PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 6, June 2018, Pages 2623–2635
http://dx.doi.org/10.1090/proc/13935
Article electronically published on February 14, 2018

UNIQUENESS FOR THE THIN-FILM EQUATION WITH A DIRAC MASS AS INITIAL DATA

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(Communicated by Catherine Sulem)

Abstract. We show the uniqueness of strong solutions for the thin-film equation $u_t + (uu_{xxx})_x = 0$ with initial data $u(0) = m\delta$, $m > 0$, where δ is the Dirac mass at the origin. In particular, the solution is the source type one obtained by Smyth and Hill. The argument is based on an entropy estimate for the equation in self-similar variables.

1. Introduction

The following one-dimensional fourth-order nonlinear degenerate parabolic equation

$$u_t + (uu_{xxx})_x = 0 \qquad \text{for } t > 0 \text{ and } x \in \{\, u(t,\cdot) > 0 \,\}, \tag{1.1a}$$

$$u = u_x = 0 \qquad \text{for } t > 0 \text{ and } x \in \partial\{\, u(t,\cdot) > 0 \,\}, \tag{1.1b}$$

$$u(t = 0) = m\delta(x = 0), \tag{1.1c}$$

arises as the particular case of the thin-film equation in the Hele-Shaw setting [4, 19, 20]. It describes the pinching of thin necks in a Hele-Shaw cell. The function $u = u(t,x) \geq 0$ represents the height of a two-dimensional viscous thin-film on a one-dimensional flat solid as a function of time $t > 0$ and the lateral variable x. One can rigorously derive equation (1.1) from the the Hele-Shaw cell in the regime of thin-films and where the dominating effects are surface tension and viscosity only [11, 16, 17].

Equation (1.1) is a particular case of the thin-film equation

$$u_t + \left(u^n u_{xxx}\right)_x = 0 \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}, \tag{1.2}$$

where $n > 0$ is the mobility exponent. For $n \in (0,3)$, the existence of the self-similar source type solution for (1.2) has been established in [3] by ODE techniques. The local structure of such a solution near the edge of the support is also studied there. See also [12]. For $n = 1$, the authors of [3] prove the uniqueness for (1.2)-(1.1c) in the class of self-similar solutions. See [3, Lemma 6.3, p. 231]. In this paper, we prove the uniqueness of source type solutions of (1.1) in a larger class.

The source type self-similar solution constructed by Smyth-Hill [21] and the Bernis-solution concept [1] (see also [7]) will be used to define a class of functions in which the uniqueness holds. The following definition of strong solutions contains

Received by the editors July 29, 2016, and, in revised form, August 5, 2017.
2010 *Mathematics Subject Classification.* Primary 74K35, 76A20, 35K65, 35K25, 35A02, 28D20, 35C06.
Key words and phrases. Thin-film equation, zero contact angle, uniqueness, self-similar variables, entropy estimates.

only the requirements needed for our uniqueness argument. We will use the notation $\mathcal{P} = \{ (t,x) \in (0,\infty)\times\mathbb{R};\ \ u(t,x) > 0 \}$ and $\tilde{\mathcal{P}} = \{ (t,x) \in (0,\infty)\times\mathbb{R};\ \ u(t,x) \geq 0 \}$.

Definition 1.1. Let $m>0$. A nonnegative function $u\in C^{1,4}(\tilde{\mathcal{P}})\cap L^\infty_{loc}((0,\infty);H^2(\mathbb{R}))$ is said to be a strong solution of the problem (1.1) if

(i) $u^{1/2}\,u_{xxx} \in L^2(\mathcal{P}_\varepsilon)$ for all $\varepsilon > 0$, where $\mathcal{P}_\varepsilon = \{(t,x) \in \mathcal{P};\ \ t \geq \varepsilon\}$,
(ii) there exists $\lambda > 0$ such that $\operatorname{supp}(u(t)) \subset [-\lambda t^{1/5}, \lambda t^{1/5}]$ for all $t > 0$,
(iii) there exists $C > 0$ such that $\|u(t)\|_{L^{\frac{3}{2}}(\mathbb{R})} \leq C t^{-\frac{1}{15}}$ for all $t > 0$,
(iv) $u(t) \rightharpoonup m\delta$ as $t \searrow 0$ in the sense of weak-* convergence of positive measures with a fixed mass, and

$$\int_0^\infty \int_{\mathbb{R}} u\varphi_t \ + \int_{\{u>0\}} u u_{xxx}\varphi_x = 0$$

for all $\varphi \in \operatorname{Lip}((0,\infty)\times\mathbb{R})$ with compact support.

Remarks 1.2.

