HAUSDORFF DIMENSION OF LIMSUP SETS OF RANDOM RECTANGLES IN PRODUCTS OF REGULAR SPACES

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ABSTRACT. The almost sure Hausdorff dimension of the limsup set of randomly distributed rectangles in a product of Ahlfors regular metric spaces is computed in terms of the singular value function of the rectangles.

1. INTRODUCTION

Limsup sets appear in various fields of mathematics. They can be traced back to the prominent works of Borel [3] and Cantelli [4] leading to the Borel-Cantelli lemma. Limsup sets are encountered, for example, in the study of Besicovitch-Eggleston sets concerning k-adic expansions of real numbers [2,6] as well as in the classical results of Khintchine [16] and Jarník [12] concerning well-approximable numbers, along with other questions related to Diophantine approximation. Recently, a considerable amount of attention has been paid to dimensional properties of random and dynamically defined limsup sets which are a class of limsup sets. Given a space X and a sequence (A_n) of subsets of X, the limsup set E consists of those points in X which are covered by infinitely many sets A_n, that is,

$$E = \limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

If the space X is a group or the sets A_n are balls, the limsup set can be made random by translating the sets A_n. In the literature there are two main lines of randomness: the sets are translated according to some probability measure or the translations are determined by a typical orbit of a measure preserving dynamical system.

The study of dimensional properties of random limsup sets was initiated by Fan and Wu [9]. They derived the following formula for the Hausdorff dimension of a typical random limsup set generated by arcs that are randomly placed on the circle according to the Lebesgue measure:

$$(1) \qquad \dim_{\mathrm{H}} E = \inf \left\{ t; \sum_n l_n^t < \infty \right\} \wedge 1,$$

where l_n is the length of the arc A_n and $x \wedge y$ is the minimum of two numbers x and y. Durand [5] generalised this result by proving a dichotomy result on Hausdorff measures defined in terms of general gauge functions. His result implies that, replacing lengths of arcs by radii of balls, formula (1) is valid for random limsup sets generated by balls which are uniformly distributed on the d-dimensional torus. As shown in [13], in the case where generating sets are balls a shorter proof of (1), which also extends to Ahlfors regular metric spaces, can be obtained using the mass transference principle proved by Beresnevich and Velani [1]. Järvenpää et al. [13] derived a formula for the almost sure Hausdorff dimension of random limsup sets induced by rectangle-like sets uniformly distributed on the d-dimensional torus. The formula is similar to (1) — only the arc lengths are replaced by the singular value functions of rectangles (for a modification of this definition see (2)). Persson [17] proved a lower bound for the Hausdorff dimension which is valid for random limsup sets generated by general open sets. Finally, in [11], Feng et al. developed a general dimension theory for random limsup sets. In [11] the results are valid in Riemann manifolds, generating sets are assumed to be only Lebesgue measurable satisfying a necessary density assumption and any distribution will do as long as it is not purely singular with respect to the Riemann volume.

Dimensional properties of random limsup sets driven by singular measures have so far been studied in the case where the generating sets are balls in Euclidean spaces or in the shift space. In [8], Fan et al. focused on dynamically defined limsup sets in the circle, where the balls are distributed along typical orbits under the angle doubling map with respect to a Gibbs measure. Seuret [19] considered the case where balls are placed randomly according to a Gibbs measure. Both instances lead to the same conclusion: formula (1) has to be modified by taking into account the multifractal spectrum of the Gibbs measure. Ekström and Persson studied in [7] limsup sets generated by balls randomly distributed according to a general measure. They derived upper and lower bounds for the Hausdorff dimension of a typical random limsup set. These bounds agree and give a formula for the Hausdorff dimension for several classes of measures.

In this paper, we initiate the study of dimensional properties of random limsup sets generated by more general sets than balls and distributed according to a singular measure. Compared to the cases where the generating sets are balls or the measure is non-singular, the situation turns out to be subtler. For example, consider in the plane a limsup set generated by rectangles $R_n = [-r_{n,1}, r_{n,1}] \times [-r_{n,2}, r_{n,2}]$ such that $r_{n,1} \leq r_{n,2}$ for all n. Suppose that the driving measure is the 1-dimensional Hausdorff measure restricted to a unit line segment making an angle $\alpha \in [0, \frac{\pi}{2}]$ with the x-axis. One easily sees that for $\alpha < \frac{\pi}{2}$, the dimension of the limsup set is determined by the sequence $(r_{n,1})$ while for $\alpha = \frac{\pi}{2}$ only the sequence $(r_{n,2})$ plays a role. Thus there is no dimension formula involving only the shapes of the generating sets and the multifractal spectrum of the measure. Therefore, more refined information is needed. We will concentrate on random limsup sets generated by rectangles in products of Ahlfors regular metric spaces distributed according to a regular measure. Our main theorem (Theorem 1) gives the almost sure Hausdorff dimension of random limsup sets in this setting. We also illustrate (Theorem 10) how our method can be applied to give a new simpler proof of the main theorem in [13].

The paper is organised as follows. The main theorem is stated and an outline of the proof is given in Section 2. In Section 3 we prove auxiliary results needed when verifying the upper bound of the dimension in Section 4. Section 5 is devoted to preliminary results aiming at the completion of the main theorem by establishing a lower bound for the dimension in Section 6. Finally, in Section 7 we give a further application of our method.

