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GHOST CLASSES IN THE COHOMOLOGY OF THE SHIMURA VARIETY ASSOCIATED TO GSp_4

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ABSTRACT. In this paper we study the existence of ghost classes in the cohomology of the Shimura variety associated to the group of symplectic similitudes GSp_4. The existence of ghost classes for the trivial coefficient system is known. We show that ghost classes only exist for the trivial coefficient system and they lie in the cohomology group in degree 2. Moreover we prove that the weight of the mixed Hodge structure associated to the space of ghost classes is the middle weight.

1. NOTATION

In this paper we work with Shimura varieties and we use the notation and definitions in [13]. We study the cohomology spaces with respect to some local systems underlying certain (complex) variations of Hodge structure (see [16] for this notion).

If G is an algebraic group, $\pi_0(G(\mathbb{R}))$ denotes the group of connected components of the Lie group $G(\mathbb{R})$. We use Kostant's theorem (see [12]) and, in this context, we denote by $\mathcal{W}^P$ the set of Weyl representatives for a standard $\mathbb{Q}$-parabolic subgroup P of G (this set depends on the choice of a maximal torus and a system of positive roots).

We denote by $\mathbb{A}$ the topological ring of adeles associated to $\mathbb{Q}$, and $\mathbb{A}_f$ denotes the topological subring of finite adeles.

2. INTRODUCTION

In this paper we study the existence of ghost classes in the cohomology of the Shimura variety associated to GSp_4. Ghost classes were first considered by A. Borel (see [1]) and have been treated by many mathematicians such as G. Harder, J. Schwermer, J. Franke, and C. Moeglin. One question that could arise from [6] is whether the only possible weight in the mixed Hodge structure in the space of ghost classes is the middle weight (we call this property the middle weight property). This paper presents a method to study the existence of ghost classes in Shimura varieties of $\mathbb{Q}$-rank 2 and provides an example in which the middle weight property is satisfied.

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In this introduction we present the definition of ghost classes for GSp_4 and the main results of this paper.

If (G, X) is a Shimura pair and (ρ, V) is an irreducible representation of G, then for every open compact subgroup $K_f \subset G(\mathbb{A}_f)$, V induces a local system $\tilde{V}$ on the corresponding level variety $S_K = Sh_K(G, X)$. Moreover $\tilde{V}$ is a variation of Hodge structure on S_K whose weight, denoted by $wt(V)$, depends on V.

We fix a maximal torus $T \subset G$ and a maximal $\mathbb{Q}$-split torus $\tilde{T} \subset G$ such that $\tilde{T} \subset T$. We choose a system of positive roots in the root systems $\Phi(G, T)$ and $\Phi(G, \tilde{T})$ associated to T and $\tilde{T}$ respectively, so that they are compatible. Let $\mathcal{P}_{\mathbb{Q}}(G)$ be the corresponding set of proper standard $\mathbb{Q}$-parabolic subgroups.

Let $\overline{S}_K$ be the Borel-Serre compactification of S_K (see [2]) and let $\partial\overline{S}_K$ be its boundary. Then the inclusion $S_K \hookrightarrow \overline{S}_K$ is a homotopy equivalence. In particular we can extend $\tilde{V}$ to a local system on $\overline{S}_K$ (also denoted by $\tilde{V}$) and we have an isomorphism $H^*(S_K, \tilde{V}) \cong H^*(\overline{S}_K, \tilde{V})$. From all of these facts we obtain a long exact sequence in cohomology

$$\ldots \to H^q_c(S_K, \tilde{V}) \to H^q(S_K, \tilde{V}) \to H^q(\partial\overline{S}_K, \tilde{V}) \to \ldots.$$

We assume, from now on, that G has semisimple $\mathbb{Q}$-rank 2. $\mathcal{P}_{\mathbb{Q}}(G)$ consists of three elements: two maximal $\mathbb{Q}$-parabolic subgroups P_1, P_2 and one minimal $\mathbb{Q}$-parabolic subgroup P_0. There is a covering $\partial\overline{S}_K = \bigcup_{P \in \mathcal{P}_{\mathbb{Q}}(G)} \partial_{P,K}$ indexed by $\mathcal{P}_{\mathbb{Q}}(G)$, and this covering defines a long exact sequence in cohomology

$$\ldots \to H^q(\partial\overline{S}_K, \tilde{V}) \to H^q(\partial_{P_1,K}, \tilde{V}) \oplus H^q(\partial_{P_2,K}, \tilde{V}) \to H^q(\partial_{P_0,K}, \tilde{V}) \to \ldots.$$

If we have open compact subgroups $K'_f \subset K_f$, then there is a canonical covering $S_{K'} \to S_K$ and a corresponding morphism $H^*(S_K, \tilde{V}) \to H^*(S_{K'}, \tilde{V})$. We can take direct limit to obtain the space

$$H^*(S, \tilde{V}) = \varinjlim_{K} H^*(S_K, \tilde{V}).$$

We consider the spaces $H^*(\partial\overline{S}, \tilde{V}), H^*_c(S, \tilde{V}), H^*(\partial_{P_i}, \tilde{V})$ defined by the corresponding direct limits and we have the long exact sequences in cohomology

$$\ldots \to H^q_c(S, \tilde{V}) \to H^q(S, \tilde{V}) \to H^q(\partial\overline{S}, \tilde{V}) \to \ldots \tag{2.1}$$

and

$$\ldots \to H^q(\partial\overline{S}, \tilde{V}) \to H^q(\partial_{P_1}, \tilde{V}) \oplus H^q(\partial_{P_2}, \tilde{V}) \to H^q(\partial_{P_0}, \tilde{V}) \to \ldots. \tag{2.2}$$

We denote by $r^* : H^*(S, \tilde{V}) \to H^*(\partial\overline{S}, \tilde{V})$ and $r^*_i : H^*(\partial\overline{S}, \tilde{V}) \to H^*(\partial_{P_i}, \tilde{V})$ (for $i \in \{0, 1, 2\}$) the canonical maps (defined by the corresponding inclusions).

