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WEAK AND STRONG A_p-A_∞ ESTIMATES FOR SQUARE FUNCTIONS AND RELATED OPERATORS

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Abstract. We prove sharp weak and strong type weighted estimates for a class of dyadic operators that includes majorants of both standard singular integrals and square functions. Our main new result is the optimal bound $[w]_{A_p}^{1/p}[w]_{A_\infty}^{1/2-1/p} \lesssim [w]_{A_p}^{1/2}$ for the weak type norm of square functions on $L^p(w)$ for $p > 2$; previously, such a bound was only known with a logarithmic correction. By the same approach, we also recover several related results in a streamlined manner.

1. Introduction

We study weighted inequalities for the (in general nonlinear) operator

$$A_{\mathcal{S}}^r(f) = \Big(\sum_{Q\in\mathcal{S}}\langle f\rangle_Q^r \mathbf{1}_Q\Big)^{\frac{1}{r}}, \qquad \langle f\rangle_Q := \frac{1}{|Q|}\int_Q f,$$

where $r > 0$ and $\mathcal{S}$ is a sparse collection of dyadic cubes, i.e., there are pairwise disjoint subsets $E(S) \subset S$ such that $|E(S)| \geq \frac{1}{2}|S|$. For $r = 1$ and $r = 2$, these operators dominate large classes of Calderón–Zygmund singular integrals and Littlewood–Paley square functions, respectively (see [12, 13] and [7] for details). Thus the various norm inequalities that we prove for $A_{\mathcal{S}}^r$ immediately translate to corresponding estimates for these classes of classical operators, recovering results like the A_2 theorem of the first author [3], and its several variants and elaborations.

More precisely, we are concerned with quantifying the dependence of various weighted operator norms on a mixture of the two-weight A_p characteristic

$$[w,\sigma]_{A_p} := \sup_Q \langle w\rangle_Q \langle\sigma\rangle_Q^{p-1}$$

and the individual A_∞ characteristics

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)}\int_Q M(1_Q w)$$

and $[\sigma]_{A_\infty}$. The study of such mixed bounds was initiated in [6]. All our estimates will be stated in a dual-weight formulation, in which the classical one-weight case

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corresponds to the choice $\sigma = w^{1-p'}$. Note that $[w,\sigma]_{A_p}$ becomes the usual one-weight A_p characteristic $[w]_{A_p} := [w, w^{1-p'}]_{A_p}$ with this choice.

Since we are dealing with dyadic operators, we also consider the dyadic versions of the weight characteristics, where the supremums above are over dyadic cubes only and M denotes the dyadic maximal operator. This is a standing convention throughout this paper without further notice. Note, however, that the domination of classical operators typically involves a sum of boundedly many $A^r_{\mathcal{S}}$'s with respect to different dyadic systems, and for this reason the nondyadic weight characteristics appear in such results.

The following strong type bound has been proved by Lacey and the second author in [8], but we shall give a new proof here.

Theorem 1.1. *Let $1 < p < \infty$ and let $r > 0$. Let w, σ be a pair of weights. Then*

$$\|A^r_{\mathcal{S}}(\cdot\sigma)\|_{L^p(\sigma)\to L^p(w)} \leq C[w,\sigma]^{\frac{1}{p}}_{A_p}([w]^{(\frac{1}{r}-\frac{1}{p})_+}_{A_\infty} + [\sigma]^{\frac{1}{p}}_{A_\infty}).$$

Here and below, we simplify case analysis by interpreting $[w]^0_{A_\infty} = 1$, whether or not $[w]_{A_\infty}$ is finite. The novelties of our approach are two-fold: we make black-box use of certain two-weight theorems, rather than adapting their proofs, and we avoid the "slicing" argument, namely, the separate consideration of families of cubes with the A_p characteristic "frozen" to a certain value $\langle w\rangle_Q \langle \sigma\rangle_Q^{p-1} \approx 2^k \leq [w,\sigma]_{A_p}$.

For $r = 1$, Theorem 1.1 (in combination with the domination of singular integrals by $A^1_{\mathcal{S}}$) is the A_p-A_∞ elaboration, by the first author and Lacey [5], of the A_2 theorem of [3]. In this case, a "slicing-free" argument was provided in [4], but we feel that the present approach is even simpler.

The benefits of this approach are best seen in the weak type estimate, for which we obtain the following new result.

Theorem 1.2. *Let $1 < p < \infty$ with $p \neq r$. Let w, σ be a pair of weights. Then*

$$\|A^r_{\mathcal{S}}(\cdot\sigma)\|_{L^p(\sigma)\to L^{p,\infty}(w)} \leq C[w,\sigma]^{\frac{1}{p}}_{A_p}[w]^{(\frac{1}{r}-\frac{1}{p})_+}_{A_\infty}.$$

