{\frac{1}{t}\mathcal{X}([f_t])(\alpha) - i(\mu,\alpha)}{l_{X_0}(\alpha)} \leq \frac{1}{t} \tag{1}$$

for all $\alpha \in \mathcal{S}$ and all $t > 0$.

To obtain the opposite inequality, we choose the universal covering of X_0 such that $B(z) = e^{-l_{X_0}(\alpha)}z$ is a cover transformation corresponding to α. Let O be the stratum of the lift $\tilde{\mu}$ of μ to the universal covering $\mathbb{H}$ that contains $e^{l_{X_0}(\alpha)}i$, and let O_1 be the stratum of $\tilde{\mu}$ that contains i. Normalize the earthquake $E^{t\tilde{\mu}}$ such that $E^{t\tilde{\mu}}|_O = id$. Then

$$B^t = E^{t\tilde{\mu}}|_{O_1} \circ B$$

is a covering transformation that corresponds to the geodesic on $f_t(X_0)$ homotopic to $f_t(\alpha)$ (cf. [12]). Denote by l_t the translation length of B_t and $l = l_{X_0}(\alpha)$ the translation length of B. Let $k_1 < 0$ and $k_2 > 0$ be the endpoints of the hyperbolic translation $E^{t\tilde{\mu}}|_{O_1}$, and let m_t be its translation length (cf. Figure 1).

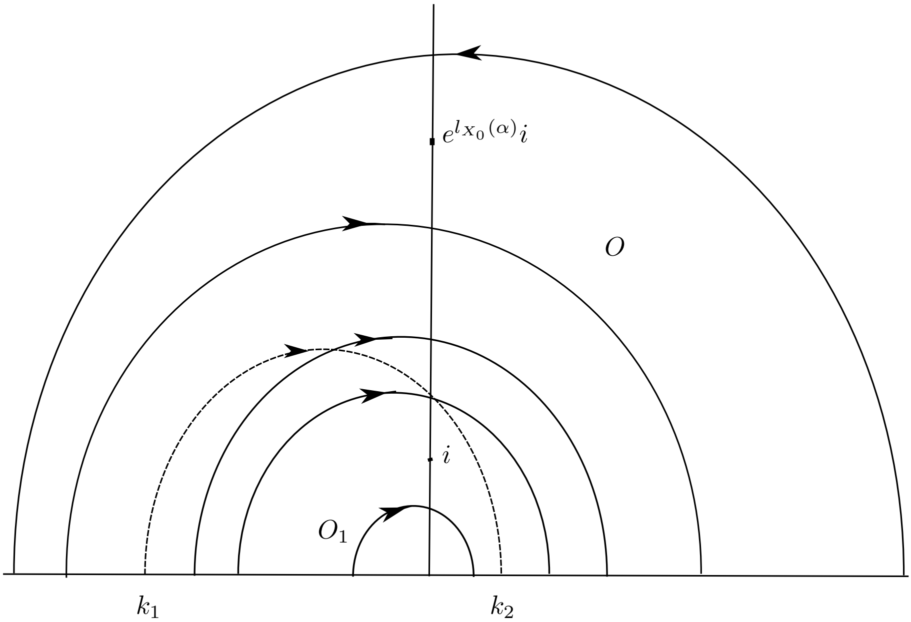

FIGURE 1. Computing $E^{t\tilde{\mu}}|_{O_1}$.

A direct computation (cf. [25]) gives

$$\operatorname{trace}(B^t) = 2\cosh\frac{m_t - l}{2} - \frac{2k_1}{k_2 - k_1}\Big(\cosh\frac{m_t + l}{2} - \cosh\frac{m_t - l}{2}\Big).$$

Consequently

$$2\cosh\frac{l_t}{2} = \operatorname{trace}(B^t) \geq 2\cosh\frac{m_t - l}{2},$$

which implies that

$$l_t \geq m_t - l.$$

Since the translation length of a composition of two hyperbolic translations (with non-intersecting axis and translating in the same direction) is at least as large as the sum of their translation lengths (cf. [27]), it follows that

$$m_t \geq ti(\mu, \alpha).$$

The above two inequalities give

$$\frac{1}{t}\frac{l_t}{l} \geq \frac{i(\mu,\alpha)}{l} - \frac{1}{t},$$

which implies that

$$\frac{1}{t}\frac{\mathcal{X}([f_t])(\alpha)}{l_{X_0}(\alpha)} - \frac{i(\mu,\alpha)}{l_{X_0}(\alpha)} \geq -\frac{1}{t}. \tag{2}$$

Then equations (1) and (2) give that, uniformly in $\alpha \in \mathcal{S}$,

$$\lim_{t\to\infty}\frac{1}{t}\frac{\mathcal{X}([f_t])(\alpha)}{l_{X_0}(\alpha)} = \frac{i(\mu,\alpha)}{l_{X_0}(\alpha)}.$$

We established that each point in $PML_{bdd}(X_0)$ is in Thurston's boundary.