The space of ghost classes $Gh^*(V)$ is the subspace of $H^*(\partial\overline{S}, \tilde{V})$ given by the intersection of the image of r^* with the kernel of each morphism r^*_i for $i \in \{0, 1, 2\}$. In this case, the space $Gh^*(V)$ can be described as the intersection of the image of the map r^* with the kernels of r^*_1 and r^*_2.

By Saito's theory of mixed Hodge modules and by the results in [9], each term in (2.1) and (2.2) is endowed with a mixed Hodge structure. Moreover, (2.1) and (2.2) are long exact sequences of mixed Hodge structure. Thus the space of ghost classes has an induced mixed Hodge structure. By using information on the cohomology spaces appearing in the aforementioned long exact sequences we can obtain information on the possible weights in the mixed Hodge structure on the spaces of ghost classes.

In this paper, the possible weights in $Gh^q(V)$ are calculated by studying the morphisms $r^q : H^q(S,\tilde{V}) \to H^q(\partial\overline{S},\tilde{V})$, $r^{q-1}_{1,0} : H^{q-1}(\partial_{P_1},\tilde{V}) \to H^{q-1}(\partial_{P_0},\tilde{V})$ and $r^{q-1}_{2,0} : H^{q-1}(\partial_{P_2},\tilde{V}) \to H^{q-1}(\partial_{P_0},\tilde{V})$. By using considerations on the weights, we obtain information about r^q and $r^{q-1}_{1,0}$. On the other hand, these arguments do not give much information about $r^{q-1}_{2,0}$, and we use the results in [5] to study this morphism.

One knows that the weights in the mixed Hodge structure associated to $H^q(S,\tilde{V})$ are greater than or equal to $q + wt(V)$, where $wt(V)$ is the weight of the variation of (complex) Hodge structure defined by $\tilde{V}$, while the weights in $H^{q+1}_c(S,\tilde{V})$ are less than or equal to $(q+1) + wt(V)$. We call $q + wt(V)$ the middle weight and we say that V satisfies the middle weight property if the only possible weight in the space of ghost classes is the middle weight.

We have a decomposition of the cohomology space of each face of the boundary of the Borel-Serre compactification of the form

$$H^q(\partial_P,\tilde{V}) = \bigoplus_{w\in\mathcal{W}^P} Ind^{G(\mathbb{A}_f)\times\pi_0(G(\mathbb{R}))}_{P(\mathbb{A}_f)\times\pi_0(P(\mathbb{R}))} H^{q-\ell(w)}(S^{M_P},\tilde{W}_{w_*(\lambda)}) \tag{2.3}$$

(compare [5] or [15]) and is obtained by using, among other things, Kostant's theorem [12]. In this decomposition $\ell(w)$ denotes the length of the element w, λ denotes the highest weight of the irreducible representation V, $W_{w_*(\lambda)}$ is the irreducible representation of the Levi subgroup M_P of P with highest weight $w_*(\lambda) = w(\lambda+\delta)-\delta$ (where δ is, as usual, half the sum of the positive roots). This decomposition is very useful for a better understanding of the mixed Hodge structures attached to the cohomology spaces $H^q(\partial_P,\tilde{V})$, as we can see in (5.5.6) of [9].

For each maximal standard $\mathbb{Q}$-parabolic subgroup P we have a decomposition of its Levi subgroup M_P into its hermitian part $G_{h,P}$ and its linear part $G_{l,P}$, and $G_{h,P}$ forms part of a Shimura datum $(G_{h,P}, h_P)$. Then the mixed Hodge structure on each direct summand $H^{q-\ell(w)}(S^{M_P},\tilde{W}_{w_*(\lambda)})$ is determined by the Shimura datum $(G_{h,P}, h_P)$ and the irreducible representation $W_{w_*(\lambda)}$ of M_P. For a non-maximal standard $\mathbb{Q}$-parabolic subgroup Q we have an associated maximal standard $\mathbb{Q}$-parabolic subgroup P (see for example [9]), and the mixed Hodge structure on $H^{q-\ell(w)}(S^{M_Q},\tilde{W}_{w_*(\lambda)})$ is determined by $(G_{h,P}, h_P)$ and the irreducible representation $W_{w_*(\lambda)}$ of M_Q.

To determine the Shimura datum $(G_{h,P}, h_P)$ we are using the definitions and results in the first pages in [8].

We now give a short summary of the contents and main results of this paper.

We work on the Shimura variety associated to the group of symplectic similitudes GSp_4. This case has been partially studied in some papers (see [11], [17], [7]). We are giving a complete treatment of this case and we calculate the weight in the space of ghost classes, proving that the middle weight property is satisfied.

In section 3, the irreducible algebraic representations of GSp_4 are parametrized by the highest weights which are given by expressions of the form $\lambda = m_1\lambda'_1 + m_2\lambda'_2 + c\kappa$, with c an integer and m_1, m_2 non-negative integers. In this setting we prove (see Theorem 3.1) that unless $m_1 = m_2 = 0$ there are no ghost classes. The existence of ghost classes in the case $m_1 = m_2 = 0$, in the degree 2 cohomology, is proved in [11] (and can also be deduced from [4]). In this paper we supplement this result by proving that the only ghost classes belong to degree 2 cohomology. Moreover, we show that the weight in the corresponding mixed Hodge structure is the middle weight (in particular we prove that in the cases $m_1 = m_2 = 0$ the corresponding local system satisfies the middle weight property).