The case $p < r$ of Theorem 1.2 was essentially known and due to Lacey and Scurry [10], and we merely repeat their one-weight proof in the two-weight case. Note that we do not say anything about the critical exponent $p = r$, as our arguments do not shed any new light into this case. For $p > r$, however, our bound

$$[w,\sigma]^{\frac{1}{p}}_{A_p}[w]^{\frac{1}{r}-\frac{1}{p}}_{A_\infty} \lesssim [w]^{\frac{1}{p}}_{A_p}[w]^{\frac{1}{r}-\frac{1}{p}}_{A_p} = [w]^{\frac{1}{r}}_{A_p}$$

is new even in the one-weight case $\sigma = w^{1-p'}$. Indeed, for $r = 2$, the previous bounds in the literature had an additional logarithmic factor, taking the form $1 + \log[w]_{A_p}$ in [10], and subsequently improved to $(1+\log[w]_{A_\infty})^{\frac{1}{2}}$ by Domingo-Salazar, Lacey, and Rey [2]. By analogy to the failure of the A_1 conjecture (see [14]), a logarithmic correction is probably necessary in the critical case $p = r$. We are able to avoid it for $p > r$ by using a proof strategy specific to this range of exponents, whereas [2, 10] treat all $p \geq r$ as one case.

Theorem 1.2 with $r = 2$ completes the picture of sharp weighted inequalities for square functions, aside from the remaining critical case of $p = 2$. Namely, $[w]^{\max(\frac{1}{p},\frac{1}{2})}_{A_p}$ is the optimal bound among all possible bounds of the form $\Phi([w]_{A_p})$ with an increasing function Φ. This was shown by Lacey and Scurry [10] in the category of power type function $\Phi(t) = t^\alpha$; a variant of their argument proves the general claim, as we show in the last section.

To prove the above results, we need the following characterization, which is essentially due to Lai [11]; we supply the necessary details to cover the cases that were not explicitly treated in that paper.

Theorem 1.3. *Let $1 < p < \infty$ and let $r > 0$. Let w, σ be a pair of weights. Then*

$$\|A_{\mathcal{S}}^r(\cdot\sigma)\|_{L^p(\sigma)\to L^p(w)}^r \simeq \begin{cases} \mathcal{T} + \mathcal{T}^*, & p > r, \\ \mathcal{T}, & 1 < p \le r, \end{cases}$$
$$\|A_{\mathcal{S}}^r(\cdot\sigma)\|_{L^p(\sigma)\to L^{p,\infty}(w)}^r \simeq \mathcal{T}^*, \qquad p > r,$$

where

$$\mathcal{T} = \sup_{R\in\mathcal{S}} \sigma(R)^{-\frac{r}{p}} \Big\| \sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \langle\sigma\rangle_Q^r \mathbf{1}_Q \Big\|_{L^{\frac{p}{r}}(w)},$$
$$\mathcal{T}^* = \sup_{R\in\mathcal{S}} w(R)^{-\frac{1}{(\frac{p}{r})'}} \Big\| \sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \langle\sigma\rangle_Q^{r-1} \langle w\rangle_Q \mathbf{1}_Q \Big\|_{L^{(\frac{p}{r})'}(\sigma)}.$$

The case $p > r$ of Theorems 1.1 and 1.2 is a consequence of Theorem 1.3 and the following, which contains the technical core of this paper.

Proposition 1.4. *Let $r > 0$ and let $1 < p < \infty$. For $\mathcal{T}$ and $\mathcal{T}^*$ as in Theorem 1.3, we have*

$$\mathcal{T} \lesssim [w,\sigma]_{A_p}^{\frac{r}{p}} [\sigma]_{A_\infty}^{\frac{r}{p}}$$

and

$$\mathcal{T}^* \lesssim [w,\sigma]_{A_p}^{\frac{r}{p}} [w]_{A_\infty}^{1-\frac{r}{p}}, \qquad p > r.$$

The plan of the paper is as follows: We start with the proof of Theorem 1.3 and proceed to the estimation of the testing constant $\mathcal{T}$ and $\mathcal{T}^*$ as in Proposition 1.4. This completes the proof of Theorems 1.1 and 1.2 in the case of $p > r$. The remaining case of Theorem 1.2 for $p < r$ is then handled in Section 4. In the final section, we discuss the sharpness of our weak type estimates by modifying the example given by Lacey and Scurry [10].

2. Proof of Theorem 1.3

As mentioned, Theorem 1.3 is essentially due to Lai [11]. Here we make a slight change to extend the range of r from $[1,\infty)$ to $(0,\infty)$. At the same time, we feel that our argument might be slightly easier, in that it makes no reference to the Rubio de Francia algorithm.

2.1. **The case $p > r$.** In this case, we first give the following lemma.