Let $\sigma \in l^{\infty}_{X_0}$ be such that its projective class $[\sigma]$ is in the length spectrum Thurston's boundary. We need to establish that $[\sigma]$ is in the closure of $PML_{bdd}(X_0)$ for the normalized supremum norm.

By assumption, there exists a sequence $[f_n] \in T(X_0)$ that converges to the projective class $[\sigma] \in Pl^{\infty}_{X_0}$. Equivalently there exists a sequence $t_n \to \infty$ as $n \to \infty$ such that $\frac{1}{t_n}\mathcal{X}([f_n]) \to \sigma$ as $n \to \infty$ in the normalized supremum norm. Necessarily we have $\sup_n \|\frac{1}{t_n}\mathcal{X}([f_n])\|^{norm}_{\infty} < \infty$.

Let f_n be represented by a sequence of earthquakes $E^{t'_n\mu_n}$ with $\|\mu_n\|_{Th} = 1$ and $t'_n > 0$. Then $t'_n \to \infty$ as $n \to \infty$ and the first part of the proof gives

$$\|\frac{1}{t'_n}\mathcal{X}([f_n]) - \mu_n\|^{norm}_{\infty} < \frac{1}{t'_n}.$$

Note that if $\|\mu_n\|_{Th} = 1$, then $\|\mu_n\|^{norm}_{\infty} \leq 2$ by the proof of Lemma 3.3. Then the above inequality implies that $\|\frac{1}{t'_n}\mathcal{X}([f_n])\|^{norm}_{\infty} \leq 3$ for all t'_n with n large enough, and the sequence $\frac{t'_n}{t_n}$ is bounded from above and below by positive numbers. By choosing a subsequence, if necessary, we can assume that $\frac{t'_n}{t_n} \to c > 0$ as $n \to \infty$. It follows that, as $n \to \infty$,

$$\|\frac{1}{t_n}\mathcal{X}([f_n]) - c\mu_n\|^{norm}_{\infty} \to 0,$$

which implies that

$$\|c\mu_n - \sigma\|^{norm}_{\infty} \to 0,$$

and the proof is completed. □

The first part of Theorem 1 from the introduction follows by the above proposition.

4. Infinite area surfaces with bounded geodesic pants decompositions

As before, X_0 is a fixed, borderless, geodesically complete hyperbolic surface equipped with a geodesic pants decomposition containing infinitely many geodesic pairs of pants. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be the family of cuffs (i.e. boundary components) that are closed geodesics (i.e. non-cusps) of the geodesic pants decomposition of X_0. We say that the geodesic pants decomposition of X_0 is *upper-bounded* if there exists $M > 0$ such that, for each $n \in \mathbb{N}$,

$$l_{X_0}(\alpha_n) \leq M$$

where $l_{X_0}(\alpha_n)$ is the length of α_n in the hyperbolic metric of X_0 (cf. [1]). Moreover, the geodesic pants decomposition is *lower-bounded* if there exists $m > 0$ such that, for each $n \in \mathbb{N}$,

$$l_{X_0}(\alpha_n) \geq m.$$

Finally, the geodesic pants decomposition of X_0 is *bounded* if it is both upper- and lower-bounded.

The next proposition establishes that the length spectrum Thurston's boundary coincides with the boundary for $T(X_0)$ introduced using the geodesic currents when X_0 has a bounded geodesic pants decomposition (cf. [10]). We note that the convergence in the two closures of $T(X_0)$ might be different.