It should be noted that when the highest weight of the irreducible representation is regular, the non-existence of ghost classes can be obtained using Theorem 4.11 in [14] and Theorem 19 in [3].

3. Ghost classes: The case GSp_4

3.1. **The Shimura variety involved.** In this section we consider the Shimura datum given by the pair (GSp_4, X), where GSp_4 is the group of symplectic similitudes. In other words,

$$GSp_4(A) = \left\{g \in GL_4(A) \mid g^t J_2 g = \nu(g) J_2, \nu(g) \in A^\times\right\} \text{ for every } \mathbb{Q}\text{-algebra } A$$

where

$$J_2 = \begin{bmatrix} 0 & S \\ -S & 0 \end{bmatrix} \text{ with } S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and X is the $GSp_4(\mathbb{R})$-conjugacy class of homomorphisms containing the element $h : \mathbb{S}(\mathbb{R}) \to GSp_4(\mathbb{R})$ given by

$$h(x+iy) = \begin{bmatrix} xI_2 & yS \\ -yS & xI_2 \end{bmatrix} \quad \forall (x+iy) \in \mathbb{S}(\mathbb{R})$$

where I_2 denotes the 2×2 identity matrix. Thus, the weight morphism $\omega_X : \mathbb{G}_m \to GSp_4$ is given by $\omega_X(t) = tI_4$ where I_4 denotes the 4×4 identity.

Let $K_f \subset GSp_4(\mathbb{A}_f)$ be an open compact subgroup. Then we denote by S_K its corresponding level variety and we denote by $S = \lim_{\overleftarrow{K}} S_K$ the Shimura variety defined by this Shimura datum.

3.2. **Maximal torus and root system.** Let H be the maximal torus on $GSp_4(\mathbb{C})$ given by the group of diagonal matrices $\left\{diag(hh_1, hh_2, h_2^{-1}, h_1^{-1}) \mid h_1, h_2, h \in \mathbb{C}^*\right\} \subset GSp_4(\mathbb{C})$. H defines a root system, and, as H is a $\mathbb{Q}$-split torus, this root system is also a $\mathbb{Q}$-root system for GSp_4.

Let $\mathfrak{h} = Lie(H)$ be the complex Lie algebra corresponding to this maximal torus. The root system $\Phi(\mathfrak{gsp}_{4,\mathbb{C}}, \mathfrak{h})$ is of type C_2.

Let $\epsilon_1, \epsilon_2, \epsilon \in \mathfrak{h}^*$ be defined as $\epsilon_1(X) = h_1$, $\epsilon_2(X) = h_2$ and $\epsilon(X) = h$ for $X = diag(h+h_1, h+h_2, -h_2, -h_1) \in \mathfrak{h}$.

Then the root system $\Phi(\mathfrak{gsp}_{4,\mathbb{C}}, \mathfrak{h})$ is given by

$$\{\epsilon+\epsilon_1+\epsilon_2, \epsilon_1-\epsilon_2, \epsilon+2\epsilon_1, \epsilon+2\epsilon_2, -\epsilon-\epsilon_1-\epsilon_2, -\epsilon_1+\epsilon_2, -\epsilon-2\epsilon_1, -\epsilon-2\epsilon_2\}.$$

We can take as positive roots $\{\epsilon+\epsilon_1+\epsilon_2, \epsilon_1-\epsilon_2, \epsilon+2\epsilon_1, \epsilon+2\epsilon_2\}$, and then the system of simple roots is $\Delta = \{\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon + 2\epsilon_2\}$.

3.3. **Standard $\mathbb{Q}$-parabolic subgroups.** The next step is to describe the standard $\mathbb{Q}$-parabolic subgroups of GSp_4 with respect to the given $\mathbb{Q}$-root system and system of positive roots.

As Δ has just two elements, we have three proper standard $\mathbb{Q}$-parabolic subgroups, one minimal and two maximal ones. The minimal parabolic is given by

$$P_0 = T_4 \cap GSp_4(\mathbb{C})$$

where T_4 denotes the subgroup of upper triangular matrices in $GL(4, \mathbb{C})$.

On the other hand, for the maximal ones we have

$$P_1 = \left\{ \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \in GL(4,\mathbb{C}) \right\} \cap GSp_4(\mathbb{C})$$

and

$$P_2 = \left\{ \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \in GL(4,\mathbb{C}) \right\} \cap GSp_4(\mathbb{C}).$$

We denote by $\overline{S}_K$ the Borel-Serre compactification of S_K and by $\partial\overline{S}_K$ its boundary. One knows that $\partial\overline{S}_K$ is the union of faces, one for each standard $\mathbb{Q}$-parabolic subgroup; then we denote each one of these faces by $\partial_{K,0}, \partial_{K,1}$ and $\partial_{K,2}$ respectively. We use the notation $\overline{S}, \partial\overline{S}, \partial_0, \partial_1$ and ∂_2 for the analogous objects at the infinite level.

3.4. **The irreducible representation.** In order to determine a notation for the finite dimensional irreducible representations of $GSp_{4,\mathbb{C}}$, we will consider the irreducible representations for $Sp_{4,\mathbb{C}}$.