Lemma 2.1. *Let w, σ be a pair of weights and let $p > r > 0$. Then*

$$\|A_{\mathcal{S}}^r(\cdot\sigma)\|_{L^p(\sigma)\to L^p(w)}^r \simeq \sup_{\|f\|_{L^p(\sigma)=1}} \Big\| \sum_{Q\in\mathcal{S}} \langle\sigma\rangle_Q^r \langle f^r\rangle_Q^\sigma \mathbf{1}_Q \Big\|_{L^{\frac{p}{r}}(w)},$$
$$\|A_{\mathcal{S}}^r(\cdot\sigma)\|_{L^p(\sigma)\to L^{p,\infty}(w)}^r \simeq \sup_{\|f\|_{L^p(\sigma)=1}} \Big\| \sum_{Q\in\mathcal{S}} \langle\sigma\rangle_Q^r \langle f^r\rangle_Q^\sigma \mathbf{1}_Q \Big\|_{L^{p/r,\infty}(w)}$$

Proof. For convenience, denote by $Y^p(w)$ the target space $L^p(w)$ or $L^{p,\infty}(w)$. We have

$$
\begin{aligned}
\|A^r_{\mathcal{S}}(\cdot\sigma)\|^r_{L^p(\sigma)\to Y^p(w)} &= \sup_{\|f\|_{L^p(\sigma)=1}} \Big\| \sum_{Q\in\mathcal{S}} \langle f\sigma\rangle^r_Q \mathbf{1}_Q \Big\|_{Y^{\frac{p}{r}}(w)} \\
&= \sup_{\|f\|_{L^p(\sigma)=1}} \Big\| \sum_{Q\in\mathcal{S}} \langle \sigma\rangle^r_Q (\langle f\rangle^\sigma_Q)^r \mathbf{1}_Q \Big\|_{Y^{\frac{p}{r}}(w)} \\
&\le \sup_{\|f\|_{L^p(\sigma)=1}} \Big\| \sum_{Q\in\mathcal{S}} \langle \sigma\rangle^r_Q \langle (M_\sigma(f))^r\rangle^\sigma_Q \mathbf{1}_Q \Big\|_{Y^{\frac{p}{r}}(w)} \\
&= \sup_{\|f\|_{L^p(\sigma)=1}} \Big\| \sum_{Q\in\mathcal{S}} \langle \sigma\rangle^r_Q \Big\langle \Big(\frac{M_\sigma(f)}{\|M_\sigma(f)\|_{L^p(\sigma)}}\Big)^r\Big\rangle^\sigma_Q \mathbf{1}_Q \Big\|_{Y^{\frac{p}{r}}(w)} \|M_\sigma(f)\|^r_{L^p(\sigma)} \\
&\lesssim \sup_{\|g\|_{L^p(\sigma)=1}} \Big\| \sum_{Q\in\mathcal{S}} \langle \sigma\rangle^r_Q \langle g^r\rangle^\sigma_Q \mathbf{1}_Q \Big\|_{Y^{\frac{p}{r}}(w)},
\end{aligned}
$$

where in the last step, we used the boundedness of M_σ on $L^p(\sigma)$, and the bound is independent of σ. For the other direction, notice that

$$
\langle f^r\rangle^\sigma_Q \le \inf_{x\in Q} M_\sigma(f^r)(x) = (\inf_{x\in Q} M_{\sigma,r}(f)(x))^r \le (\langle M_{\sigma,r}(f)\rangle^\sigma_Q)^r,
$$

where $M_{\sigma,r}(f) := (M_\sigma(f^r))^{1/r}$. With this observation, we have

$$
\begin{aligned}
&\sup_{\|f\|_{L^p(\sigma)=1}} \Big\| \sum_{Q\in\mathcal{S}} \langle \sigma\rangle^r_Q \langle f^r\rangle^\sigma_Q \mathbf{1}_Q \Big\|_{Y^{\frac{p}{r}}(w)} \\
&\le \sup_{\|f\|_{L^p(\sigma)=1}} \Big\| \sum_{Q\in\mathcal{S}} \langle \sigma\rangle^r_Q (\langle M_{\sigma,r} f\rangle^\sigma_Q)^r \mathbf{1}_Q \Big\|_{Y^{\frac{p}{r}}(w)} \\
&\le \sup_{\|f\|_{L^p(\sigma)=1}} \|A^r_{\mathcal{S}}(\cdot\sigma)\|^r_{L^p(\sigma)\to Y^p(w)} \|M_{\sigma,r} f\|^r_{L^p(\sigma)} \\
&\lesssim \|A^r_{\mathcal{S}}(\cdot\sigma)\|^r_{L^p(\sigma)\to Y^p(w)},
\end{aligned}
$$

where in the last step, we use the boundedness of $M_{\sigma,r}$ on $L^p(\sigma)$ since $p > r$, and the bound is independent of σ. This completes the proof of Lemma 2.1. □

Now suppose that C_1 is the best constant such that

$$
\Big\| \sum_{Q\in\mathcal{S}} \langle \sigma\rangle^r_Q \langle f^r\rangle^\sigma_Q \mathbf{1}_Q \Big\|_{Y^{\frac{p}{r}}(w)} \le C_1 \|f\|^r_{L^p(\sigma)},
$$

i.e.,

$$
\Big\| \sum_{Q\in\mathcal{S}} \langle \sigma\rangle^r_Q \langle f\rangle^\sigma_Q \mathbf{1}_Q \Big\|_{Y^{\frac{p}{r}}(w)} \le C_1 \|f\|_{L^{\frac{p}{r}}(\sigma)}. \tag{2.2}
$$

Then

$$
\|A^r_{\mathcal{S}}(\cdot\sigma)\|_{L^p(\sigma)\to Y^p(w)} \simeq C_1^{\frac{1}{r}}.
$$

Hence, we have reduced the problem to study (2.2). We need the following result given by Lacey, Sawyer and Uriarte-Tuero [9].