2. Main result

Let $X_1, \ldots, X_d$ be metric spaces and consider the product space $X_1^d = \bigtimes_{i=1}^d X_i$ with the metric

$$d(x, \widetilde{x}) = \max_{1 \leq i \leq d} d_i(x_i, \widetilde{x}_i).$$

The closed *rectangle* in X_1^d with centre $x = (x_1, \ldots, x_d)$ and "side radii" $r = (r_1, \ldots, r_d)$ is the set

$$\overline{R}(x, r) = \bigtimes_{i=1}^d \overline{B}(x_i, r_i),$$

where $\overline{B}(x_i, r_i)$ is the closed ball in X_i with radius r_i centred at x_i. If $\underline{x} = (x_n)$ is a sequence of points $x_n = (x_{n,1}, \ldots, x_{n,d})$ in X_1^d and $\underline{r} = (r_n)$ is a sequence of d-tuples $r_n = (r_{n,1}, \ldots, r_{n,d})$ of positive numbers, define

$$E_{\underline{r}}(\underline{x}) = \limsup_n \overline{R}(x_n, r_n).$$

Next, let μ be a Borel probability measure on X_1^d and define the probability space $(\Omega, \mathbf{P}_\mu)$ by $\Omega = (X_1^d)^{\mathbf{N}}$ and $\mathbf{P}_\mu = \mu^{\mathbf{N}}$. A sample $\omega = (\omega_n)$ from this probability space represents a sequence of points $\omega_n = (\omega_{n,1}, \ldots, \omega_{n,d})$ in X_1^d, chosen independently according to μ. The Hausdorff dimension of $E_{\underline{r}}$ is almost surely constant with respect to $\mathbf{P}_\mu$, since the event that it is less than any given value is a tail event. (For the proof of measurability of this event, see [14, Lemma 3.1].) Let

$$f_\mu(\underline{r}) = \mathbf{P}_\mu\text{-a.s. value of } \dim_{\mathrm{H}} E_{\underline{r}}.$$

For $r = (r_1, \ldots, r_d)$ and $s = (s_1, \ldots, s_d)$ such that $r_i > 0$ and $s_i \geq 0$ for every i, let τ be a permutation of $\{1, \ldots, d\}$ such that $r_{\tau(1)} \geq \ldots \geq r_{\tau(d)}$ and define the *singular value function* Φ_r^s on $[0, s_1 + \ldots + s_d]$ by

$$(2) \quad \Phi_r^s(t) = r_{\tau(i)}^{t - \sum_{j=1}^{i-1} s_{\tau(j)}} \cdot \prod_{j=1}^{i-1} r_{\tau(j)}^{s_{\tau(j)}} \text{ for } t \in [s_{\tau(1)} + \ldots + s_{\tau(i-1)}, s_{\tau(1)} + \ldots + s_{\tau(i)}].$$

The product $\Phi_r^s(t)$ is formed by distributing a "total exponent" t among the bases $r_1, \ldots, r_d$, giving an exponent of at most s_i to the base r_i, and prioritising larger bases over smaller ones.

A Borel measure μ on a metric space X is *s-regular* if there is a constant $c \geq 1$ such that $c^{-1} r^s \leq \mu(B(x, r)) \leq c r^s$ for every x in X and r in $[0, 2|X|]$.

Theorem 1. *For $i = 1, \ldots, d$, let X_i be a compact metric space and let μ_i be an s_i-regular Borel probability measure on X_i. Let $\mu = \bigtimes_{i=1}^d \mu_i$. Then*

$$f_\mu(\underline{r}) = \inf \left\{ t; \sum_n \Phi_{r_n}^s(t) < \infty \right\} \wedge (s_1 + \ldots + s_d).$$

In case $r_{n,i}$ is decreasing in both n and i, and there is a positive number α such that $r_{n,i} \leq n^{-\alpha}$ for every i for all but finitely many n, a bit of computation shows that the expression for $f_\mu(\underline{r})$ in the theorem equals

$$\min_{1 \leq i \leq d} \limsup_{n \to \infty} \left(\frac{\log n}{-\log r_{n,i}} + \sum_{j=1}^{i-1} s_j \left(1 - \frac{\log r_{n,j}}{\log r_{n,i}} \right) \right).$$

Theorem 1 can be applied also for limsup sets driven by purely singular measures. For example, let $s, t \in [0,1]$ with $s \wedge t < 1$ and let $X, Y \subset [0,1]$ be compact sets such that the restrictions of the s-dimensional and t-dimensional Hausdorff measures to X and Y are s-regular and t-regular, respectively. Then the product measure is singular with respect to the 2-dimensional Lebesgue measure. Since $X \times Y \subset [0,1]^2$ is closed, the limsup set defined in $[0,1]^2$ is included in $X \times Y$ provided (r_n) tends to zero. Thus Theorem 1 can be employed.

Overview of the proof. The upper bound for f_μ is obtained in a standard way by showing that each rectangle $\overline{R}(x_n, r_n)$ has a cover by balls whose contribution to the sum in the definition of the t-dimensional Hausdorff measure is comparable to $\Phi^s_{r_n}(t)$. See Lemma 5 below.