$H' = H \cap Sp_4(\mathbb{C})$ is a Cartan subgroup of $Sp_4(\mathbb{C})$ and can be described as

$$H' = \left\{ diag(x_1, x_2, x_2^{-1}, x_1^{-1}) \mid x_1, x_2 \in \mathbb{C}^\times \right\} \subset Sp_4(\mathbb{C}).$$

H' has Lie algebra

$$\mathfrak{h}' = \{ diag(h_1, h_2, -h_2, -h_1) \mid h_1, h_2 \in \mathbb{C} \} \subset \mathfrak{sp}_4(\mathbb{C}).$$

We define the linear functionals $\varepsilon_1, \varepsilon_2 : \mathfrak{h}' \to \mathbb{C}$ by

$$\varepsilon_1(diag(h_1, h_2, -h_2, -h_1)) = h_1 \text{ and } \varepsilon_2(diag(h_1, h_2, -h_2, -h_1)) = h_2.$$

The associated root system $\Phi(\mathfrak{sp}_{4,\mathbb{C}}, \mathfrak{h}')$ (with the usual system of positive roots) has fundamental weights $\lambda_1, \lambda_2 : \mathfrak{h}' \to \mathbb{C}$ given by $\lambda_1 = \varepsilon_1$ and $\lambda_2 = \varepsilon_1 + \varepsilon_2$. Thus the irreducible finite dimensional representations of $Sp_{4,\mathbb{C}}$ are determined by their highest weights, which in this case are the linear functionals of the form $m_1\lambda_1 + m_2\lambda_2$ with m_1, m_2 non-negative integers.

The Lie algebra $\mathfrak{h}$ of H can be described as the direct sum $\mathfrak{h} = \mathfrak{z} \oplus \mathfrak{h}'$. We extend $\varepsilon_1, \varepsilon_2, \lambda_1$ and λ_2 by zero in the previous decomposition, obtaining linear functionals on $\mathfrak{h}$ denoted by $\varepsilon_1', \varepsilon_2', \lambda_1'$ and λ_2' respectively. We denote by κ the linear functional on $\mathfrak{h}$ given by the projection to the first component in the aforementioned decomposition.

One can see that an irreducible finite dimensional representation of $GSp_{4,\mathbb{C}}$ is determined by a linear functional of the form $m_1\lambda_1' + m_2\lambda_2' + c\kappa$ where m_1, m_2 are non-negative integers and c is an integer congruent to $(m_1 + 2m_2)$ module 2.

We fix an irreducible algebraic representation (ρ, V_λ) of GSp_4 with highest weight $\lambda = m_1\lambda_1' + m_2\lambda_2' + c\kappa$.

3.5. **The decomposition for each $H^q(\partial_i, \tilde{V}_\lambda)$.** Our first task is to understand the weights of the mixed Hodge structure in

$$H^{q-1}(\partial_0, \tilde{V}_\lambda)/i(H^{q-1}(\partial_1, \tilde{V}_\lambda) \oplus H^{q-1}(\partial_2, \tilde{V}_\lambda))$$

where $i : H^{q-1}(\partial_1, \tilde{V}_\lambda) \oplus H^{q-1}(\partial_2, \tilde{V}_\lambda) \to H^{q-1}(\partial_0, \tilde{V}_\lambda)$ is the morphism in the long exact sequence (2.2).

In order to apply the decompositions (2.3) we need to understand the Weyl group $\mathcal{W}$. Let $w_{\alpha_1}, w_{\alpha_2} \in \mathcal{W}$ be the simple reflections associated to α_1 and α_2 respectively. Table 1 gives a description of each element of $\mathcal{W}$.

TABLE 1. The Weyl group of GSp_4

w	$w^{-1}(\alpha_1)$	$w^{-1}(\alpha_2)$	$\ell(w)$	$w_*(\lambda) = w(\lambda+\delta)-\delta$
$w_0 = 1$	$\varepsilon_1' - \varepsilon_2'$	$2\varepsilon_2'$	0	(m_1+m_2, m_2)
$w_1 = w_{\alpha_1}$	$-\varepsilon_1' + \varepsilon_2'$	$2\varepsilon_1'$	1	(m_2-1, m_1+m_2+1)
$w_2 = w_{\alpha_2}$	$\varepsilon_1' + \varepsilon_2'$	$-2\varepsilon_2'$	1	$(m_1+m_2, -m_2-2)$
$w_3 = w_{\alpha_1} \circ w_{\alpha_2}$	$-\varepsilon_1' - \varepsilon_2'$	$2\varepsilon_1'$	2	$(-m_2-3, m_1+m_2+1)$
$w_4 = w_{\alpha_2} \circ w_{\alpha_1}$	$\varepsilon_1' + \varepsilon_2'$	$-2\varepsilon_1'$	2	$(m_2-1, -m_1-m_2-3)$
$w_5 = w_{\alpha_1} \circ w_{\alpha_2} \circ w_{\alpha_1}$	$-\varepsilon_1' - \varepsilon_2'$	$2\varepsilon_2'$	3	$(-m_1-m_2-4, m_2)$
$w_6 = w_{\alpha_2} \circ w_{\alpha_1} \circ w_{\alpha_2}$	$\varepsilon_1' - \varepsilon_2'$	$-2\varepsilon_1'$	3	$(-m_2-3, -m_1-m_2-3)$
$w_7 = w_{\alpha_1} \circ w_{\alpha_2} \circ w_{\alpha_1} \circ w_{\alpha_2}$	$-\varepsilon_1' + \varepsilon_2'$	$-2\varepsilon_2'$	4	$(-m_1-m_2-4, -m_2-2)$

In the last column, each pair (a,b) denotes the elements $a\varepsilon_1' + b\varepsilon_2' + c\kappa$ and $\delta = 2\varepsilon_1' + \varepsilon_2'$.

Clearly $\mathcal{W}^{P_0} = \mathcal{W}$, and, by using Table 1, one can see that $\mathcal{W}^{P_1} = \{w_0, w_1, w_3, w_5\}$ and $\mathcal{W}^{P_2} = \{w_0, w_2, w_4, w_6\}$.