Proposition 2.3. *Let $\tau = \{\tau_Q : Q \in \mathcal{Q}\}$ be nonnegative constants, let w, σ be weights and let T be the linear operator defined by*

$$
T_\tau(f) := \sum_{Q\in\mathcal{Q}} \tau_Q \langle f\rangle_Q \mathbf{1}_Q.
$$

Then for $1 < p < \infty$, *there holds*

$$\begin{aligned} \|T_\tau(\cdot\sigma)\|_{L^p(\sigma)\to L^{p,\infty}(w)} &\simeq \sup_{R\in\mathcal{Q}} w(R)^{-\frac{1}{p'}} \Big\| \sum_{\substack{Q\in\mathcal{Q}\\ Q\subset R}} \tau_Q\langle w\rangle_Q \mathbf{1}_Q\Big\|_{L^{p'}(\sigma)} \\ \|T_\tau(\cdot\sigma)\|_{L^p(\sigma)\to L^{p}(w)} &\simeq \sup_{R\in\mathcal{Q}} w(R)^{-\frac{1}{p'}} \Big\| \sum_{\substack{Q\in\mathcal{Q}\\ Q\subset R}} \tau_Q\langle w\rangle_Q \mathbf{1}_Q\Big\|_{L^{p'}(\sigma)} \\ &\quad + \sup_{R\in\mathcal{Q}} \sigma(R)^{-\frac{1}{p}} \Big\| \sum_{\substack{Q\in\mathcal{Q}\\ Q\subset R}} \tau_Q\langle \sigma\rangle_Q \mathbf{1}_Q\Big\|_{L^{p}(w)}. \end{aligned}$$

Observing that

$$LHS(2.2) = \|T_\tau(f\sigma)\|_{Y^{\frac{p}{r}}(w)}$$

with $\tau_Q = \langle\sigma\rangle_Q^{r-1}$, Theorem 1.3 follows immediately from Proposition 2.3.

2.2. **The case** $1 < p \le r$**.** In this case, making use of the usual construction principal cubes $\mathcal{F}$ of (f,σ), we have

$$\begin{aligned} \|A^r_{\mathcal{S}}(f\sigma)\|_{L^p(\sigma)\to L^p(w)} &= \Big\|\Big(\sum_{Q\in\mathcal{S}}\langle f\sigma\rangle_Q^r\mathbf{1}_Q\Big)^{\frac{1}{r}}\Big\|_{L^p(w)} \\ &\lesssim \Big\|\Big(\sum_{F\in\mathcal{F}}(\langle f\rangle_F^\sigma)^r \sum_{\substack{Q\in\mathcal{S}\\ \pi(Q)=F}}\langle \sigma\rangle_Q^r\mathbf{1}_Q\Big)^{\frac{1}{r}}\Big\|_{L^p(w)} \\ &\le \Big(\sum_{F\in\mathcal{F}}(\langle f\rangle_F^\sigma)^p \Big\|\Big(\sum_{\substack{Q\in\mathcal{S}\\ \pi(Q)=F}}\langle \sigma\rangle_Q^r\mathbf{1}_Q\Big)^{\frac{1}{r}}\Big\|_{L^p(w)}^p\Big)^{\frac{1}{p}} \\ &\le \Big(\sum_{F\in\mathcal{F}}(\langle f\rangle_F^\sigma)^p \mathcal{T}^{\frac{p}{r}}\sigma(F)\Big)^{\frac{1}{p}} \lesssim \mathcal{T}^{\frac{1}{r}}\|f\|_{L^p(\sigma)}. \end{aligned}$$

On the other hand, it is obvious that

$$\mathcal{T}^{\frac{1}{r}} \le \|A^r_{\mathcal{S}}(\cdot\sigma)\|_{L^p(\sigma)\to L^p(w)}.$$

Therefore, $\|A^r_{\mathcal{S}}(\cdot\sigma)\|_{L^p(\sigma)\to L^p(w)} \simeq \mathcal{T}^{\frac{1}{r}}$.

3. Proof of Proposition 1.4

We recall the following proposition.

Proposition 3.1 ([1, Proposition 2.2])**.** *Let* $1 < s < \infty$, *let* σ *be a positive Borel measure and let*

$$\phi = \sum_{Q\in\mathcal{D}}\alpha_Q\mathbf{1}_Q, \qquad \phi_Q = \sum_{Q'\subset Q}\alpha_{Q'}\mathbf{1}_{Q'}.$$

Then

$$\|\phi\|_{L^s(\sigma)} \eqsim \Big(\sum_{Q\in\mathcal{D}}\alpha_Q(\langle\phi_Q\rangle_Q^\sigma)^{s-1}\sigma(Q)\Big)^{1/s}.$$

We also need the following result, whose proof is based on the Kolmogorov's inequality.

Proposition 3.2 ([4, Lemma 5.2])**.** *Let $\gamma \in [0,1)$. Then*

$$\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \langle w\rangle_Q^\gamma |Q| \lesssim \langle w\rangle_R^\gamma |R|.$$

Now we can estimate the two testing constants.