Proposition 4.1. *Assume that X_0 has a bounded geodesic pants decomposition. Then the length spectrum Thurston's boundary is equal to the space of projective bounded measured laminations $PML_{bdd}(X_0)$ on X_0.*

Proof. Consider a sequence of points $[f_k]\in T(X_0)$ that converge to (the projective class) $[\sigma]\in Pl^{\infty}_{X_0}$ in the length spectrum Thurston's boundary of $T(X_0)$. Then there exists a sequence $t_k\to\infty$ as $k\to\infty$ such that $\frac{1}{t_k}\mathcal{X}([f_k])\to\sigma$ in $l^{\infty}_{X_0}-\{\bar{0}\}$, where $\bar{0}(\alpha)=0$ for all $\alpha\in\mathcal{S}$. Let $E^{t_k\beta_k}$ be a sequence of earthquakes of X_0 that represent the equivalence class $[f_k]$, where $\|\beta_k\|_{Th}<\infty$ (cf. [27]).

The proof of the above proposition gives

$$\Big|\frac{1}{t_k}\frac{\mathcal{X}(E^{t_k\beta_k})(\alpha)}{l_{X_0}(\alpha)}-\frac{i(\beta_k,\alpha)}{l_{X_0}(\alpha)}\Big|\leq\frac{1}{t_k}\tag{3}$$

for all $\alpha\in\mathcal{S}$.

Since $\frac{1}{t_k}\mathcal{X}([f_k])\to\sigma$, the above inequality implies that

$$\left|\frac{i(\beta_k,\alpha)}{l_{X_0}(\alpha)}-\frac{\sigma(\alpha)}{l_{X_0}(\alpha)}\right|\to 0$$

as $k\to\infty$ uniformly in $\alpha\in\mathcal{S}$. Define

$$\|\beta\|_{ls}=\sup_{\alpha\in\mathcal{S}}\frac{i(\beta,\alpha)}{l_{X_0}(\alpha)}$$

for any $\beta\in ML_{bdd}(X_0)$. The above convergence gives

$$\sup_{k\in\mathbb{N}}\|\beta_k\|_{ls}=N<\infty.$$

We use the assumption that X_0 has a bounded geodesic pants decomposition in order to prove that $\|\beta_k\|_{Th}$ is bounded in k. Indeed, let $\{\alpha_n\}_{n\in\mathbb{N}}$ be (non-cusp) cuffs of the geodesic pants decomposition of X_0. Then there exists $M>1$ with

$$\frac{1}{M}\leq l_{X_0}(\alpha_n)\leq M$$

for all $n\in\mathbb{N}$. Let P^i be a geodesic pair of pants in the above decomposition with the cuffs α_{i_j}, for $j=1,2,3$. Assume first that α_{i_j}, for $j=1,2,3$, are different geodesics of X_0. Denote by P_{i_j}, $j=1,2,3$, adjacent pairs of pants to P^i with common cuff α_{i_j}. Then there exists a simple closed geodesic $\alpha^*_{i_j}$ in $P_{i_j}\cup P^i$ that intersects α_{i_j} in two points such that $l_{X_0}(\alpha^*_{i_j})$ is bounded from above and below by positive constants depending only on $M>0$. The components of $P^i-\bigcup_{j=1}^3(\alpha_{i_j}\cup\alpha^*_{i_j})$ are simply connected for each i (cf. Figure 2). If two of α_{i_j}, for $j=1,2,3$, is the same geodesic, then a similar construction yields $\alpha^*_{i_j}$ such that components of $P_i-\bigcup_{j=1}^3(\alpha_{i_j}\cup\alpha^*_{i_j})$ are simply connected and $l_{X_0}(\alpha^*_{i_j})$ is bounded in terms of M.

The above convergence of β_k to σ and boundedness of the lengths of α_{i_j} and $\alpha^*_{i_j}$ on X_0 imply that

$$i(\beta_k,\alpha_{i_j}),i(\beta_k,\alpha^*_{i_j})<C(M)$$

for some constant $C=C(M)$ and for all $i,k\in\mathbb{N}$ and $j=1,2,3$. Since $X_0-\bigcup_i\bigcup_{j=1}^3\{\alpha_{i_j},\alpha^*_{i_j}\}$ has simply connected and uniformly bounded components (that are polygons with at most six sides) whose boundaries are subarcs of $\alpha_{i_j},\alpha^*_{i_j}$, we conclude that the supremum over all k and over all above components of the β_k-mass of the geodesics intersecting components is finite. Since each geodesic arc of length