The proof of the lower bound is done by induction on d. For $d = 1$, the statement of Theorem 1 reduces to the well-known result that if X is a compact metric space and μ is an s-regular Borel probability measure on X, then for $\mathbf{P}_\mu$-a.e. ω

$$\dim_{\mathrm{H}} \left(\limsup_n \overline{B}(\omega_n, r_n) \right) = \inf \left\{ t; \sum_n r_n^t < \infty \right\} \wedge s$$

(see [14, Theorem 2.1] and [13, Proposition 4.7]). Proving the lower bound for $d \geq 2$, it may be assumed that every r_n is decreasing, that is, that $r_{n,1} \geq \ldots \geq r_{n,d}$, since there are only finitely many ways in which a d-tuple $r = (r_1, \ldots, r_d)$ can be ordered (see the proof of the theorem in Section 6 for details). Under this assumption, the induction step is as follows. Let π be the projection $X_1^d \to X_1^{d-1}$, and if a is something d-dimensional (a tuple, a sequence of tuples or a product measure), let a' be the image of a under π. Then for $\mathbf{P}_\mu$-a.e. ω,

$$\dim_{\mathrm{H}} E_{\underline{r}}(\omega) \geq \dim_{\mathrm{H}} \pi \left(E_{\underline{r}}(\omega) \right) \geq \dim_{\mathrm{H}} E_{\underline{r}'}(\omega')$$

(see Lemma 9 below), and $\Phi^s_r(t) = \Phi^{s'}_{r'}(t)$ for $t \in [0, s_1 + \ldots + s_{d-1}]$. Thus if

$$\sum_n \Phi^s_{r_n}(s_1 + \ldots + s_{d-1}) < \infty,$$

then the theorem for $d-1$ immediately implies the theorem for d.

Otherwise, there is some $u \in [0, s_d]$ such that

$$\sum_n \Phi^s_{r_n}(s_1 + \ldots + s_{d-1} + u) = \infty.$$

From this it can be deduced, using the theorem for $d = 1$, that for any fixed $x' \in X_1^{d-1}$, the fibre

$$E_{\underline{r}}^{x'}(\omega) = \left\{ x_d \in X_d; \, (x', x_d) \in E_{\underline{r}}(\omega) \right\}$$

almost surely has Hausdorff dimension greater than or equal to u. By Fubini's theorem it follows that for $\mathbf{P}_\mu$-a.e. ω

$$\mu'\left(\left\{x';\ \dim_{\mathrm{H}} E_{\underline{r}}^{x'}(\omega) \geq u\right\}\right) = 1.$$

Since the support of μ' has dimension $s_1 + \ldots + s_{d-1}$, this implies (by Lemma 7 below) that

$$\dim_{\mathrm{H}} E_{\underline{r}}(\omega) \geq s_1 + \ldots + s_{d-1} + u,$$

and then the theorem is proved by taking supremum over u.

Notation and conventions. The open and closed balls with centre x and radius r are denoted by $B(x,r)$ and $\overline{B}(x,r)$, respectively. All rectangles that appear are closed, and denoted as $\overline{R}(x,r) = \times_{i=1}^{d} \overline{B}(x_i, r_i)$, where $x = (x_1, \ldots, x_d)$ and $r = (r_1, \ldots, r_d)$. All measures that appear are Borel measures, but this will not be explicitly stated every time — thus "a measure" should be read as "a Borel measure".

3. Regular spaces

A *regular space* is a metric space X together with a measure μ on X for which there exist a constant $c \geq 1$ and a non-negative number s such that for every $x \in X$ and $r \in [0, 2|X|]$,

$$c^{-1} r^s \leq \mu(B(x,r)) \leq c r^s.$$

When the exponent or the constant and the exponent need to be emphasised, X will be called s-regular or (c,s)-regular. The inequalities in the definition are satisfied for all r if and only if they are satisfied for all r with $\overline{B}(x,r)$ in place of $B(x,r)$.

A subset A of a metric space is *r-sparse* if the distance between any two distinct points in A is greater than or equal to r. If A is a maximal (with respect to inclusion) r-sparse subset of B, then $B \subset \bigcup_{x \in A} \overline{B}(x,r)$.

Lemma 2. *Let X be a (c,s)-regular space and let r and R be positive numbers such that $r \leq 2R$ and $R \leq |X|$. Let $x_0 \in X$. Then there exists a maximal r-sparse subset A of $\overline{B}(x_0, R)$, and every such A satisfies*

$$c^{-2}\left(\frac{R}{r}\right)^s \leq \#A \leq 4^s c^2 \left(\frac{R}{r}\right)^s.$$

Proof. First consider *any* r-sparse subset A of $\overline{B}(x_0, R)$. Then since the balls $\{B(x, r/2)\}_{x \in A}$ are disjoint and included in $\overline{B}(x_0, 2R)$,

$$\#A \cdot c^{-1}(r/2)^s \leq \sum_{x \in A} \mu(B(x, r/2)) \leq \mu(\overline{B}(x_0, 2R)) \leq c(2R)^s.$$