For each $K_f \subset GSp_4(\mathbb{A}_f)$ open compact subgroup, $H^k(S_K^{M_{P_0}}, \widetilde{W}_{w_*(\lambda)}) = 0$ for $k > 0$, thus

$$H^{q-1}(\partial_0, \tilde{V}_\lambda) = \bigoplus_{\substack{w \in \mathcal{W}^{P_0} \\ \ell(w) = q-1}} Ind_{P_0(\mathbb{A}^f) \times \pi_0(P_0(\mathbb{R}))}^{GSp_4(\mathbb{A}^f) \times \pi_0(GSp_4(\mathbb{R}))} H^0(S^{M_{P_0}}, \widetilde{W}_{w_*(\lambda)}).$$

3.6. **Weights.** In this subsection we determine, for each maximal standard $\mathbb{Q}$-parabolic subgroup P of GSp_4, the homomorphism $h_P : \mathbb{S} \to G_{P,h}$ where $G_{P,h}$ is the hermitian part of the Levi subgroup M_P of P and $(G_{P,h}, h_P)$ defines a Shimura datum. These calculations will give some information about the mixed Hodge structure on the cohomology spaces on the faces of the boundary of the Borel-Serre compactification.

For the case P_1 we follow the first pages of [8], in the order: first we calculate the subgroup A_{P_1} of the Levi subgroup M_{P_1}, then we calculate the admissible Cayley morphism $w_1 : \mathbb{G}_m \to A_{P_1}$ and finally we compute the morphism $h_{P_1} : \mathbb{S} \to G_{P_1,h}$.

One can see that A_{P_1}, which is defined to be the maximal $\mathbb{Q}$-split torus in the center of M_{P_1} times the center of $GSp_{4,\mathbb{C}}$, is given by

$$A_{P_1} = \left\{ \begin{bmatrix} h_1h_2 & & & \\ & h_1 & & \\ & & h_1 & \\ & & & h_1(h_2)^{-1} \end{bmatrix} \mid h_1, h_2 \in \mathbb{C}^* \right\}.$$

By the properties of the admissible Cayley morphism described in [8] we can determine $w_1 : \mathbb{G}_m \to A_{P_1}$. Indeed, let W_1 be the unipotent radical of P_1 and U_1 its center, and let $\mathfrak{w}_1$ and $\mathfrak{u}_1$ be their corresponding Lie algebras. Then denoting by $\mathfrak{g}_i$ the subspace of $\mathfrak{g}$ in which the action (by conjugation) of $w_1(t)$ is given by multiplication by t^i, we have that $\mathfrak{g}_{-2} = \mathfrak{u}_1$ and $\mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} = \mathfrak{w}_1$ (see [8]). Thus one can see that

$$w_1(t) = \begin{bmatrix} t^{k-2} & & & \\ & t^{k-1} & & \\ & & t^{k-1} & \\ & & & t^k \end{bmatrix} \text{ with } k \in \mathbb{Z}.$$

We apply the description on page 330 of [8] to calculate the morphism h_{P_1}, taking $V = \mathbb{R}^4$ and $\rho : GSp_4 \to GL_4$ to be the natural inclusion. We consider the canonical base $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{R}^4$. The filtration defined by the admissible Cayley morphism w_1 is given by $W_{k-2} = \mathbb{R}e_1, W_{k-1} = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3$ and $W_k = \mathbb{R}^4$. We take h the morphism given in subsection 3.1. Then we know (by another property of the admissible Cayley morphism; see [8, p. 327]) that the decreasing filtration defined by $\rho \circ h$, together with the increasing filtration $W_\bullet$, defines a mixed Hodge structure on V. The Hodge filtration defined by $\rho \circ h$ is given by

$$F_h^{-1} V_{\mathbb{C}} = V_{\mathbb{C}}, \quad F_h^0 V_{\mathbb{C}} = \mathbb{C}(e_1 - ie_4) \oplus \mathbb{C}(e_2 - ie_3) \text{ and } F_h^1 V_{\mathbb{C}} = 0.$$

The only possible value for k to obtain such a mixed Hodge structure is 0. Then

$$w_1(t) = \begin{bmatrix} t^{-2} & & & \\ & t^{-1} & & \\ & & t^{-1} & \\ & & & 1 \end{bmatrix},$$

and the mixed Hodge structure on V has types $(-1,-1), (-1,0), (0,-1)$ and $(0,0)$; moreover the morphism h_{P_1} is given by

$$h_{P_1}(z) = \begin{bmatrix} |z|^2 & & & \\ & Re(z) & Im(z) & \\ & -Im(z) & Re(z) & \\ & & & 1 \end{bmatrix} \quad \forall z \in \mathbb{S}(\mathbb{R}).$$

In particular the weight morphism of this Shimura datum $(G_{P_1,h}, h_{P_1})$ is

$$w_{P_1}(t) = \begin{bmatrix} t^2 & & & \\ & t & & \\ & & t & \\ & & & 1 \end{bmatrix}.$$

Thus, if $w_*(\lambda) = d_1\varepsilon'_1 + d_2\varepsilon'_2 + c\kappa$ for $w \in \mathcal{W}^{P_1}$, then the weight of the local system $(\widetilde{W_{w_*(\lambda)}})_h$ induced by $W_{w_*(\lambda)}$ on the Shimura variety associated to the hermitian part $G_{P_1,h}$ of M_{P_1} is $-c - d_1$.

We make the same calculations for the case P_2, obtaining that the weight morphism of the corresponding Shimura datum $(G_{P_2,h}, h_{P_2})$ is

$$w_{P_2}(t) = \begin{bmatrix} t^2 & & & \\ & t^2 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

As the hermitian part of M_{P_0} is that of M_{P_2}, we have that if $w_*(\lambda) = d_1\varepsilon'_1 + d_2\varepsilon'_2 + c\kappa$ for $w \in \mathcal{W}^{P_0}$, then the weight of the local system $(\widetilde{W_{w_*(\lambda)}})_h$ induced by $W_{w_*(\lambda)}$ on the Shimura variety associated to the hermitian part $G_{P_2,h}$ of M_{P_0} is $-c - d_1 - d_2$.