1) A sequence of Radon measures $(\mu_n)_{n\in\mathbb{N}}$ is said to converge weak-* in the sense of measures to μ if

$$\int_{\mathbb{R}} \varphi(x)d\mu_n(x) \longrightarrow \int_{\mathbb{R}} \varphi(x)d\mu(x)$$

for all $\varphi \in C_b(\mathbb{R})$. If in addition $\mu_n(\mathbb{R}) = m > 0$ for all $n \in \mathbb{N}$, the convergence holds in the sense of positive measures with a fixed mass.
2) Unlike in [1,7] where the condition $u^{1/2}\,u_{xxx} \in L^2(\mathcal{P})$ is required in the definition of solutions of (1.1a) with H^1 initial data, this is not the case for (1.3) below. Fortunately, with this weaker condition (i), some properties of the solution stated in [7] are still satisfied.
3) The assumption (ii) together with $u(t) \in H^2(\mathbb{R})$ implies the boundary conditions (1.1b).
4) The condition $u \in C^{1,4}(\tilde{\mathcal{P}})$ is needed to write an equivalent form of the equation (1.1a) in self-similar variables on the support. See (3.12) below.
5) In many previous works, see for instance [1,7,9], it is requested that $u \in C^{\frac{1}{2},\frac{1}{8}}([0,\infty)\times\mathbb{R})$, for a solution of (1.1a) instead of $u \in C^{1,4}(\tilde{\mathcal{P}})$. Note that if the initial data is in $H^1(\mathbb{R})$ with compact support, then a solution $u \in C^{\frac{1}{2},\frac{1}{8}}([0,\infty)\times\mathbb{R})$ of (1.1a) is also in $C^{1,4}(\mathcal{P})$ and verifies (i). See [9, Remark 3, p. 160].
6) The conditions (ii)-(iii) are not so restrictive. In fact, the solution of (1.1a) with a nonnegative Radon measure with finite mass as initial data constructed in [9, Theorm 4, iv) and vi), p. 160], satisfies (ii)-(iii).

The existence of nonnegative weak solutions to the Cauchy problem related to the thin-film equation (1.2) was obtained first by Bernis-Friedman [2]. Some uniqueness results are given in [2] for $n \geq 4$. Other existence results for (1.1) were obtained by Bertozzi-Pugh [5]. Their class of existence includes the source type solutions constructed in [3,21]. For positive Radon measure with finite mass as initial data, the existence of weak solutions was proved by Dal Passo-Garcke [9].

To the best of our knowledge, no uniqueness result is known if the initial data is a Radon measure with finite mass, in particular for the Dirac mass. Recently, a uniqueness result was proved by John [13] but only for weak solutions and without considering Dirac mass as initial data.

Here we are concerned with uniqueness for strong solutions. Our main result is the following.

Theorem 1.3. *Let $m > 0$ and let u be a strong solution of* (1.1) *in the sense of Definition* 1.1. *Then*

$$u(t,x) = \frac{1}{(5t)^{1/5}}\rho_m\left(\frac{x}{(5t)^{1/5}}\right), \tag{1.3}$$

where

$$\rho_m(\eta) = \frac{1}{24}(A^2-\eta^2)_+^2, \quad A = \left(\frac{45}{2}m\right)^{1/5}. \tag{1.4}$$

Remarks 1.4.

1) The constant $A = A(m)$ verifies $\int_{\mathbb{R}} \rho_m(\eta)d\eta = m$.
2) The function u given by (1.3) is a solution of (1.1) in the sense of Definition 1.1.
3) One can use the previous uniqueness result to study the long time asymptotics or the behavior near the origin for a perturbation of the equation (1.1a). This fact was already observed and used by Kamin in [14, 15] for the porous medium equation.
4) The asymptotic behavior for the equation (1.1a) was studied in [7] using the entropy method.

To prove Theorem 1.3, we first reformulate equation (1.1) into an equivalent one using self-similar variables. See Problem (2.2) and Remark 2.4 below. This leads to an equivalent version of Theorem 1.3. See Theorem 2.5 below. The proof of Theorem 2.5 uses an entropy method performed in [7] for instance and some ideas from [10]. The fact that equation (2.2) has an equivalent form (see (3.12) below) is essential in our approach.

The rest of the text is organized as follows. In the next section, we reformulate the equation (1.1) in self-similar variables. Section 3 is devoted to the proof of the main result.

2. Reformulation of the problem

In this section we reformulate the equation (1.1) and state an equivalent version of Theorem 1.3. We consider the self-similar variables transformation

$$w(\tau,\eta) = (5t)^{1/5}u(t,x),\ \tau = \frac{1}{5}\log(5t),\ \eta = \frac{x}{(5t)^{1/5}}. \tag{2.1}$$

Then u is the solution of (1.1a)-(1.1b) if and only if w is the solution of the problem

$$w_\tau + \left(w(w_{\eta\eta\eta}-\eta)\right)_\eta = 0 \qquad \text{for } \tau\in\mathbb{R} \text{ and } \eta \in \left\{ w(\tau,\cdot) > 0 \right\}, \tag{2.2a}$$

$$w = w_\eta = 0 \qquad \text{for } \tau\in\mathbb{R} \text{ and } \eta \in \partial\left\{ w(\tau,\cdot) > 0 \right\}. \tag{2.2b}$$

The condition (1.1c) is reformulated via the following proposition.