The upper bound for $\#A$ follows. It also follows that starting with $A = \emptyset$ and repeatedly adding points from $\overline{B}(x_0, R)$ such that A is always r-sparse, the process must end after a bounded number of steps. At that point, a maximal r-sparse subset of $\overline{B}(x_0, R)$ is obtained. Now, assuming that A is maximal,

$$c^{-1} R^s \leq \mu(B(x_0, R)) \leq \sum_{x \in A} \mu(\overline{B}(x,r)) \leq \#A \cdot c r^s,$$

from which the lower bound for $\#A$ follows. $\qquad\square$

Lemma 3. *Let X be a (c,s)-regular space and let r and R be positive numbers such that $r \le 2R$. Let B be a closed ball of radius R in X. Then B can be covered by $4^s c^2 (R/r)^s$ closed balls of radius r.*

Proof. It may be assumed that $R \le |X|$. Let A be a maximal r-sparse subset of B. Then $B \subset \bigcup_{x \in A} \overline{B}(x,r)$, and $\#A \le 4^s c^2 (R/r)^s$ by Lemma 2. $\square$

A *cube* in X_1^d is a rectangle whose sides are balls of the same radius—this is the same as a ball in X_1^d.

Lemma 4. *For $i = 1, \ldots, d$ let X_i be a (c_i, s_i)-regular space, and consider a rectangle $R = \times_{i=1}^d \overline{B}(x_i, r_i)$ in X_1^d. Let $r > 0$ and let $I = \{i;\ r_i > r\}$. Then R can be covered by M cubes of radius r, where*

$$M = \prod_{i \in I} 4^{s_i} c_i^2 \left(\frac{r_i}{r} \right)^{s_i}$$

with the interpretation $M = 1$ if $I = \emptyset$.

Proof. By Lemma 3, there is for each $i \in I$ a subset A_i of X_i such that

$$\overline{B}(x_i, r_i) \subset \bigcup_{a_i \in A_i} \overline{B}(a_i, r) \quad \text{and} \quad \#A_i \le 4^{s_i} c_i^2 \left(\frac{r_i}{r} \right)^{s_i}.$$

For $i \notin I$, let $A_i = \{x_i\}$. Then R is covered by the set of cubes of the form $\times_{i=1}^d \overline{B}(a_i, r)$, where $a_i \in A_i$ for each i, and the number of such cubes is less than or equal to M. $\square$

4. Upper bound

The following lemma gives the upper bound for f_μ in Theorem 1.

Lemma 5. *For $i = 1, \ldots, d$ let X_i be a (c_i, s_i)-regular space, and let $\underline{x} = (x_n)$ be a sequence of points in X_1^d. Let $\underline{r} = (r_n)$ be a sequence of d-tuples of positive numbers. Then*

$$\dim_{\mathrm{H}} E_{\underline{r}}(\underline{x}) \le \inf \left\{ t;\ \sum_n \Phi_{r_n}^s(t) < \infty \right\} \wedge (s_1 + \ldots + s_d).$$

Proof. It is clear that $\dim_{\mathrm{H}} E_{\underline{r}}(\underline{x}) \le \dim_{\mathrm{H}} X_1^d = s_1 + \ldots + s_d$. Suppose that $t < s_1 + \ldots + s_d$ is such that $\sum_n \Phi_{r_n}^s(t) < \infty$. Let $\delta > 0$. Then there is some $n_0 = n_0(\delta)$ such that $r_{n,i} \le \delta$ for every i and every $n \ge n_0$, and the limsup set is included in $\bigcup_{n \ge n_0} \overline{R}(x_n, r_n)$.

For each n, let τ_n be a permutation of $\{1, \ldots, d\}$ such that $r_{n,\tau_n(1)} \ge \ldots \ge r_{n,\tau_n(d)}$ and let i_n be such that $t \in [s_{\tau_n(1)} + \ldots + s_{\tau_n(i_n-1)},\ s_{\tau_n(1)} + \ldots + s_{\tau_n(i_n)}]$. By Lemma 4 there is a cover of $\overline{R}(x_n, r_n)$ by M_n cubes of radius $r_{n,\tau_n(i_n)}$, where

$$M_n \le C \prod_{j=1}^{i_n-1} \left(\frac{r_{n,\tau_n(j)}}{r_{n,\tau_n(i_n)}} \right)^{s_{\tau_n(j)}}$$

for some constant C that is independent of n. Note that if $r_{n,\tau_n(j)} = r_{n,\tau_n(i_n)}$ for some $j < i_n$, Lemma 4 gives an upper bound where the product is up to the largest

j_0 such that $r_{n,\tau_n(j_0)} > r_{n,\tau_n(i_n)}$. The product can be extended up to $i_n - 1$ by noting that the extra terms added to the product are equal to 1. Thus

$$\mathcal{H}^t_\delta(E_{\underline{r}}(\underline{x})) \leq 2^t \sum_{n=n_0}^{\infty} M_n r^t_{n,\tau_n(i_n)} \leq 2^t C \sum_{n=1}^{\infty} \Phi^s_{r_n}(t).$$

Since the last expression is finite and independent of δ the limsup set has finite $\mathcal{H}^t$-measure, and hence Hausdorff dimension less than or equal to t. $\square$

5. Auxiliary lemmas

The following standard lemma will be used in the proof of Theorem 1.