3.1. **Estimate of $\mathcal{T}$.** Let us first note that the case $p \geq r+1$ implies the general case. Indeed, suppose the mentioned case is already proven, and consider $p < r+1$. Let $\mathcal{T}_r$ denote the testing constant related to a given value of r. Now in particular $r > p-1$, and hence

$$\begin{aligned}\mathcal{T}_r^{1/r} &= \sup_{R\in\mathcal{S}} \sigma(R)^{-\frac{1}{p}} \Big\| \Big(\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \langle\sigma\rangle_Q^r 1_Q \Big)^{\frac{1}{r}} \Big\|_{L^p(w)} \\ &\leq \sup_{R\in\mathcal{S}} \sigma(R)^{-\frac{1}{p}} \Big\| \Big(\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \langle\sigma\rangle_Q^{p-1} 1_Q \Big)^{\frac{1}{p-1}} \Big\|_{L^p(w)} = (\mathcal{T}_{p-1})^{\frac{1}{p-1}}.\end{aligned}$$

Since $p \geq (p-1)+1$, we know by assumption that $(\mathcal{T}_{p-1})^{\frac{1}{p-1}} \leq [w,\sigma]_{A_p}^{\frac{1}{p}} [\sigma]_{A_\infty}^{\frac{1}{p}}$, and this gives the required bound for $\mathcal{T}_r$.

So we concentrate on $p \geq r+1$. By Proposition 3.1, we have

$$\begin{aligned}&\Big\| \sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \langle\sigma\rangle_Q^r \mathbf{1}_Q \Big\|_{L^{\frac{p}{r}}(w)} \\ &\eqsim \Big(\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \langle\sigma\rangle_Q^r w(Q) \Big(\frac{1}{w(Q)} \sum_{\substack{Q'\in\mathcal{S}\\ Q'\subset Q}} \langle\sigma\rangle_{Q'}^r w(Q') \Big)^{\frac{p}{r}-1} \Big)^{\frac{r}{p}} \\ &= \Big(\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \langle\sigma\rangle_Q^r w(Q) \Big(\frac{1}{w(Q)} \sum_{\substack{Q'\in\mathcal{S}\\ Q'\subset Q}} \langle\sigma\rangle_{Q'}^r \langle w\rangle_{Q'}^{\frac{r}{p-1}} \langle w\rangle_{Q'}^{1-\frac{r}{p-1}} |Q'| \Big)^{\frac{p}{r}-1} \Big)^{\frac{r}{p}} \\ &\leq [w,\sigma]_{A_p}^{\frac{r}{p-1}(1-\frac{r}{p})} \Big(\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \langle\sigma\rangle_Q^r w(Q) \Big(\frac{1}{w(Q)} \sum_{\substack{Q'\in\mathcal{S}\\ Q'\subset Q}} \langle w\rangle_{Q'}^{1-\frac{r}{p-1}} |Q'| \Big)^{\frac{p}{r}-1} \Big)^{\frac{r}{p}} \\ &\lesssim [w,\sigma]_{A_p}^{\frac{r}{p-1}(1-\frac{r}{p})} \Big(\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \langle\sigma\rangle_Q^r w(Q) \Big(\frac{1}{w(Q)} \langle w\rangle_{Q}^{1-\frac{r}{p-1}} |Q| \Big)^{\frac{p}{r}-1} \Big)^{\frac{r}{p}} \\ &= [w,\sigma]_{A_p}^{\frac{r}{p-1}(1-\frac{r}{p})} \Big(\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \langle\sigma\rangle_Q^r \langle w\rangle_Q^{\frac{r-1}{p-1}} |Q| \Big)^{\frac{r}{p}} \\ &\leq [w,\sigma]_{A_p}^{\frac{r}{p-1}(1-\frac{r}{p})+\frac{r-1}{p-1}\cdot\frac{r}{p}} \Big(\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \langle\sigma\rangle_Q |Q| \Big)^{\frac{r}{p}} \lesssim [w,\sigma]_{A_p}^{\frac{r}{p}} [\sigma]_{A_\infty}^{\frac{r}{p}} \sigma(R)^{\frac{r}{p}}.\end{aligned}$$

Therefore,

$$\mathcal{T} \lesssim [w,\sigma]_{A_p}^{\frac{r}{p}} [\sigma]_{A_\infty}^{\frac{r}{p}}. \tag{3.3}$$

3.2. **Estimate of $\mathcal{T}^*$.** Recall that we only consider $p > r$ in this case. For simplicity, we denote $s = (\frac{p}{r})'$. By Proposition 3.1 again, we have

$$\begin{aligned}
&\Big\| \sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \langle\sigma\rangle_Q^{r-1}\langle w\rangle_Q \mathbf{1}_Q \Big\|_{L^{(\frac{p}{r})'}(\sigma)} \\
&\approx \Big(\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \langle\sigma\rangle_Q^{r-1}\langle w\rangle_Q \Big(\frac{1}{\sigma(Q)} \sum_{\substack{Q'\in\mathcal{S}\\ Q'\subset Q}} \langle\sigma\rangle_{Q'}^{r-1}\langle w\rangle_{Q'}\sigma(Q')\Big)^{s-1}\sigma(Q)\Big)^{1/s}.
\end{aligned} \tag{3.4}$$