Proposition 2.1. *Let w, u τ, η, x, t be given by* (2.1). *Assume that*

(i) *There exists $m > 0$ such that $\|w(\tau)\|_{L^1(\mathbb{R})} = m$ for a.e. $\tau \in \mathbb{R}$,*

(ii) $\displaystyle\lim_{R\to\infty}\limsup_{\tau\to-\infty}\int_{|\eta|>R} w(\tau,\eta)d\eta = 0.$

Then

$$u(t) \rightharpoonup m\delta \quad as \quad t \searrow 0 \quad in \quad \mathcal{D}'(\mathbb{R})\,. \tag{2.3}$$

Moreover if $w \in L^\infty\left(\mathbb{R},\ L^{3/2}(\mathbb{R})\right)$, *then* (2.3) *holds in the sense of weak-* convergence of positive measures with a fixed mass.*

Remark 2.2. It is known that (i) and (2.3) imply that the convergence holds in the sense of measures. See [6, Chapter 4, p. 115]. The additional requirement on w (which is included in the class of uniqueness) enables us to give a direct and simple proof of this fact.

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R})$, the space of C^∞-functions with compact support. Then for $R > 0$ we have

$$\begin{aligned} J(\tau) \;&:=\; \int_{\mathbb{R}} u(t,x)\varphi(x)dx - m\varphi(0) \\ &=\; \int_{\mathbb{R}} w(\tau,\eta)\left(\varphi(\mathrm{e}^\tau\eta) - \varphi(0)\right)d\eta \\ &=\; \int_{|\eta|\le R} w(\tau,\eta)\left(\varphi(\mathrm{e}^\tau\eta) - \varphi(0)\right)d\eta + \int_{|\eta|>R} w(\tau,\eta)\left(\varphi(\mathrm{e}^\tau\eta) - \varphi(0)\right)d\eta \\ &=\; J_1(\tau) + J_2(\tau). \end{aligned}$$

By the mean value theorem, we have that

$$|J_1(\tau)| \le \|\varphi'\|_{L^\infty(\mathbb{R})} m\, R\, \mathrm{e}^\tau.$$

On the other hand

$$|J_2(\tau)| \le 2\|\varphi\|_{L^\infty(\mathbb{R})} \int_{|\eta|>R} w(\tau,\eta)d\eta.$$

Hence

$$|J(\tau)| \le m\, R\, \mathrm{e}^\tau \|\varphi'\|_{L^\infty(\mathbb{R})} + 2\|\varphi\|_{L^\infty(\mathbb{R})} \int_{|\eta|>R} w(\tau,\eta)d\eta.$$

By taking the limit sup as $\tau \to -\infty$, we get

$$\limsup_{\tau\to-\infty} |J(\tau)| \le 2\|\varphi\|_{L^\infty(\mathbb{R})} \limsup_{\tau\to-\infty} \int_{|\eta|>R} w(\tau,\eta)d\eta.$$

To conclude we take the limit as $R \to \infty$, and use assumption (ii).

Now, if $w \in L^\infty\left(\mathbb{R},\ L^{3/2}(\mathbb{R})\right)$ we may take $\varphi \in C_b(\mathbb{R})$, the set of continuous and bounded functions on $\mathbb{R}$. Indeed, for such test functions φ, we have by Hölder's inequality

$$|J_1(\tau)| \le \|w\|_{L^\infty(L^{3/2})}\, \|\varphi(\mathrm{e}^\tau\cdot) - \varphi(0)\|_{\mathrm{L}^3(\{|\eta|\le \mathrm{R}\})}.$$

Hence, by the Lebesgue theorem, $\lim\limits_{\tau\to-\infty} |J_1(\tau)| = 0$ and the the conclusion follows as in the previous case. ∎

According to Definition 1.1, the transformation (2.1) and using Proposition 2.1, we look for solutions w of (2.2) in the following sense.

Definition 2.3. A nonnegative function $w \in C^{1,4}(\{w \ge 0\}) \cap L^\infty_{loc}(\mathbb{R}; H^2(\mathbb{R}))$ is said to be a strong solution of the problem (2.2) if

(i) $\mathrm{e}^{-\frac{3}{2}\tau} w^{1/2}\, w_{\eta\eta\eta} \in L^2\Big(\{(\tau,\eta);\ \ \tau \ge a,\ \ w(\tau,\eta) > 0\}\Big)$ for all $a \in \mathbb{R}$,

(ii) there exists $\lambda > 0$ such that $\mathrm{supp}\,(w(\tau)) \subset [-\lambda, \lambda]$ for all $\tau \in \mathbb{R}$,

(iii) $w \in L^\infty\left(\mathbb{R},\ L^{3/2}(\mathbb{R})\right)$, and

$$\int_{\mathbb{R}^2} w\psi_\tau \ - \int_{\{w>0\}} \eta w\psi_\eta + \int_{\{w>0\}} ww_{\eta\eta\eta}\psi_\eta = 0$$

for all $\psi \in \mathrm{Lip}(\mathbb{R}^2)$ with compact support.

Remark 2.4. The equation (2.2) has mass conservation. That is, $\|w(\tau)\|_{L^1(\mathbb{R})} = m$ for all $\tau \in \mathbb{R}$ and for some constant $m > 0$. Also, the assumption (ii) implies that

$$\lim_{R\to\infty} \limsup_{\tau\to-\infty} \int_{|\eta|>R} w(\tau,\eta)d\eta = 0.$$

Then it follows by Proposition 2.1 that (1.1c) holds.