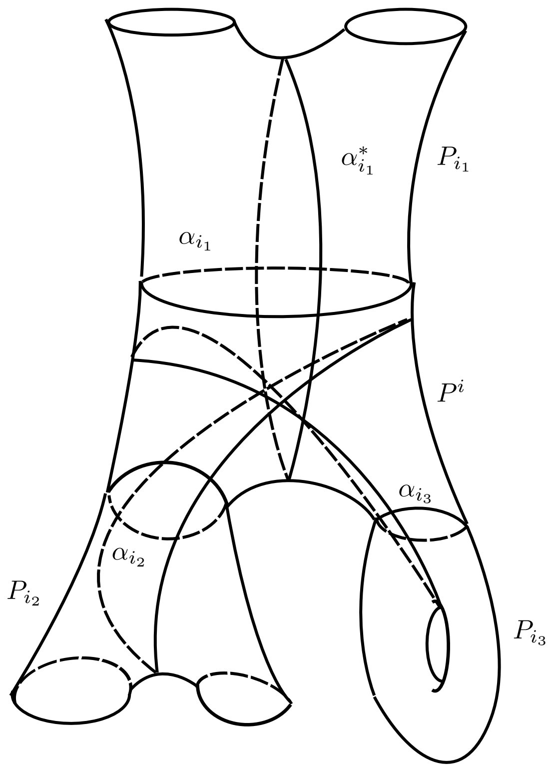

FIGURE 2. Decomposition of X_0 into bounded polygons.

1 on X_0 can intersect at most finitely many components of $X_0 - \bigcup_i \bigcup_{j=1}^3 \{\alpha_{i_j}, \alpha^*_{i_j}\}$, it follows that $\sup_{k\in\mathbb{N}} \|\beta_k\|_{Th} < \infty$.

By $\sup_{k\in\mathbb{N}} \|\beta_k\|_{Th} < \infty$, there exists a subsequence β_{k_j} and $\beta^* \in ML_{bdd}(X_0)$ such that $\beta_{k_j} \to \beta^*$ as $j \to \infty$ in the weak* topology. (The weak* topology is described in terms of the lifts of the measured laminations β_k to the universal covering $\mathbb{H}$.) Then

$$\sigma(\alpha) = \beta^*(\alpha)$$

for all $\alpha \in \mathcal{S}$ and

$$\|\beta^*\|_{Th} < \infty.$$

Thus any point in the length spectrum Thurston's boundary is in $PML_{bdd}(X_0)$. The above proposition gives that all points in $PML_{bdd}(X_0)$ are also in the length spectrum Thurston's boundary for $T(X_0)$. □

5. Infinite hyperbolic surfaces with upper-bounded geodesic pants decompositions

Let X_0 be a fixed, borderless, geodesically complete hyperbolic surface equipped with an upper-bounded geodesic pants decomposition. Namely, if $\{\alpha_n\}_{n\in\mathbb{N}}$ are cuffs (that are not cusps) of the geodesic pants decomposition, then

$$\sup_n l_{X_0}(\alpha_n) = M < \infty.$$

In addition, we assume that there exists a subsequence $\{\alpha_{n_j}\}_j$ with $l_{X_0}(\alpha_{n_j}) > 0$ and $l_{X_0}(\alpha_{n_j}) \to 0$ as $j \to \infty$. Let P_n^1 and P_n^2 be the geodesic pairs of pants in $\mathcal{P}$ with a common cuff α_n (possibly $P_n^1 = P_n^2$). Let γ_n be a shortest closed geodesic in $P_n^1 \cup P_n^2$ that intersects α_n in either one point (when $P_n^1 = P_n^2$) or in two points (when $P_n^1 \neq P_n^2$). We have that (cf. [1])

$$\frac{l_{X_0}(\gamma_n)}{\max\{1, |\log l_{X_0}(\alpha_n)|\}} = O(1),$$

where $O(1)$ is a function pinched between two positive constants.