We consider $r < p < r+1$ and $p > r+1$ separately. If $r < p < r+1$, then

$$\begin{aligned}
&RHS(3.4)\\
&= \Big(\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \langle\sigma\rangle_Q^{r-1}\langle w\rangle_Q \Big(\frac{1}{\sigma(Q)} \sum_{\substack{Q'\in\mathcal{S}\\ Q'\subset Q}} \langle\sigma\rangle_{Q'}^{p-1}\langle w\rangle_{Q'}\langle\sigma\rangle_{Q'}^{r+1-p}|Q'|\Big)^{s-1}\sigma(Q)\Big)^{1/s}\\
&\le [w,\sigma]_{A_p}^{\frac{s-1}{s}} \Big(\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \langle\sigma\rangle_Q^{r-1}\langle w\rangle_Q \Big(\frac{1}{\sigma(Q)} \sum_{\substack{Q'\in\mathcal{S}\\ Q'\subset Q}} \langle\sigma\rangle_{Q'}^{r+1-p}|Q'|\Big)^{s-1}\sigma(Q)\Big)^{1/s}\\
&\lesssim [w,\sigma]_{A_p}^{\frac{s-1}{s}} \Big(\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \langle\sigma\rangle_Q^{r-1}\langle w\rangle_Q \Big(\frac{1}{\sigma(Q)} \langle\sigma\rangle_{Q}^{r+1-p}|Q|\Big)^{s-1}\sigma(Q)\Big)^{1/s}\\
&= [w,\sigma]_{A_p}^{\frac{r}{p}} \Big(\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \langle w\rangle_Q|Q|\Big)^{1/s} \lesssim [w,\sigma]_{A_p}^{\frac{r}{p}}[w]_{A_\infty}^{1-\frac{r}{p}} w(R)^{1/(\frac{p}{r})'}.
\end{aligned}$$

If $p \ge r+1$, then

$$\begin{aligned}
&RHS(3.4)\\
&= \Big(\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \langle\sigma\rangle_Q^{r-1}\langle w\rangle_Q \Big(\frac{1}{\sigma(Q)} \sum_{\substack{Q'\in\mathcal{S}\\ Q'\subset Q}} \langle\sigma\rangle_{Q'}^{r}\langle w\rangle_{Q'}^{\frac{r}{p-1}}\langle w\rangle_{Q'}^{1-\frac{r}{p-1}}|Q'|\Big)^{s-1}\sigma(Q)\Big)^{1/s}\\
&\le [w,\sigma]_{A_p}^{\frac{r^2}{(p-1)p}} \Big(\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \langle\sigma\rangle_Q^{r-1}\langle w\rangle_Q \Big(\frac{1}{\sigma(Q)} \sum_{\substack{Q'\in\mathcal{S}\\ Q'\subset Q}} \langle w\rangle_{Q'}^{1-\frac{r}{p-1}}|Q'|\Big)^{s-1}\sigma(Q)\Big)^{1/s}\\
&\le [w,\sigma]_{A_p}^{\frac{r^2}{(p-1)p}} \Big(\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \langle\sigma\rangle_Q^{r-1}\langle w\rangle_Q \Big(\frac{1}{\sigma(Q)} \langle w\rangle_{Q}^{1-\frac{r}{p-1}}|Q|\Big)^{s-1}\sigma(Q)\Big)^{1/s}\\
&= [w,\sigma]_{A_p}^{\frac{r^2}{(p-1)p}} \Big(\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \langle w\rangle_Q^{1+\frac{(p-1-r)r}{(p-1)(p-r)}}\langle\sigma\rangle_Q^{\frac{(p-1-r)r}{p-r}}|Q|\Big)^{1/s}\\
&\le [w,\sigma]_{A_p}^{\frac{r^2}{(p-1)p}+\frac{(p-1-r)r}{p(p-1)}} \Big(\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \langle w\rangle_Q|Q|\Big)^{1/s} \lesssim [w,\sigma]_{A_p}^{\frac{r}{p}}[w]_{A_\infty}^{1-\frac{r}{p}} w(R)^{1/(\frac{p}{r})'}.
\end{aligned}$$

Therefore, in both cases,

$$\mathcal{T}^* \lesssim [w,\sigma]_{A_p}^{\frac{r}{p}}[w]_{A_\infty}^{1-\frac{r}{p}}. \tag{3.5}$$

Combining (3.3) and (3.5), we have completed the proof of Proposition 1.4. Together with Theorem 1.3, this yields Theorem 1.1 as well as Theorem 1.2 in the case that $p > r$.