Clearly Theorem 1.3 is equivalent to the following result.

Theorem 2.5. *Let $m > 0$ and ρ_m be given by* (1.4)*. Let w be a strong solution of* (2.2) *of mass m. Then*

$$w(\tau) = \rho_m \quad \textit{for all} \quad \tau \in \mathbb{R}.$$

3. Proof of the main result

This section is devoted to the proof of Theorem 2.5. We introduce the entropy functional that is useful for the proof. Let $f : \mathbb{R} \to (0,\infty)$ be a continuous function. We define the entropy

$$H(f) = \int_{\mathbb{R}} \left(\frac{\eta^2}{2} f(\eta) + \sqrt{\frac{8}{3}} f^{3/2}(\eta) \right) d\eta. \tag{3.1}$$

We recall the following result due to [22].

Theorem 3.1 ([22])**.** *Let*

$$F_m = \left\{ f \geq 0,\ f \in L^1(\mathbb{R}),\ \int_{\mathbb{R}} f(\eta)d\eta = m \right\}.$$

Then

$$\min_{f\in F_m} H(f) = H(\rho_m).$$

Moreover, $H(f) = H(\rho_m)$ if and only if $f = \rho_m$.

Let $f \in F_m$. The relative entropy between f and ρ_m is given by the quantity

$$H(f|\rho_m) = H(f) - H(\rho_m). \tag{3.2}$$

We quote the following Csiszár-Kullback and Sobolev type inequalities.

Theorem 3.2 ([8])**.** *Let $f \in F_m$. Then the following holds:*

$$C_m\|f - \rho_m\|^2_{L^1(\mathbb{R})} \leq H(f|\rho_m) \leq \frac{1}{2} I(f), \tag{3.3}$$

where

$$I(f) = \int_{\mathbb{R}} f(\eta) \left[\left(\frac{\eta^2}{2} + \sqrt{6} f^{1/2}(\eta) \right)_\eta \right]^2 d\eta, \tag{3.4}$$

and

$$C_m = \left(\frac{16}{3} \int_{\mathbb{R}} \sqrt{\rho_m(\eta)}\, d\eta \right)^{-1}.$$

The first inequality in (3.3) is that of Csiszár-Kullback, see [8, Case d), p. 71]. It is used in the study of the asymptotic behavior ([7]). The second one in (3.3) is the Sobolev inequality; see [8, Inequality (120), p. 78]. It is crucial in our proof.

The main ingredient for the proof of Theorem 2.5 is the following.

Proposition 3.3. *Let w be a strong solution of (2.2). Let H be the entropy of w given by*

$$H(w(\tau)) = \int_{\mathbb{R}} \left(\frac{\eta^2}{2} w(\tau,\eta) + \sqrt{\frac{8}{3}} w^{3/2}(\tau,\eta) \right) d\eta \,.$$

Then, the function $\tau \longmapsto H(w(\tau))$ is absolutely continuous on $\mathbb{R}$ and verifies

$$\frac{d}{d\tau} H(w(\tau)) := -D(w(\tau)), \tag{3.5}$$

where $D(w(\tau))$ is given by

$$\begin{aligned} D(w(\tau)) \; &= \; \int_{\mathbb{R}} \eta^2 w \, d\eta - \frac{3}{2} \int_{\mathbb{R}} w_\eta^2 \, d\eta - \sqrt{\frac{2}{3}} \int_{\mathbb{R}} w^{3/2} \, d\eta \\ &\quad + \sqrt{\frac{3}{2}} \int_{\{w>0\}} w^{1/2} \, w_{\eta\eta}^2 \, d\eta + \frac{1}{8} \sqrt{\frac{2}{3}} \int_{\{w>0\}} w^{-3/2} \, w_\eta^4 \, d\eta. \end{aligned} \tag{3.6}$$

In particular,

$$\frac{d}{d\tau} H(w(\tau)) = \int_{\mathbb{R}} \frac{d}{d\tau} \left(\frac{\eta^2}{2} w(\tau,\eta) + \sqrt{\frac{8}{3}} w^{3/2}(\tau,\eta) \right) d\eta \,. \tag{3.7}$$

To prove the previous proposition, we need the following Hardy type lemma.

Lemma 3.4. *Let $v \in H^2(\mathbb{R})$, $v \geq 0$ and* $\operatorname{supp}(v) \subset [-1,1]$. *Then, for any $0 < \beta < 1$, we have $v^{\beta-2}(v')^4 \in L^1(\mathbb{R})$ and the following inequality holds:*

$$\int_{\mathbb{R}} v^{\beta-2} (v')^4 dx \leq \frac{9}{(1-\beta)^2} \int_{\mathbb{R}} v^{\beta} (v'')^2 \, dx. \tag{3.8}$$

For the case $v > 0$ in $[-1,1]$, the proof of this lemma is given in [1, Lemma 9.1, p. 363]. In the case $v \geq 0$, (3.8) was stated in [18, Lemma 3.1, p. 728] but without proving that $v^{\beta-2}(v')^4 \in L^1((-1,1))$. For completeness we give the proof.