On the other hand we can see that the weight of the local system $\tilde{V}_\lambda$ defined by V_λ on the Shimura variety associated to GSp_4 is $-c$.

3.7. **Ghost classes.** We will study the subspace $i(H^{q-1}(\partial_1, \tilde{V}_\lambda) \oplus H^{q-1}(\partial_2, \tilde{V}_\lambda)) \subset H^{q-1}(\partial_0, \tilde{V}_\lambda)$.

We will abbreviate $Ind_P^{GSp_4} = Ind_{P(\mathbb{A}_f)\times\pi_0(P(\mathbb{R}))}^{GSp_4(\mathbb{A}_f)\times\pi_0(GSp_4(\mathbb{R}))}$ for every P standard parabolic $\mathbb{Q}$-subgroup of GSp_4.

Considering that

$$\begin{aligned}
H^0(\partial_0, \tilde{V}_\lambda) &= Ind_{P_0}^{GSp_4} H^0(S^{M_{P_0}}, \widetilde{W}_\lambda),\\
H^1(\partial_0, \tilde{V}_\lambda) &= Ind_{P_0}^{GSp_4}(H^0(S^{M_{P_0}}, \widetilde{W}_{(w_1)_*(\lambda)}) \oplus H^0(S^{M_{P_0}}, \widetilde{W}_{(w_2)_*(\lambda)})),\\
H^2(\partial_0, \tilde{V}_\lambda) &= Ind_{P_0}^{GSp_4}(H^0(S^{M_{P_0}}, \widetilde{W}_{(w_3)_*(\lambda)}) \oplus H^0(S^{M_{P_0}}, \widetilde{W}_{(w_4)_*(\lambda)})),\\
H^3(\partial_0, \tilde{V}_\lambda) &= Ind_{P_0}^{GSp_4}(H^0(S^{M_{P_0}}, \widetilde{W}_{(w_5)_*(\lambda)}) \oplus H^0(S^{M_{P_0}}, \widetilde{W}_{(w_6)_*(\lambda)})),\\
H^4(\partial_0, \tilde{V}_\lambda) &= Ind_{P_0}^{GSp_4} H^0(S^{M_{P_0}}, \widetilde{W}_{(w_7)_*(\lambda)}),
\end{aligned}$$

we analyze these eight spaces to study in which cases they could contribute to ghost classes and we arrive at the following result.

Theorem 3.1. *For the Shimura variety associated to GSp_4 and the local system defined by the irreducible finite dimensional representation with highest weight $\lambda = m_1\lambda'_1 + m_2\lambda'_2 + c\kappa$, we have the following results:*

- *If $m_1 > 0$ or $m_2 > 0$, then there is no ghost class in the cohomology of the boundary of the Borel-Serre compactification.*
- *If $m_1 = m_2 = 0$, then there exist ghost classes only in degree 2 cohomology, and their weight, in the corresponding mixed Hodge structure, is the middle weight.*

In particular, the middle weight property is satisfied in all cases.

Proof. For each non-negative integer q, we will denote by $r_{1,0}^q : H^q(\partial_1, \tilde{V}_\lambda) \to H^q(\partial_0, \tilde{V}_\lambda)$ and $r_{2,0}^q : H^q(\partial_2, \tilde{V}_\lambda) \to H^q(\partial_0, \tilde{V}_\lambda)$ the canonical morphisms.

Running over the elements $w \in \mathcal{W}^{P_0}$ we analyze whether or not the space $H^0(S^{M_{P_0}}, \widetilde{W}_{w_*(\lambda)})$ can possibly contribute to the space of ghost classes.

- *Case* $w = 1$: $Ind_{P_0}^{GSp_4} H^0(S^{M_{P_0}}, \widetilde{W}_\lambda) \subset H^0(\partial_0, \tilde{V}_\lambda)$. Thus in order to contribute to a ghost class the image of this space under the morphism

$$H^0(\partial_0, \tilde{V}_\lambda) \to H^1(\partial\overline{S}, \tilde{V}_\lambda)$$

  should have weight greater than or equal to the middle weight of $H^1(S, \tilde{V}_\lambda)$, which is $1-c$. But we can check that the weight of $Ind_{P_0}^{GSp_4} H^0(S^{M_{P_0}}, \widetilde{W}_\lambda)$ is $-c-m_1-2m_2$, so we conclude that $H^0(S^{M_{P_0}}, \widetilde{W}_\lambda)$ does not contribute to ghost classes.
- *Case* $w = w_1$: One can check by the same procedure that the subspace

$$Ind_{P_0}^{GSp_4} H^0(S^{M_{P_0}}, \widetilde{W}_{(w_1)_*(\lambda)}) \subset H^1(\partial_0, \tilde{V}_\lambda)$$

  does not contribute to ghost classes.
- *Case* $w = w_4$: M_{P_1} coincides with its hermitian part. The restriction of the morphism $r_{1,0}^2$ to the subspace $Ind_{P_1}^{GSp_4} H^1(S^{M_{P_1}}, \widetilde{W}_{(w_1)_*(\lambda)}) \subset H^2(\partial_1, \tilde{V}_\lambda)$ has image in the subspace $Ind_{P_0}^{GSp_4} H^0(S^{M_{P_0}}, \widetilde{W}_{(w_4)_*(\lambda)}) \subset H^2(\partial_0, \tilde{V}_\lambda)$, and as the

parabolic induction is exact we can study the image of this morphism by studying the morphism