Lemma 6. *Let (ξ_n) be a sequence of independent random variables taking values in $[0,1]$, such that $\sum_n \mathbf{E}\,\xi_n = \infty$. Then almost surely, $\sum_n \xi_n = \infty$.*

Proof. If $M \leq \frac{1}{2} \sum_{n=1}^{N} \mathbf{E}\,\xi_n$, then, using Markov's inequality and the assumption that the variables ξ_n are independent and take values in $[0,1]$,

$$\mathbf{P}\left\{\sum_{n=1}^{N} \xi_n \leq M\right\} = \mathbf{P}\left\{\sum_{n=1}^{N}(\mathbf{E}\,\xi_n - \xi_n) \geq \sum_{n=1}^{N}\mathbf{E}\,\xi_n - M\right\}$$

$$\leq \mathbf{P}\left\{\left(\sum_{n=1}^{N}(\mathbf{E}\,\xi_n - \xi_n)\right)^2 \geq \left(\sum_{n=1}^{N}\mathbf{E}\,\xi_n - M\right)^2\right\}$$

$$\leq \frac{\mathbf{E}\left(\sum_{n=1}^{N}(\mathbf{E}\,\xi_n - \xi_n)\right)^2}{\left(\sum_{n=1}^{N}\mathbf{E}\,\xi_n - M\right)^2} = \frac{\sum_{n=1}^{N}\left(\mathbf{E}\,\xi_n^2 - (\mathbf{E}\,\xi_n)^2\right)}{\left(\sum_{n=1}^{N}\mathbf{E}\,\xi_n - M\right)^2}$$

$$\leq \frac{4\sum_{n=1}^{N}\mathbf{E}\,\xi_n^2}{\left(\sum_{n=1}^{N}\mathbf{E}\,\xi_n\right)^2} \leq \frac{4\sum_{n=1}^{N}\mathbf{E}\,\xi_n}{\left(\sum_{n=1}^{N}\mathbf{E}\,\xi_n\right)^2} = \frac{4}{\sum_{n=1}^{N}\mathbf{E}\,\xi_n} \leq \frac{2}{M}.$$

Thus

$$\mathbf{P}\left\{\sum_{n=1}^{\infty}\xi_n < \infty\right\} = \mathbf{P}\left(\bigcup_{M=1}^{\infty}\bigcap_{N=1}^{\infty}\left\{\sum_{n=1}^{N}\xi_n \leq M\right\}\right)$$

$$= \lim_{M\to\infty}\lim_{N\to\infty}\mathbf{P}\left\{\sum_{n=1}^{N}\xi_n \leq M\right\} = 0. \qquad \square$$

The following lemma is a consequence of Corollary 2.10.27 in Federer's book [10], using that a complete regular space is boundedly compact (meaning that every closed bounded subset of the space is compact, or equivalently, that the space is complete and every bounded subset is totally bounded).

Lemma 7 ([10, Corollary 2.10.27])**.** *Let X and Y be complete regular metric spaces and let A be a subset of $X \times Y$ such that*

$$\dim_{\mathrm{H}}\{x \in X;\, \dim_{\mathrm{H}} A^x \geq \beta\} \geq \alpha,$$

where

$$A^x = \{y \in Y;\, (x,y) \in A\}.$$

Then $\dim_{\mathrm{H}} A \geq \alpha + \beta$.

According to the following lemma, typical limsup sets are dense.

Lemma 8. *Let X be a complete separable metric space, let μ be a fully supported probability measure on X and let $\underline{r}$ be a sequence of positive numbers. Then for $\mathbf{P}_\mu$-a.e. ω, the set $E_{\underline{r}}(\omega) = \limsup_n \overline{B}(\omega_n, r_n)$ is dense in X. In particular, $E_{\underline{r}}(\omega) \neq \emptyset$.*

Proof. If B is an open ball in X, then $\mu(B) > 0$, and thus $\#\{n;\ \omega_n \in B\} = \infty$ for $\mathbf{P}_\mu$-a.e. ω by the Borel–Cantelli lemma (or Lemma 6). Since X is separable there is a basis for the topology on X consisting of a countable family of open balls. It follows that $\mathbf{P}_\mu$-a.e. ω is such that $\#\{n;\ \omega_n \in B\} = \infty$ for every open ball B. Fix such an ω. Let B_0 be an open ball in X and let $n_0 = 0$. Assuming that the open ball B_{k-1} and the natural number n_{k-1} have been defined, let $n_k > n_{k-1}$ be such that $\omega_{n_k} \in B_{k-1}$ and let B_k be an open ball such that $\overline{B}_k \subset B_{k-1} \cap B(\omega_{n_k}, r_{n_k})$. Then

$$\emptyset \neq \bigcap_{k=1}^{\infty} \overline{B}_k \subset B_0 \cap \bigcap_{k=1}^{\infty} B(\omega_{n_k}, r_{n_k}) \subset B_0 \cap E_{\underline{r}}(\omega).$$

Since B_0 is arbitrary, this shows that $E_{\underline{r}}(\omega)$ is dense. $\qquad\square$