Proof. The fact that $v \in H^2(\mathbb{R}) \hookrightarrow C^1(\mathbb{R})$ and $\operatorname{supp}(v) \subset [-1,1]$ implies that $v'(\pm 1) = 0$. Let $\varepsilon > 0$ and put $v_\varepsilon = v + \varepsilon$. Since $v \geq 0$ and $v \in H^2(\mathbb{R})$, $v_\varepsilon \geq \varepsilon > 0$ on $[-1,1]$. Then, by integrating by parts and using $(v_\varepsilon)' = v'$, $v''(v')^2 = \frac{1}{3}\left((v')^3\right)'$, we get

$$\begin{aligned} \int_{-1}^{1} (v_\varepsilon)^{\beta-1} \, (v_\varepsilon)'' \, \left((v_\varepsilon)'\right)^2 \, dx \; &= \; \frac{1}{3} \int_{-1}^{1} (v_\varepsilon)^{\beta-1} \left((v')^3\right)' \, dx \\ &= \; \frac{1-\beta}{3} \int_{-1}^{1} (v_\varepsilon)^{\beta-2} \, (v')^4 \, dx. \end{aligned}$$

Hence, by the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \int_{-1}^{1} (v_\varepsilon)^{\beta-2} \, (v')^4 \, dx \; &= \; \frac{3}{1-\beta} \int_{-1}^{1} \left((v_\varepsilon)^{\frac{\beta}{2}-1} (v')^2 \right) \left((v_\varepsilon)^{\frac{\beta}{2}} v'' \right) dx \\ &\leq \; \frac{3}{1-\beta} \left(\int_{-1}^{1} (v_\varepsilon)^{\beta-2} (v')^4 \, dx \right)^{1/2} \left(\int_{-1}^{1} (v_\varepsilon)^{\beta} \, (v'')^2 \, dx \right)^{1/2}. \end{aligned}$$

This leads to

$$\int_{-1}^{1} (v_\varepsilon)^{\beta-2}\,(v')^4\,dx \le \frac{9}{(1-\beta)^2}\int_{-1}^{1}(v_\varepsilon)^{\beta}\,(v'')^2\,dx \quad \text{for all} \quad \varepsilon>0\,.$$

To conclude, we apply Fatou's lemma and the monotone convergence theorem

$$\begin{aligned}\int_{-1}^{1} v^{\beta-2}\,(v')^4\,dx &= \int_{-1}^{1}\lim_{\varepsilon\to 0}\left((v_\varepsilon)^{\beta-2}\,(v')^4\right)\,dx\\ &\le \liminf_{\varepsilon\to 0}\int_{-1}^{1}(v_\varepsilon)^{\beta-2}\,(v')^4\,dx\\ &\le \frac{9}{(1-\beta)^2}\liminf_{\varepsilon\to 0}\int_{-1}^{1}(v_\varepsilon)^{\beta}\,(v'')^2\,dx\\ &= \frac{9}{(1-\beta)^2}\int_{-1}^{1}v^{\beta}\,(v'')^2\,dx\,.\end{aligned}$$

□

Remarks 3.5.

1) Inequality (3.8) is true for every $\beta\ge 0$ with $\beta\ne 1$.
2) For any $\beta<0$, the function $v(x)=(1-x^2)^\alpha$, $\alpha>3/2$, $\alpha(2+\beta)\le 3$, satisfies $v\in H^2(-1,1)$, $v(\pm 1)=v'(\pm 1)=0$ and

$$\int_{-1}^{1} v^{\beta}\,(v'')^2\,dx=+\infty\,.$$

Hence the inequality (3.8) is not interesting in this case.

We now give the proof of Proposition 3.3.

Proof of Proposition 3.3. We use some calculations done in [7]. Taking $\lambda=\frac{1}{2}$ in [7, equality (3.25), p. 560] and using Lemma 3.4 with $\beta=\frac{1}{2}$, we get

$$-\frac{4}{3}\frac{d}{d\tau}\int_{\mathbb{R}} w^{3/2}d\eta+\frac{2}{3}\int_{\mathbb{R}} w^{3/2}d\eta=\int_{\{w>0\}} w^{1/2}w_{\eta\eta}^2 d\eta+\frac{1}{12}\int_{\{w>0\}} w^{-3/2}w_\eta^4 d\eta.$$

This gives

$$\begin{aligned}\frac{d}{d\tau}\int_{\mathbb{R}}\sqrt{\frac{8}{3}}w^{3/2}d\eta &= \frac{1}{2}\sqrt{\frac{8}{3}}\int_{\mathbb{R}} w^{3/2}d\eta-\frac{3}{4}\sqrt{\frac{8}{3}}\int_{\{w>0\}} w^{1/2}w_{\eta\eta}^2 d\eta\\ &\quad -\frac{1}{16}\sqrt{\frac{8}{3}}\int_{\{w>0\}} w^{-3/2}w_\eta^4 d\eta.\end{aligned} \tag{3.9}$$

In addition, by [7, equality (3.28), p. 560], we have

$$\frac{d}{d\tau}\int_{\mathbb{R}}\frac{\eta^2}{2}w d\eta=-\int_{\mathbb{R}}\eta^2 w d\eta+\frac{3}{2}\int_{\mathbb{R}} w_\eta^2 d\eta. \tag{3.10}$$

Clearly, (3.9)-(3.10) prove (3.5).