4. Proof of the weak type bound for $1 < p < r$

We are left to prove Theorem 1.2 in the case that $1 < p < r$. Actually, Lacey and Scurry [10] have investigated the one-weight case. Following their method, it is easy to give the two-weight estimate as well. For completeness, we give the details. We want to bound the following inequality:

$$\sup_{t>0} tw(\{x \in \mathbb{R}^n : A^r_{\mathcal{S}}(f\sigma) > t\})^{\frac{1}{p}} \le C\|f\|_{L^p(\sigma)}.$$

By scaling it suffices to give a uniform estimate for

$$t_0 w(\{x \in \mathbb{R}^n : A^r_{\mathcal{S}}(f\sigma) > t_0\})^{\frac{1}{p}},$$

where t_0 is some constant to be determined later. It is also free to further sparsify $\mathcal{S}$ such that

$$\Big|\bigcup_{\substack{Q' \subsetneq Q \\ Q', Q \in \mathcal{S}}} Q'\Big| \le \frac{1}{4}|Q|.$$

Now set

$$\mathcal{S}_l := \{Q \in \mathcal{S} : 2^{-l-1} < \langle f\sigma\rangle_Q \le 2^{-l}\}, \qquad l \ge 0$$

and

$$\mathcal{S}_{-1} := \{Q \in \mathcal{S} : \langle f\sigma\rangle_Q > 1\}.$$

Then for $Q \in \mathcal{S}_l$, $l \ge 0$, denote by $\mathrm{ch}_{\mathcal{S}_l}(\mathrm{Q})$ the maximal subcubes of Q in $\mathcal{S}_l$ and $E_Q = Q \setminus (\bigcup_{Q' \in \mathrm{ch}_{\mathcal{S}_l}(\mathrm{Q})} Q')$. We have

$$\begin{aligned}
\langle f\sigma \mathbf{1}_{E_Q}\rangle_Q &= \frac{1}{|Q|}\int_Q f\sigma dx - \frac{1}{|Q|}\sum_{Q' \in \mathrm{ch}_{\mathcal{S}_l}(\mathrm{Q})} \int_{Q'} f\sigma dx \\
&= \frac{1}{|Q|}\int_Q f\sigma dx - \sum_{Q' \in \mathrm{ch}_{\mathcal{S}_l}(\mathrm{Q})} \frac{|Q'|}{|Q|}\frac{1}{|Q'|}\int_{Q'} f\sigma dx \\
&\ge \frac{1}{|Q|}\int_Q f\sigma dx - \frac{1}{4}2^{-l} \ge \frac{1}{2}\langle f\sigma\rangle_Q.
\end{aligned}$$

Since

$$\begin{aligned}
&w(\{x \in \mathbb{R}^n : A^r_{\mathcal{S}}(f\sigma) > t_0\}) \\
&= w(\{x \in \mathbb{R}^n : \sum_{Q \in \mathcal{S}} \langle f\sigma\rangle^r_Q \mathbf{1}_Q > t_0^r\}) \\
&\le w(\{x \in \mathbb{R}^n : \sum_{l \ge 0}\sum_{Q \in \mathcal{S}_l} \langle f\sigma\rangle^r_Q \mathbf{1}_Q > \frac{t_0^r}{2}\}) \\
&\qquad + w(\{x \in \mathbb{R}^n : \sum_{Q \in \mathcal{S}_{-1}} \langle f\sigma\rangle^r_Q \mathbf{1}_Q > \frac{t_0^r}{2}\}) =: I_1 + I_2,
\end{aligned}$$

it is easy to see that

$$I_2 \leq w\left(\bigcup_{S\in\mathcal{S}_{-1}} S\right) \leq w(\{M(f\sigma)>1\}) \lesssim [w,\sigma]_{A_p}\|f\|^p_{L^p(\sigma)}.$$

Let $\frac{t_0^r}{2}=\sum_{l\geq 0}2^{-\epsilon l}$, where $\epsilon := (r-p)/2$. We have

$$\begin{aligned}
I_1 &\leq \sum_{l\geq 0} w(\{x\in\mathbb{R}^n : \sum_{Q\in\mathcal{S}_l}\langle f\sigma\rangle_Q^r \mathbf{1}_Q > 2^{-\epsilon l}\})\\
&\leq \sum_{l\geq 0} w(\{x\in\mathbb{R}^n : \sum_{Q\in\mathcal{S}_l}\langle f\sigma\rangle_Q^p \mathbf{1}_Q > 2^{(r-p)l}2^{-\epsilon l}\})\\
&\leq \sum_{l\geq 0} w(\{x\in\mathbb{R}^n : \sum_{Q\in\mathcal{S}_l}\langle f\sigma\mathbf{1}_{E_Q}\rangle_Q^p \mathbf{1}_Q > 2^{-p}2^{(r-p)l}2^{-\epsilon l}\})\\
&\leq \sum_{l\geq 0} 2^{(p+\epsilon-r)l+p}\int_{\mathbb{R}^n}\sum_{Q\in\mathcal{S}_l}\langle f\sigma\mathbf{1}_{E_Q}\rangle_Q^p\mathbf{1}_Q \mathrm{d}w\\
&\leq \sum_{l\geq 0} 2^{(p+\epsilon-r)l+p}\sum_{Q\in\mathcal{S}_l}\frac{w(Q)}{|Q|^p}\sigma(Q)^{p-1}\int_{E_Q} f^p\mathrm{d}\sigma\\
&\lesssim [w,\sigma]_{A_p}\|f\|^p_{L^p(\sigma)}.
\end{aligned}$$