$$H^1(S^{M_{P_1}}, \widetilde{W}_{(w_1)_*(\lambda)}) \to Ind_{P_0^1}^{M_{P_1}} H^0(S^{M_{P_0}}, \widetilde{W}_{(w_4)_*(\lambda)})$$

where P_0^1 denotes the $\mathbb{Q}$-parabolic subgroup $P_0 \cap M_{P_1}$ of M_{P_1}. In the long exact sequence

$$\dots \to H^1(S^{M_{P_1}}, \widetilde{W}_{(w_1)_*(\lambda)}) \to Ind_{P_0^1}^{M_{P_1}} H^0(S^{M_{P_0}}, \widetilde{W}_{(w_4)_*(\lambda)})$$

$$\to H_c^2(S^{M_{P_1}}, \widetilde{W}_{(w_1)_*(\lambda)}) \to \dots$$

the weights in $H_c^2(S^{M_{P_1}}, \widetilde{W}_{(w_1)_*(\lambda)})$ are $\leq 3 - c - m_2$, while the weight in the space $Ind_{P_0^1}^{M_{P_1}} H^0(S^{M_{P_0}}, \widetilde{W}_{(w_4)_*(\lambda)})$ is $4 - c + m_1$, so we conclude that the space $Ind_{P_0}^{GSp_4} H^0(S^{M_{P_0}}, \widetilde{W}_{(w_4)_*(\lambda)})$ is inside the image of $r_{1,0}^2$ and does not contribute to ghost classes.

- *Case* $w = w_6$: One can prove, by the same procedure as in the case $w = w_4$, that the corresponding space $Ind_{P_0}^{GSp_4} H^0(S^{M_{P_0}}, \widetilde{W}_{(w_6)_*(\lambda)}) \subset H^3(\partial_0, \tilde{V}_\lambda)$ does not contribute to ghost classes.
- *Case* $w = w_3$: We analyze the restriction of the morphism $r_{2,0}^2$ to the subspace $Ind_{P_2}^{GSp_4} H^1(S^{M_{P_2}}, \widetilde{W}_{(w_2)_*(\lambda)})$ by using the results in [5] and as in the previous items we just need to consider the morphism

$$H^1(S^{M_{P_2}}, \widetilde{W}_{(w_2)_*(\lambda)}) \to Ind_{P_0^2}^{M_{P_2}} H^0(S^{M_{P_0}}, \widetilde{W}_{(w_3)_*(\lambda)})$$

where P_0^2 denotes the $\mathbb{Q}$-parabolic subgroup $P_0 \cap M_{P_2}$ of M_{P_2}. In the context of Theorem 2 in [5], we are in the unbalanced case. More precisely, we are in the case (1) of the aforementioned theorem. As a conclusion, the space $Ind_{P_0}^{GSp_4} H^0(S^{M_{P_0}}, \widetilde{W}_{(w_3)_*(\lambda)})$ is in the image of $r_{2,0}^2$ and does not contribute to ghost classes.
- *Case* $w = w_5$: We can use the same arguments as in the previous item to prove that the space $Ind_{P_0}^{GSp_4} H^0(S^{M_{P_0}}, \widetilde{W}_{(w_5)_*(\lambda)})$ is in the image of $r_{2,0}^3$ and does not contribute to ghost classes.
- *Case* $w = w_7$: If $m_1 > 0$, then by using the same arguments as in the case $w = w_3$ we conclude that the space $Ind_{P_0}^{GSp_4} H^0(S^{M_{P_0}}, \widetilde{W}_{(w_7)_*(\lambda)})$ is in the image of $r_{2,0}^4$ and does not contribute to ghost classes.

  On the other hand, using the same arguments as in the case $w = w_4$ we conclude that unless $m_2 = 0$, the aforementioned space is in the image of $r_{1,0}^4$ and does not contribute to ghost classes.

  In conclusion the only case in which the space $Ind_{P_0}^{GSp_4} H^0(S^{M_{P_0}}, \widetilde{W}_{(w_7)_*(\lambda)})$ could contribute to ghost classes is when $m_1 = m_2 = 0$. Assume $m_1 = m_2 = 0$. First of all this space is mapped to the degree 5 cohomology space of the boundary of the Shimura variety, so in order to contribute to ghost classes we need its image in $H^5(\partial\overline{S}, \tilde{V}_\lambda)$ to have non-trivial intersection with the image of $H^5(S, \tilde{V}_\lambda) \to H^5(\partial\overline{S}, \tilde{V}_\lambda)$. Consider the case of a finite level variety, defined by an open compact subgroup $K_f \subset GSp_4(\mathbb{A}_f)$. We have the long exact sequence in cohomology

$$\dots \to H^5(S_K, \tilde{V}_\lambda) \to H^5(\partial\overline{S}_K, \tilde{V}_\lambda) \to H_c^6(S_K, \tilde{V}_\lambda) \to \dots$$

and by Poincaré duality we have the exact sequence

$$\dots \to H^0(S_K, \tilde{V}_\lambda^\vee) \to H^0(\partial\overline{S}_K, \tilde{V}_\lambda^\vee) \to H_c^1(S_K, \tilde{V}_\lambda^\vee) \to \dots$$

(where $\tilde{V}_\lambda^\vee$ will be also one dimensional) and, as $H^0(S_K, \tilde{V}_\lambda^\vee) \to H^0(\partial\overline{S}_K, \tilde{V}_\lambda^\vee)$ is an isomorphism (this follows from the fact that the boundary of the Borel-Serre compactification of each connected component of S_K is connected) we conclude that $H^5(\partial\overline{S}_K, \tilde{V}_\lambda) \to H_c^6(S_K, \tilde{V}_\lambda)$ is also an isomorphism. As a conclusion $H^5(S_K, \tilde{V}_\lambda) \to H^5(\partial\overline{S}_K, \tilde{V}_\lambda)$ is the zero morphism. This is then true also in the infinite level and it is then impossible to obtain ghost classes in degree 5 cohomology, so $H^0(S^{M_{P_0}}, \widetilde{W}_{(w_7)_*(\lambda)})$ does not contribute to ghost classes.