Finally, let us justify the assertion (3.7). We have

$$\begin{aligned}\int_{\mathbb{R}}\left(\frac{\eta^2}{2}w(\tau,\eta)+\sqrt{\frac{8}{3}}w^{3/2}(\tau,\eta)\right)_\tau d\eta &= \int_{\mathbb{R}}\frac{\eta^2}{2}w_\tau\,d\eta+\sqrt{6}\int_{\mathbb{R}} w^{1/2}w_\tau d\eta\\ &:= I(w)+J(w)\,.\end{aligned}$$

Using (2.2) and integrating by parts, we obtain

$$\begin{aligned}
I(w) &= \int_{\mathbb{R}} \frac{\eta^2}{2}\Big[(\eta w)_\eta - (w w_{\eta\eta\eta})_\eta\Big] d\eta \\
&= -\int_{\mathbb{R}} \eta^2 w d\eta + \int_{\mathbb{R}} \eta w w_{\eta\eta\eta} d\eta \\
&= -\int_{\mathbb{R}} \eta^2 w d\eta - \int_{\mathbb{R}} w w_{\eta\eta} d\eta - \int_{\mathbb{R}} \eta w_\eta w_{\eta\eta} d\eta \\
&= -\int_{\mathbb{R}} \eta^2 w d\eta + \int_{\mathbb{R}} w_\eta^2 d\eta - \frac{1}{2}\int_{\mathbb{R}} \eta (w_\eta^2)_\eta d\eta \\
&= -\int_{\mathbb{R}} \eta^2 w d\eta + \frac{3}{2}\int_{\mathbb{R}} w_\eta^2 d\eta.
\end{aligned}$$

Similarly, using (2.2) we have

$$\begin{aligned}
J(w) &= \sqrt{6}\int_{\mathbb{R}} w^{1/2}(\eta w)_\eta d\eta - \sqrt{6}\int_{\mathbb{R}} w^{1/2}(w w_{\eta\eta\eta})_\eta d\eta \\
&:= J_1(w) + J_2(w).
\end{aligned}$$

By integration by parts, we get

$$\begin{aligned}
J_1(w) &= -\frac{\sqrt{6}}{2}\int_{\mathbb{R}} \eta w^{1/2} w_\eta d\eta \\
&= -\frac{\sqrt{6}}{3}\int_{\mathbb{R}} \eta (w^{3/2})_\eta d\eta \\
&= \sqrt{\frac{2}{3}}\int_{\mathbb{R}} w^{3/2} d\eta.
\end{aligned}$$

For J_2, we use integration by parts and Lemma 3.4 to obtain

$$\begin{aligned}
J_2(w) &= \frac{\sqrt{6}}{2}\int_{\mathbb{R}} \eta w^{1/2} w_\eta w_{\eta\eta\eta} d\eta \\
&= -\frac{\sqrt{6}}{2}\int_{\mathbb{R}} w^{1/2} w_{\eta\eta}^2 d\eta - \frac{\sqrt{6}}{4}\int_{\mathbb{R}} w^{-1/2} w_\eta^2 w_{\eta\eta} d\eta \\
&= -\frac{\sqrt{6}}{2}\int_{\mathbb{R}} w^{1/2} w_{\eta\eta}^2 d\eta - \frac{\sqrt{6}}{12}\int_{\mathbb{R}} w^{-1/2} (w_\eta^3)_\eta d\eta \\
&= -\frac{\sqrt{6}}{2}\int_{\mathbb{R}} w^{1/2} w_{\eta\eta}^2 d\eta - \frac{\sqrt{6}}{24}\int_{\mathbb{R}} w^{-3/2} w_\eta^4 d\eta.
\end{aligned}$$

In conclusion, using (3.5), (3.6) we get (3.7). □

We now give the proof of the main theorem.

Proof of Theorem 2.5. Let w be a solution of (2.2) satisfying the hypotheses of Theorem 2.5. We consider the relative entropy

$$h(\tau) = H(w(\tau)|\rho_m) = H(w(\tau)) - H(\rho_m),$$

and the partial entropy dissipation $I(\tau) = I(w(\tau))$ defined in (3.4). By Proposition 3.3, we have

$$h'(\tau) = \frac{d}{d\tau} H(w(\tau)) = -D(w(\tau)). \tag{3.11}$$

Let us now write the equation (2.2a) in another equivalent form. Such a form is derived in [7]. For completeness we give the details. Recall that from (2.2a), we have

$$w_\tau = (\eta w)_\eta - (w w_{\eta\eta\eta})_\eta\,,\ \eta \in \{w > 0\}.$$

On the one hand, since

$$(w w_{\eta\eta\eta})_\eta = 2\left[w^{3/2}\left(w^{1/2}\right)_{\eta\eta}\right]_{\eta\eta},$$

we have

$$w_\tau = (\eta w)_\eta - 2\left[w^{3/2}\left(w^{1/2}\right)_{\eta\eta}\right]_{\eta\eta}.$$