Combining the above, we get

$$\|A^r_{\mathcal{S}}(f\sigma)\|_{L^{p,\infty}(w)} \lesssim [w,\sigma]_{A_p}^{\frac{1}{p}}\|f\|_{L^p(\sigma)}.$$

5. Sharpness of the weak type bounds

In this section, let

$$Sf := \Big(\sum_{I\in\mathcal{D}}\frac{1_I}{|I|}|\langle h_I, f\rangle|^2\Big)^{1/2}$$

denote the Haar square function, and $\sigma := w^{1-p'}$ will always be the L^p dual-weight of w for a fixed value of $p\in(1,\infty)$. We show that the norm bound $\|S\|_{L^p(w)\to L^{p,\infty}(w)} \lesssim [w]_{A_p}^{\max(\frac{1}{p},\frac{1}{2})}$ is unimprovable. Actually, a lower bound with the exponent $\frac{1}{p}$ holds uniformly over all weights, which is the content of the next (straightforward) proposition. The optimality of the exponent $\frac{1}{2}$ is slightly more tricky and is based on a (standard) example of a specific weight.

Proposition 5.1. *For any weight w, we have*

$$\|S\|_{L^p(w)\to L^{p,\infty}(w)} \geq [w]_{A_p}^{\frac{1}{p}}.$$

Proof. Let $N := \|S\|_{L^p(w)\to L^{p,\infty}(w)}$ and consider $f=\operatorname{sgn}(h_I)|f|$. Then $Sf\geq 1_I|I|^{-1/2}\langle |h_I|, |f|\rangle = 1_I\langle |f|\rangle_I$. Thus

$$N\|f\|_{L^p(w)} \geq \|1_I\langle|f|\rangle_I\|_{L^{p,\infty}(w)} = \frac{w(I)^{1/p}}{|I|}\int_I |f| = \frac{w(I)^{1/p}}{|I|}\int_I |f|w^{-1}w$$

for all positive functions $|f|$ on I. By the converse to Hölder's inequality, this shows that

$$N\geq \frac{w(I)^{1/p}}{|I|}\|w^{-1}\|_{L^{p'}(w)} = \frac{w(I)^{1/p}\sigma(I)^{1/p'}}{|I|},$$

and taking the supremum over all I proves the claim. □

Proposition 5.2. *Let Φ be an increasing function such that*

$$\|S\|_{L^p(w)\to L^{p,\infty}(w)} \le \Phi([w]_{A_p})$$

for all $w\in A_p$. Then $\Phi(t)\ge ct^{1/2}$.

Lacey and Scurry [10] showed this result in the class of power functions, namely, they proved that there cannot be a bound of the form $\Phi(t)=t^{1/2-\eta}$ for $\eta>0$. The stronger claim above follows by an elaboration of their argument.

Proof. Following the same arguments as those in [10], the assumption implies

$$\Big\|\Big(\sum_Q \langle a_Q\cdot w\rangle_Q^2 \mathbf{1}_Q\Big)^{1/2}\Big\|_{L^{p'}(\sigma)} \lesssim \Phi([w]_{A_p})\Big\|(\sum_Q a_Q^2)^{1/2}\Big\|_{L^{p',1}(w)}$$

for all sequences of measurable functions a_Q. For $\varepsilon>0$, we consider $w(x)=|x|^{\varepsilon-1}$ and a sequence of functions

$$a_{[0,2^{-k})}(x):=a_k(x):=\varepsilon^{\frac12}\sum_{j=k+1}^{\infty}2^{-\varepsilon(j-k)}\mathbf{1}_{[2^{-j},2^{-j+1})}(x),\quad k\in\mathbb{N}.$$

Then it is easy to check that $[w]_{A_p}\simeq w([0,1])\simeq\varepsilon^{-1}$ and $\sum_k a_k(x)^2\lesssim \mathbf{1}_{[0,1]}$ so that

$$\Big\|(\sum_{k=1}^{\infty}a_k(x)^2)^{1/2}\Big\|_{L^{p',1}(w)}\lesssim w([0,1])^{1/p'}.$$

On the other hand,

$$\langle a_k\cdot w\rangle_{[0,2^{-k})}\simeq \varepsilon^{\frac12}2^k\sum_{j=k+1}^{\infty}2^{-\varepsilon(j-k)}2^{-\varepsilon j}\simeq\varepsilon^{-\frac12}2^{k(1-\varepsilon)}.$$

It follows that

$$\int_{[0,1]}\Big(\sum_{k=1}^{\infty}\langle a_k\cdot w\rangle_{[0,2^{-k})}^2\mathbf{1}_{[0,2^{-k})}\Big)^{p'/2}\mathrm{d}\sigma\simeq\varepsilon^{-p'/2-1}\simeq\varepsilon^{-p'/2}w([0,1]).$$

By assumption, this implies $\varepsilon^{-1/2}\lesssim\Phi([w]_{A_p})\le\Phi(c\varepsilon^{-1})$, and hence $\Phi(t)\gtrsim t^{1/2}$. □

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