Proposition 5.1. *Let X_0 be a geodesically complete infinite area hyperbolic surface with an upper-bounded geodesic pants decomposition with (non-cusp) cuffs $\{\alpha_n\}_{n\in\mathbb{N}}$ such that a subsequence of cuffs α_{n_j} have lengths going to zero. Then the length spectrum Thurston's boundary of $T(X_0)$ is strictly larger than $PML_{bdd}(X_0)$.*

Proof. We use the description of the closure of $T(X_0)$ in the Fenchel-Nielsen coordinates for the pants decomposition with (non-cusp) cuffs $\{\alpha_n\}_{n\in\mathbb{N}}$. Namely, a marked surface $f : X_0 \to X$ is in $T(X_0)$ if and only if the corresponding Fenchel-Nielsen coordinates $\{(\frac{l_X(\alpha_n)}{l_{X_0}(\alpha_n)}, t_X(\alpha_n))\}_{n\in\mathbb{N}}$ are uniformly bounded; $f : X_0 \to X$ is in the closure of $T(X_0)$ if and only if $\{\frac{l_X(f(\alpha_n))}{l_{X_0}(\alpha_n)}\}_n$ is bounded and $|t_X(\alpha_n)| = o(\max\{1, |\log l_{X_0}(\alpha_n)|\})$ for all n (cf. [23]).

Define a measured lamination $\mu = \sum_j w_j \alpha_{n_j}$ for some $w_j = o(|\log l_{X_0}(\alpha_{n_j})|)$ with $w_j \to \infty$ as $j \to \infty$. Then μ is not Thurston bounded and $E^{t\mu}(X_0) = X^t$ is in the closure of $T(X_0)$ for the length spectrum metric (cf. [23]). The proof of Proposition 3.4 extends to μ to get $\frac{1}{t}\mathcal{X}(X^t) \to \mu$ as $t \to \infty$ in the normalised supremum norm. Since each X^t is a limit of points in $T(X_0)$, it follows that μ is in Thurston's boundary and the proof is completed. $\square$

Theorem 1 from the introduction is established by Propositions 3.4, 4.1 and 5.1.

6. Two infinite surfaces with unbounded geodesic pants decompositions

The first surface X_1 that we consider was introduced by Kinjo [17]. Let Γ' be the hyperbolic triangle group of signature $(2, 4, 8)$. Let T' be the triangle fundamental polygon for Γ' with angles $\pi/2$, $\pi/4$ and $\pi/8$. Then $\Gamma'(T')$ tiles the hyperbolic plane $\mathbb{H}$. Let T be the union of T' and $\gamma_0'(T')$, where $\gamma_0' \in \Gamma'$ is a reflection in the geodesic containing the side of T' which subtends the angles $\pi/2$ and $\pi/8$ of T'. Denote the vertices of T by a, b and c; the vertex b is where T' has angle $\pi/8$ (cf. [17, Figure 2]). We choose three points a', b' and c' close to a, b and c, respectively, in the interior of the triangle T such that b' is on the side of T' containing b. The surface X_1 is obtained by puncturing the hyperbolic plane at the points $\Gamma'\{a', b', c'\}$ (cf. [17, Figures 2, 3]). Kinjo [17] proved that the length spectrum distance induces the same topology on $T(X_0)$ as the Teichmüller distance.

Let $\{\gamma_i\}_{i=1,\dots,8}$ be the elements of Γ' that fix a. Let l_a be the simple closed geodesic which separates the eight points $\{\gamma_i(a')\}_{i=1,\dots,8}$ from the other punctures of X_1. We similarly define curves l_b and l_c, and then extend the definition using Γ' to all other groups of eight cusps. The lengths of all $\Gamma'(l_a)$ are the same, as well as the lengths of all $\Gamma'(l_b)$, as well as the lengths of all $\Gamma'(l_c)$.

For the triangle T, we denote by $l_{a',b'}$ the simple closed geodesic which is homotopic to a simple closed curve in T that separates a', b' from c'. We similarly extend the definition to $l_{b',c'}$ and $l_{c',a'}$, and then extend it to all triangles using the invariance under Γ'. Note that the lengths of $\Gamma'(l_{a',b'})$ are the same, as well as the lengths of all $\Gamma'(l_{b',c'})$, and the lengths of all $\Gamma'(l_{c',a'})$.

The lengths of the family of geodesics $\Gamma'(l_a)\cup\Gamma'(l_b)\cup\Gamma'(l_c)\cup\Gamma'(l_{a',b'})\cup\Gamma'(l_{b',c'})\cup\Gamma'(l_{c',a'})$ are bounded from below and from above, and this family separates the surface X_1 into finite bounded polygons with a uniformly bounded number of sides. Then the proof of Proposition 5.1 extends to show that the length spectrum Thurston's boundary coincides with $PML_{bdd}(X_1)$.