On the other hand, since

$$\left(w^{3/2}\right)_{\eta\eta} = \left[w^{3/2}\left(\frac{\eta^2}{2}\right)_{\eta\eta}\right]_{\eta\eta},$$

we write, for some $\mu > 0$, to be chosen later,

$$\begin{aligned}
w_\tau &= (\eta w)_\eta + 2\mu\left(w^{3/2}\right)_{\eta\eta} - 2\mu\left(w^{3/2}\right)_{\eta\eta} - 2\left[w^{3/2}\left(w^{1/2}\right)_{\eta\eta}\right]_{\eta\eta}\\
&= (\eta w)_\eta + 2\mu\left(w^{3/2}\right)_{\eta\eta} - 2\mu\left[w^{3/2}\left(\frac{\eta^2}{2}\right)_{\eta\eta}\right]_{\eta\eta} - 2\left[w^{3/2}\left(w^{1/2}\right)_{\eta\eta}\right]_{\eta\eta}\\
&= \left[\eta w + 2\mu\left(w^{3/2}\right)_\eta\right]_\eta - 2\left[w^{3/2}\left(\mu\frac{\eta^2}{2} + w^{1/2}\right)_{\eta\eta}\right]_{\eta\eta}\\
&= \left[\eta w + 3\mu w^{1/2} w_\eta\right]_\eta - 2\left[w^{3/2}\left(\mu\frac{\eta^2}{2} + w^{1/2}\right)_{\eta\eta}\right]_{\eta\eta}\\
&= \left[w\left(\frac{\eta^2}{2} + 6\mu w^{1/2}\right)_\eta\right]_\eta - 2\mu\left[w^{3/2}\left(\frac{\eta^2}{2} + \frac{1}{\mu} w^{1/2}\right)_{\eta\eta}\right]_{\eta\eta}.
\end{aligned}$$

Choose now $\mu > 0$ such that $6\mu = \frac{1}{\mu}$, that is, $\mu = \frac{1}{\sqrt{6}}$. We get that the equation (2.2a) can be written in the following form:

(3.12)
$$w_\tau = \left[w\left(\frac{\eta^2}{2} + \sqrt{6} w^{1/2}\right)_\eta\right]_\eta - \frac{2}{\sqrt{6}}\left[w^{3/2}\left(\frac{\eta^2}{2} + \sqrt{6} w^{1/2}\right)_{\eta\eta}\right]_{\eta\eta},\ \eta \in \{w > 0\}.$$

Note that to derive such a form we need that $w \in C^{1,4}(\{w \geq 0\})$.

Using the second form (3.12) of the equation (2.2a), we derive another form of the derivative of the entropy H, where H is defined by (3.1). In fact, since $\frac{d}{d\tau} H(w(\tau))$

is well defined by Proposition 3.3 we may write, using (3.7),

$$\begin{aligned}\frac{d}{d\tau}H(w(\tau)) &= \int_{\mathbb{R}}\left(\frac{\eta^2}{2}+\sqrt{6}w^{1/2}\right)w_\tau d\eta\\ &= -\frac{2}{\sqrt{6}}\int_{\mathbb{R}}\left(\frac{\eta^2}{2}+\sqrt{6}w^{1/2}\right)\left[w^{3/2}\left(\frac{\eta^2}{2}+\sqrt{6}w^{1/2}\right)_{\eta\eta}\right]_{\eta\eta}d\eta\\ &\quad+\int_{\mathbb{R}}\left(\frac{\eta^2}{2}+\sqrt{6}w^{1/2}\right)\left[w\left(\frac{\eta^2}{2}+\sqrt{6}w^{1/2}\right)_{\eta}\right]_{\eta}d\eta\\ &:= \mathcal{A}(w(\tau))+\mathcal{B}(w(\tau)).\end{aligned}$$

Since $w=0$ on the boundary, then using integration by parts we get

$$\mathcal{B}(w(\tau)) = -\int w\left[\left(\frac{\eta^2}{2}+\sqrt{6}w^{1/2}\right)_{\eta}\right]^2 d\eta = -I(w(\tau)),$$

where I is given by (3.4).