6. Lower bound

This section involves several d-dimensional objects, whose $(d-1)$-dimensional counterparts will be denoted by appending a prime. Thus if a is something d-dimensional, then $a' = \pi a$, where π is the appropriate projection to the first $d-1$ coordinates. For example, if

$$s = (s_1, \ldots, s_d), \qquad \underline{r} = \big((r_{n,1}, \ldots, r_{n,d})\big)_{n \in \mathbf{N}}, \qquad \mu = \mu_1 \times \ldots \times \mu_d,$$

then

$$s' = (s_1, \ldots, s_{d-1}), \qquad \underline{r}' = \big((r_{n,1}, \ldots, r_{n,d-1})\big)_{n \in \mathbf{N}}, \qquad \mu' = \mu_1 \times \ldots \times \mu_{d-1}.$$

According to the next lemma, the dimension $f_\mu(\underline{r})$ of a typical limsup set in d-dimensional product space is bounded from below by the corresponding number $f_{\mu'}(\underline{r}')$ in $(d-1)$-dimensional product space.

Lemma 9. *Let X_i be a complete separable metric space and μ_i be a probability measure on X_i for $i = 1, \ldots, d$. Let $\underline{r} = (r_n)$ be a sequence of d-tuples $r_n = (r_{n,1}, \ldots, r_{n,d})$. Set $\mu = \bigtimes_{i=1}^{d} \mu_i$. Then $f_\mu(\underline{r}) \geq f_{\mu'}(\underline{r}')$.*

Proof. Let

$$G = \big\{\omega;\ \dim_{\mathrm{H}} E_{\underline{r}}(\omega) \geq f_{\mu'}(\underline{r}')\big\},$$

and for each $\omega' \in \Omega' = \big(X_1^{d-1}\big)^{\mathbf{N}}$ let

$$G^{\omega'} = \Big\{\sigma \in (X_d)^{\mathbf{N}};\ (\omega', \sigma) \in G\Big\}.$$

Since $\mathbf{P}_\mu$ can be identified with $\mathbf{P}_{\mu'} \times \mathbf{P}_{\mu_d}$ and G is universally measurable [13, Lemma 3.1], Fubini's theorem gives

$$\mathbf{P}_\mu(G) = \int \mathbf{P}_{\mu_d}\big(G^{\omega'}\big) \, d\mathbf{P}_{\mu'}(\omega').$$

Thus it suffices to show that for every ω', the set of σ such that

$$\dim_{\mathrm{H}} E_{\underline{r}}(\omega', \sigma) \geq \dim_{\mathrm{H}} E_{\underline{r}'}(\omega')$$

has full $\mathbf{P}_{\mu_d}$-measure.

So fix ω' and let $s < \dim_{\mathrm{H}} E_{\underline{r}'}(\omega')$. Then by [18, Theorems 48 and 57] there is a compact subset K of $E_{\underline{r}'}(\omega')$ such that the measure $\theta = \mathcal{H}^s|_K$ is positive and finite. Let π be the projection $X_1^d \to X_1^{d-1}$ and for $\xi \in X_1^{d-1}$ let $(n_k(\xi))$ be the increasing enumeration of $\{n; \xi \in \overline{R}(\omega_n', r_n')\}$. (Note that this set is infinite for every $\xi \in E_{\underline{r}'}(\omega')$ and thus for θ-a.e. ξ.) Then

$$\int \theta\big(\pi(E_{\underline{r}}(\omega', \sigma)))\big)\, \mathrm{d}\mathbf{P}_{\mu_d}(\sigma) = \iint \chi_{\pi(E_{\underline{r}}(\omega', \sigma))}(\xi)\, \mathrm{d}\theta(\xi)\, \mathrm{d}\mathbf{P}_{\mu_d}(\sigma)$$

$$= \iint \chi_{\pi(E_{\underline{r}}(\omega', \sigma))}(\xi)\, \mathrm{d}\mathbf{P}_{\mu_d}(\sigma)\, \mathrm{d}\theta(\xi)$$

$$= \int \mathbf{P}_{\mu_d}\left\{\limsup_k \overline{B}\big(\sigma_{n_k(\xi)}, r_{n_k(\xi),d}\big) \neq \emptyset\right\}\, \mathrm{d}\theta(\xi)$$

$$= \theta(K),$$

using Lemma 8 in the last step. Thus for $\mathbf{P}_{\mu_d}$-a.e. σ the set $\pi(E_{\underline{r}}(\omega', \sigma))$ has full θ-measure, and in particular positive $\mathcal{H}^s$-measure. For such σ,

$$\dim_{\mathrm{H}} E_{\underline{r}}(\omega', \sigma) \geq \dim_{\mathrm{H}} \pi(E_{\underline{r}}(\omega', \sigma)) \geq s.$$

Letting $s \to \dim_{\mathrm{H}} E_{\underline{r}'}(\omega')$ along a countable set concludes the proof. $\square$

Proof of Theorem 1. The upper bound for $f_\mu(\underline{r})$ claimed by the theorem follows from Lemma 5, so it only remains to prove the lower bound. Assume first that

(3) $$r_{n,1} \geq \ldots \geq r_{n,d} \qquad \text{for every } n;$$

then the proof is by induction on d. The theorem is known to hold for $d = 1$; see [14, Theorem 2.1] and [13, Proposition 4.7]. For $d \geq 2$ there are two cases.