- *Case* $w = w_2$: If $m_2 > 0$, we can prove by the same arguments as in the case $w = w_4$ that the space $Ind_{P_0}^{GSp_4} H^0(S^{M_{P_0}}, \widetilde{W}_{(w_2)_*(\lambda)})$ is inside the image of $r^1_{1,0}$ and does not contribute to ghost classes.

  On the other hand, if $m_1 > 0$ we can prove by the same arguments as in the case $w = 1$ that the space $Ind_{P_0}^{GSp_4} H^0(S^{M_{P_0}}, \widetilde{W}_{(w_2)_*(\lambda)})$ does not contribute to ghost classes.

  On the other hand, when $m_1 = m_2 = 0$, there is proof of the existence of a one dimensional space of ghost classes in degree 2 cohomology (which therefore should come from the contribution of the space $H^0(S^{M_{P_0}}, \widetilde{W}_{(w_2)_*(\lambda)}))$ in section 14.1 of [11] (another proof of this can be deduced from [4]). Moreover, in this case one can verify that the weight in its corresponding mixed Hodge structure is equal to the middle weight. □

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References

[1] A. Borel, *Cohomology and spectrum of an arithmetic group*, Operator algebras and group representations, Vol. I (Neptun, 1980), Monogr. Stud. Math., vol. 17, Pitman, Boston, MA, 1984, pp. 28–45. MR731761

[2] A. Borel and J.-P. Serre, *Corners and arithmetic groups*, avec un appendice: Arrondissement des variétés à coins, par A. Douady et L. Hérault, Comment. Math. Helv. **48** (1973), 436–491, DOI 10.1007/BF02566134. MR0387495

[3] Jens Franke, *Harmonic analysis in weighted L_2-spaces* (English, with English and French summaries), Ann. Sci. École Norm. Sup. (4) **31** (1998), no. 2, 181–279, DOI 10.1016/S0012-9593(98)80015-3. MR1603257

[4] Neven Grbac and Harald Grobner, *The residual Eisenstein cohomology of Sp_4 over a totally real number field*, Trans. Amer. Math. Soc. **365** (2013), no. 10, 5199–5235, DOI 10.1090/S0002-9947-2013-05796-0. MR3074371

[5] G. Harder, *Eisenstein cohomology of arithmetic groups. The case* GL_2, Invent. Math. **89** (1987), no. 1, 37–118, DOI 10.1007/BF01404673. MR892187

[6] Günter Harder, *Eisensteinkohomologie und die Konstruktion gemischter Motive* (German), Lecture Notes in Mathematics, vol. 1562, Springer-Verlag, Berlin, 1993. MR1285354

[7] Günter Harder, *The Eisenstein motive for the cohomology of* $\mathrm{GSp}_2(\mathbb{Z})$ (English, with English and German summaries), Geometry and arithmetic, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012, pp. 143–164, DOI 10.4171/119-1/10. MR2987659

[8] Michael Harris, *Arithmetic vector bundles and automorphic forms on Shimura varieties. II*, Compositio Math. **60** (1986), no. 3, 323–378. MR869106

[9] Michael Harris and Steven Zucker, *Boundary cohomology of Shimura varieties. II. Hodge theory at the boundary*, Invent. Math. **116** (1994), no. 1-3, 243–308, DOI 10.1007/BF01231562. MR1253194

[10] M. Harris and S. Zucker, *Erratum: "Boundary cohomology of Shimura varieties. II. Hodge theory at the boundary" [Invent. Math.* **116** *(1994), no. 1-3, 243–308; MR1253194 (95f:14041)]*, Invent. Math. **121** (1995), no. 2, 437, DOI 10.1007/BF01884308. MR1346216

[11] A. Kewenig and T. Rieband, *Geisterklassen im Bild der Borelabbildung fur symplektische und orthogonale Gruppen*, Diplomarbeit, Mathematisches Institut der Rheinisch Friedrich-Wilhelms-Universität Bonn, Bonn, 1997 (unpublished).

[12] Bertram Kostant, *Lie algebra cohomology and the generalized Borel-Weil theorem*, Ann. of Math. (2) **74** (1961), 329–387, DOI 10.2307/1970237. MR0142696

[13] J. S. Milne, *Introduction to Shimura varieties*, Harmonic analysis, the trace formula, and Shimura varieties, Clay Math. Proc., vol. 4, Amer. Math. Soc., Providence, RI, 2005, pp. 265–378. MR2192012

[14] Joachim Schwermer, *Kohomologie arithmetisch definierter Gruppen und Eisensteinreihen* (German), Lecture Notes in Mathematics, vol. 988, Springer-Verlag, Berlin, 1983. MR822473

[15] Joachim Schwermer, *Cohomology of arithmetic groups, automorphic forms and L-functions*, Cohomology of arithmetic groups and automorphic forms (Luminy-Marseille, 1989), Lecture Notes in Math., vol. 1447, Springer, Berlin, 1990, pp. 1–29, DOI 10.1007/BFb0085724. MR1082960

[16] Steven Zucker, *Locally homogeneous variations of Hodge structure*, Enseign. Math. (2) **27** (1981), no. 3-4, 243–276 (1982). MR659151

[17] Steven Zucker, *On the boundary cohomology of locally symmetric varieties*, Vietnam J. Math. **25** (1997), no. 4, 279–318. MR1679475

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