To conclude, we will show that $\mathcal{A}(w(\tau))\le 0$. We have

$$\left(\frac{\eta^2}{2}+\sqrt{6}w^{1/2}\right)\left[w^{3/2}\left(\frac{\eta^2}{2}+\sqrt{6}w^{1/2}\right)_{\eta\eta}\right]_{\eta} = 0, \quad \eta\in\partial\{w>0\}, \tag{3.13}$$

which follows since $w=0$ on the boundary, the regularity of w and the fact that

$$\left[w^{3/2}\left(\frac{\eta^2}{2}+\sqrt{6}w^{1/2}\right)_{\eta\eta}\right]_{\eta} = \frac{3}{2}w_\eta w^{1/2}+\frac{\sqrt{6}}{2}w_{\eta\eta\eta}w. \tag{3.14}$$

Hence, doing a first integration by parts, we get that

$$\mathcal{A}(w(\tau)) = \frac{2}{\sqrt{6}}\int_{\mathbb{R}}\left(\frac{\eta^2}{2}+\sqrt{6}w^{1/2}\right)_{\eta}\left[w^{3/2}\left(\frac{\eta^2}{2}+\sqrt{6}w^{1/2}\right)_{\eta\eta}\right]_{\eta}d\eta. \tag{3.15}$$

If one can do a second integration by parts we reach the desired conclusion, that is, $\mathcal{A}(w(\tau))\le 0$. But, unfortunately, a second integration by parts will require the following boundary condition:

$$\left(\frac{\eta^2}{2}+\sqrt{6}w^{1/2}\right)_{\eta} w^{3/2}\left(\frac{\eta^2}{2}+\sqrt{6}w^{1/2}\right)_{\eta\eta} = 0, \quad \eta\in\partial\{w>0\}, \tag{3.16}$$

which reads

$$w_\eta^3 w^{-1/2} = 0 \quad \eta\in\partial\{w>0\}, \tag{3.17}$$

and it is not clear that this last condition is satisfied under the hypotheses on the solution w. To clarify this we replace w by $w+\epsilon$ and then take the limit as $\epsilon\to 0^+$.

For $\epsilon>0$, set

$$\begin{aligned}\mathcal{A}^\epsilon(w) &:= \mathcal{A}(w+\epsilon)\\ &= \frac{2}{\sqrt{6}}\int_{\mathbb{R}}\left(\frac{\eta^2}{2}+\sqrt{6}(w+\epsilon)^{1/2}\right)_{\eta}\left[(w+\epsilon)^{3/2}\left(\frac{\eta^2}{2}+\sqrt{6}(w+\epsilon)^{1/2}\right)_{\eta\eta}\right]_{\eta}d\eta.\end{aligned}$$

By Definition 2.3, we have that $w\in H^2$, $w^{1/2}w_{\eta\eta\eta}\in L^2$ and the additional regularity $(w^{1/2})_\eta\in L^4$ (see [7, Assertion (3.7), p. 557]). This in particular implies, using (ii) of Definition 2.3, that $w^{1/2}\in H^1$. We may now apply the dominated convergence theorem to obtain

$$\lim_{\epsilon\to 0}\mathcal{A}^\epsilon(w) = \mathcal{A}(w).$$

So now it suffices to show that $\mathcal{A}^\epsilon(w) \le 0$. Owing to the fact that $w_\eta^3\,(w+\epsilon)^{-1/2} = 0$ on the boundary, we obtain after integration by parts that

$$\mathcal{A}^\epsilon(w) = -\frac{2}{\sqrt{6}} \int_{\mathbb{R}} (w+\epsilon)^{3/2} \left[\left(\frac{\eta^2}{2} + \sqrt{6}(w+\epsilon)^{1/2} \right)_{\eta\eta} \right]^2 d\eta \le 0.$$

We then conclude that $\mathcal{A}(w) \le 0$. Using the fact that

$$\frac{d}{d\tau} H(w(\tau)) = \mathcal{A}(w) + \mathcal{B}(w) \le \mathcal{B}(w) = -I(\tau),$$

we get by (3.11),

$$h'(\tau) \le -I(\tau) \quad \text{for all} \quad \tau \in \mathbb{R}.$$

It follows, using (3.3), that

$$h'(\tau) \le -2h(\tau) \quad \text{for all} \quad \tau \in \mathbb{R}.$$

Hence

$$0 \le h(\tau) \le \mathrm{e}^{-2(\tau-s)} h(s) \quad \text{for all} \quad \tau \ge s.$$

By (ii)-(iii) in Definition 2.3, there exists a positive constant K such that

$$0 \le h(s) \le K, \ s \in \mathbb{R}.$$

This leads to

$$0 \le h(\tau) \le K\mathrm{e}^{-2(\tau-s)} \quad \text{for all} \quad \tau \ge s.$$

By letting $s \to -\infty$, we obtain $h(\tau) = 0$ for all $\tau \in \mathbb{R}$. This proves that

$$H(w(\tau)) = H(\rho_m).$$

Then, by Theorem 3.1 and the fact that $\|w(\tau)\|_{L^1(\mathbb{R})} = m$, we get $w(\tau) = \rho_m$ for all $\tau \in \mathbb{R}$ which concludes the proof of the theorem. □

4. Conclusion

In this work, we proved the uniqueness of source type solutions for the equation (1.1) without a self-similarity assumption. In particular, this provides a larger class for uniqueness.

In the framework of self-similar solutions, the uniqueness proof used ODE techniques; see [3]. To broaden the class for uniqueness of source type solutions of (1.1), we use an entropy estimate for (1.1) rewritten in self-similar variables.

Acknowledgments

The authors are grateful to the anonymous referees for a careful reading of the manuscript and for their constructive comments, which led to a marked improvement of the paper.

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