Case 1. Suppose that $\sum_n \Phi_{r_n'}^{s'}(s_1 + \ldots + s_{d-1}) < \infty$. Then

$$f_\mu(\underline{r}) \geq f_{\mu'}(\underline{r}') = \inf\left\{t; \sum_n \Phi_{r_n'}^{s'}(t) < \infty\right\} = \inf\left\{t; \sum_n \Phi_{r_n}^{s}(t) < \infty\right\}.$$

Here the inequality is by Lemma 9, the first equality is by the induction hypothesis and the second equality holds since $\Phi_{r_n'}^{s'}(t) = \Phi_{r_n}^{s}(t)$ for $t \in [0, s_1 + \ldots + s_{d-1}]$ by the assumption (3).

Case 2. Suppose that $u \in [0, s_d]$ is such that $\sum_n \Phi_{r_n}^{s}(s_1 + \ldots + s_{d-1} + u) = \infty$. For $\omega \in \Omega$ and $x' \in X_1^{d-1}$ let

$$E_{\underline{r}}^{x'}(\omega) = \left\{x_d \in X_d; (x', x_d) \in E_{\underline{r}}(\omega)\right\}$$

and let $\big(n_k^{x'}(\omega)\big)$ be the increasing enumeration of $\{n; x' \in \overline{R}(\omega_n', r_n')\}$. Note that this set is infinite if $x' \in E_{\underline{r}'}(\omega')$. Then

$$E_{\underline{r}}^{x'}(\omega) = \limsup_k \overline{B}\left(\omega_{n_k^{x'}(\omega),d}, r_{n_k^{x'}(\omega),d}\right).$$

Summing the u:th powers of the radii of the balls in the last expression yields

(4) $$\sum_k r_{n_k^{x'}(\omega),d}^u = \sum_n \chi_{\overline{R}(\omega_n', r_n')}(x')\, r_{n,d}^u = \sum_n \chi_{\overline{R}(x', r_n')}(\omega_n')\, r_{n,d}^u.$$

The last sum is a sum of independent random variables, and the sum of their expectations is bounded from below by

$$\sum_n \left(\prod_{i=1}^{d-1} c_i^{-1} r_{n,i}^{s_i} \right) r_{n,d}^u = c \sum_n \Phi_{r_n}^s (s_1 + \ldots + s_{d-1} + u) = \infty.$$

Lemma 6 then implies that (4) diverges for $\mathbf{P}_{\mu'}$-a.e. ω'. For such ω', the inequality $\dim_{\mathrm{H}} E_{\underline{r}}^{x'}(\omega', \sigma) \geq u$ is valid for $\mathbf{P}_{\mu_d}$-a.e. σ by the theorem for $d = 1$.

Thus, for each fixed x', Fubini's theorem implies

$$\mathbf{P}_\mu \left\{ \dim_{\mathrm{H}} E_{\underline{r}}^{x'} \geq u \right\} = \mathbf{P}_{\mu'} \times \mathbf{P}_{\mu_d} \left\{ \dim_{\mathrm{H}} E_{\underline{r}}^{x'} \geq u \right\} = 1.$$

By Fubini's theorem it follows that

$$\mathbf{P}_\mu \times \mu' \left\{ (\omega, x'); \dim_{\mathrm{H}} E_{\underline{r}}^{x'}(\omega) \geq u \right\} = 1,$$

and thus $\mathbf{P}_\mu$-a.e. ω is such that

$$\mu' \left\{ x'; \dim_{\mathrm{H}} E_{\underline{r}}^{x'}(\omega) \geq u \right\} = 1.$$

For such ω,

$$\dim_{\mathrm{H}} E_{\underline{r}}(\omega) \geq s_1 + \ldots + s_{d-1} + u$$

by Lemma 7, since μ' is $(s_1 + \ldots + s_{d-1})$-regular. Thus

$$f_\mu(\underline{r}) \geq s_1 + \ldots + s_{d-1} + \sup \left\{ u \in [0, s_d]; \sum_n \Phi_{r_n}^s (s_1 + \ldots + s_{d-1} + u) = \infty \right\}$$

$$= \sup \left\{ t \leq s_1 + \ldots + s_d; \sum_n \Phi_{r_n}^s (t) = \infty \right\}$$

$$= \inf \left\{ t; \sum_n \Phi_{r_n}^s (t) < \infty \right\} \wedge (s_1 + \ldots + s_d).$$

The theorem is now proved in the case when r_n is decreasing for every n. It follows that the theorem holds when all the r_n are ordered in the same way, that is, when there is a permutation τ of $\{1, \ldots, d\}$ such that $r_{n,\tau(1)} \geq \ldots \geq r_{n,\tau(d)}$ for every n, since the singular value function and the Hausdorff dimension do not change if the spaces $\{(X_i, \mu_i)\}$ are relabelled. In general, there is for each n a permutation τ_n such that $r_{n,\tau_n(1)} \geq \ldots \geq r_{n,\tau_n(d)}$. Given a permutation τ, let $N(\tau)$ be the set of n such that $\tau_n = \tau$ and let $E_{\underline{r}}^\tau(\omega) = \limsup_{n \in N(\tau)} \overline{R}(\omega_n, r_n)$.