Denote by X_2 an infinite hyperbolic surface defined by Shiga [26] that has geodesic pants decomposition with cuff lengths converging to infinity. The surface X_2 contains a sequence γ_n of simple closed geodesics with $l_{X_2}(\gamma_n)\to\infty$ as $n\to\infty$ such that for each closed geodesic δ we have

$$l_{X_2}(\delta) \geq \sum_{k=1}^{\infty} k l_{X_2}(\gamma_k) i(\gamma_k,\delta), \tag{4}$$

where only finitely many terms are non-zero. Shiga [26] proved that a sequence of full Dehn twists f_n around the curve γ_n diverges in the Teichmüller metric and it converges to the identity in the length spectrum distance. Thus the two metrics produce different topologies on $T(X_2)$.

We define β_n to be a measured lamination whose support is $\{\gamma_k\}_{k=1,\dots,n}$ such that, for $k=1,\dots,n$,

$$\beta_n|_{\gamma_k} = l_{X_2}(\gamma_k).$$

The projective class $[\beta_n]$ is in $PML_{bdd}(X_2)$. Define β_* to be a measured lamination on X_2 whose support is $\{\gamma_k\}_{k=1}^{\infty}$ such that, for all $k=1,2,\dots,$

$$\beta_*|_{\gamma_k} = l_{X_2}(\gamma_k).$$

It is clear that the projective class $[\beta_*]$ is not in $PML_{bdd}(X_2)$.

We prove that $[\beta_n]\to[\beta_*]$ as $n\to\infty$ in the normalized supremum norm. Indeed, let δ be a simple closed geodesic in X_2. Then

$$\frac{|i(\beta_n,\delta)-i(\beta_*,\delta)|}{l_{X_2}(\delta)} = \sum_{k=n+1}^{\infty}\frac{i(\beta_k,\delta)}{l_{X_2}(\delta)} = \frac{\sum_{k=n+1}^{\infty} i(\delta,\gamma_k) l_{X_2}(\gamma_k)}{\sum_{k=1}^{\infty} k i(\delta,\gamma_k) l_{X_2}(\gamma_k)} \leq \frac{1}{n+1},$$

and $[\beta_*]$ is in the length spectrum Thurston's boundary of $T(X_2)$. Therefore the boundary is larger than $PML_{bdd}(X_2)$.

7. Proof of Theorem 2

Let $[h]\in Pl^{\infty}_{X_0}$ be a point in the length spectrum Thurston's closure of $T(X_0)$. Let $[f_n]\in T(X_0)$ be a sequence which converges to $[h]$. Thus there exists $t_n\to\infty$ as $n\to\infty$ such that

$$\sup_{\alpha\in\mathcal{S}}\left|\frac{1}{t_n}\frac{l_{f_n(X_0)}(f_n(\alpha))}{l_{X_0}(\alpha)} - \frac{h(\alpha)}{l_{X_0}(\alpha)}\right| \to 0$$

as $n\to\infty$.

Let $g \in MCG_{qc}(X_0)$. We need to prove that $[f_n \circ g^{-1}] \to [h \circ g^{-1}]$ as $n \to \infty$. Note that

$$\begin{aligned}&\sup_{\alpha\in\mathcal{S}}\left|\frac{1}{t_n}\frac{l_{f_n(g^{-1}(X_0))}(f_n(g^{-1}(\alpha)))}{l_{X_0}(\alpha)}-\frac{h(g^{-1}(\alpha))}{l_{X_0}(\alpha)}\right|\\ &\leq\ K\sup_{g^{-1}(\alpha)\in\mathcal{S}}\left|\frac{1}{t_n}\frac{l_{f_n((X_0))}(f_n(g^{-1}(\alpha)))}{l_{X_0}(g^{-1}(\alpha))}-\frac{h(g^{-1}(\alpha))}{l_{X_0}(g^{-1}(\alpha))}\right|\end{aligned}$$

where g is a K-quasiconformal map, because $\frac{l_{X_0}(g^{-1}(\alpha))}{l_{X_0}(\alpha)} \leq K$, $g^{-1}(X_0) = X_0$ and g^{-1} is a bijection between homotopy classes of closed curves on X_0. The theorem follows by letting $n \to \infty$ in the above inequality.

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