Then for $\mathbf{P}_\mu$-a.e. ω,

$$
\begin{aligned}
\dim_{\mathrm{H}} E_{\underline{r}}(\omega) = \dim_{\mathrm{H}} \left(\bigcup_\tau E_{\underline{r}}^\tau(\omega) \right) &= \max_\tau \left(\dim_{\mathrm{H}} E_{\underline{r}}^\tau(\omega) \right) \\
&= \max_\tau \left(\sup \left\{ t \le s_1 + \ldots + s_d; \; \sum_{n \in N(\tau)} \Phi_{r_n}^s(t) = \infty \right\} \right) \\
&= \sup \left(\bigcup_\tau \left\{ t \le s_1 + \ldots + s_d; \; \sum_{n \in N(\tau)} \Phi_{r_n}^s(t) = \infty \right\} \right) \\
&= \sup \left\{ t \le s_1 + \ldots + s_d; \; \sum_n \Phi_{r_n}^s(t) = \infty \right\}. \qquad \square
\end{aligned}
$$

7. Convex bodies uniformly distributed in the unit cube

In this section it is shown how the method can be applied to give a new shorter proof of the main result of [13]. Let X_1^d be the unit cube in $\mathbf{R}^d$ and let μ be the uniform probability measure on X_1^d. If $\underline{K} = (K_n)$ is a sequence of convex bodies in $\mathbf{R}^d$ containing 0, let

$$
g(\underline{K}) = \mathbf{P}_\mu\text{-a.s. value of } \dim_{\mathrm{H}} \left(\limsup_n \left(\omega_n + K_n \right) \right).
$$

By John's theorem [15] there is a sequence $\underline{E} = (E_n)$ of ellipsoids such that $E_n \subset K_n \subset \widehat{E}_n$, where $\widehat{E}_n$ is the ellipsoid obtained by dilating E_n by a factor of d around its centre. Then $g(\underline{E}) \le g(\underline{K}) \le g(\widehat{E})$. For each n let $r = (r_1, \ldots, r_d)$ be the lengths of the semiaxes of E_n in decreasing order. Define $\mathbf{1} = (1, \ldots, 1)$.

Theorem 10. *In the setting above,*

$$
g(\underline{K}) = \inf \left\{ t; \; \sum_n \Phi_{r_n}^{\mathbf{1}}(t) < \infty \right\} \wedge d.
$$

Proof. It is clear that $g(\widehat{E}) \le d$, and a slight reformulation of Lemma 5 implies that if $\sum_n \Phi_{r_n}^{\mathbf{1}}(t) < \infty$, then $g(\widehat{E}) \le t$. Conversely, if $t \le d$ and $\sum_n \Phi_{r_n}^{\mathbf{1}}(t) = \infty$, then $g(\underline{E}) \ge t$. This can be proved by induction on d. For $d = 1$ the statement reduces to a known result as in Section 6.

Let $d \ge 2$. For each n there is some $i = i(n)$ such that $|v_n \cdot e_i| \ge |v_n|/\sqrt{d}$, where v_n is the shortest semiaxis of E_n. Let $N(i) = \{n; i(n) = i\}$. Then there must be some i such that $\sum_{n \in N(i)} \Phi_{r_n}^{\mathbf{1}}(t) = \infty$. By considering a subsequence of $\underline{E}$ and relabelling the coordinate axes, it may thus be assumed that $|v_n \cdot e_d| \ge |v_n|/\sqrt{d}$ for every n. It follows that $|\pi(w)| \ge |w|/\sqrt{d}$ for every vector w in $\mathbf{R}^d$ orthogonal to v_n, where π is the projection $X_1^d \to X_1^{d-1}$. In particular, the lengths of the semiaxes of $\pi(E_n)$ are comparable to $(r_{n,1}, \ldots, r_{n,d-1})$. Now there are two cases, just as in Section 6.

Case 1. If $t \le d - 1$, then $g(\underline{E}) \ge t$ by the induction hypothesis and an obvious modification of Lemma 9.

Case 2. If $t = d - 1 + u$ where $u \in [0, 1]$, let E'_n be the ellipsoid in X_1^{d-1} obtained by dilating $\pi(E_n)$ by a factor of $1/2$ around its centre. Then the lengths of the semiaxes of E'_n are comparable to $r'_n = (r_{n,1}, \ldots, r_{n,d-1})$, and for $x' \in E'_n$ the diameter of $E_n^{x'} = \{x_d \in X_d; (x', x_d) \in E_n\}$ is comparable to $r_{n,d}$. Thus for every fixed $x' \in X_1^{d-1}$,

$$\sum_n \mathbf{E} \, l_n(\omega)^u \geq c \sum_n \Phi^{\mathbf{1}}_{r_n}(t) = \infty,$$

where $l_n(\omega) = |(\omega_n + E_n)^{x'}|$ and c is a constant. Using Lemma 6, Fubini's theorem and Lemma 7 as in Section 6 then shows that

$$\dim_{\mathrm{H}} \left(\limsup_n \left(\omega_n + E_n \right) \right) \geq d - 1 + u = t$$

for $\mathbf{P}_\mu$-a.e. ω.

$\square$

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