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NO CRITICAL NONLINEAR DIFFUSION IN 1D QUASILINEAR FULLY PARABOLIC CHEMOTAXIS SYSTEM

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ABSTRACT. This paper deals with the fully parabolic 1d chemotaxis system with diffusion $1/(1+u)$. We prove that the above mentioned nonlinearity, despite being a natural candidate, is not critical. It means that for such a diffusion any initial condition, independently on the magnitude of mass, generates the global-in-time solution. In view of our theorem one sees that the one-dimensional Keller-Segel system is essentially different from its higher-dimensional versions. In order to prove our theorem we establish a new Lyapunov-like functional associated to the system. The information we gain from our new functional (together with some estimates based on the well-known classical Lyapunov functional) turns out to be rich enough to establish global existence for the initial-boundary value problem.

1. Introduction

We consider the one-dimensional version of the following quasilinear Keller-Segel problem:

(1.1)
$$\begin{cases} \partial_t u = \partial_x \left(a(u)\partial_x u - u\partial_x v \right) & \text{in } (0,T)\times(0,1), \\ \partial_t v = \partial_x^2 v - v + u & \text{in } (0,T)\times(0,1), \\ \partial_x A(u)(t,0) = \partial_x A(u)(t,1) = \partial_x v(t,0) = \partial_x v(t,1) = 0, & t\in(0,T), \\ u(0,\cdot) = u_0, \quad v(0,\cdot) = v_0 & \text{in } (0,1), \end{cases}$$

where a is a positive function $a \in C^1(0,\infty)\cap C[0,\infty)$. The function A is an indefinite integral of a and $u_0 \in W^{1,\infty}(0,1)$ such that $u_0 \geq 0$ in $(0,1)$. Furthermore we assume $0 \leq v_0 \in W^{1,\infty}(0,1)$.

The particular nonlinear diffusion $a(u) = 1/(1+u)$ is a natural candidate for a critical one in the one-dimensional setting. Namely in dimensions $n \geq 2$ $a(u) = (1+u)^{1-\frac{2}{n}}$ it is critical in the sense that it distinguishes between the global-in-time existence for any initial data for stronger diffusions (see [16]) and finite-time blowups when the diffusion is weaker; see [10]. Next, in the particular case of diffusion given by $a(u) = (1+u)^{1-\frac{2}{n}}$, global solutions exist for small mass data while they blow up in finite time for initial masses large enough; see [13] in dimension 2 and [12] in dimensions 3 and 4. An interested reader might find more details in [1].

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In dimension 1 a situation is similar, when diffusion is subcritical, namely $a(u) = (1+u)^{-p}, p < 1$ solutions emanating from any data (regardless the magnitude of mass) exist globally in time and are bounded, see [4], while in the supercritical case ($p > 1$) solutions blowing up in finite time (only for large masses and under some additional restrictions) have been constructed in [7]. In the critical case $a(u) = 1/(1+u)$ so far only existence of global solutions for initial masses small enough are known ([4]). Our aim is to study this case further. Before we introduce our main result let us mention that similar results are known also in the parabolic-elliptic version of quasilinear Keller-Segel system. The higher-dimensional problem in bounded domain is treated in [11, 15] (global existence in the subcritical and critical cases for small masses, respectively) and [11] (finite-time blowups in supercritical case for initial mass of any magnitude), [8] (blowup in the critical case for mass large enough). The same problem in the whole space has been solved in [3, 17, 18]. The one-dimensional case has been solved in [9], where a peculiar change of variables was used by the authors. As a consequence both blowup in the supercritical case and global existence in the subcritical one have been obtained. Surprisingly, in [9], also global existence for any initial data in the critical case $a(u) = 1/(1+u)$ was obtained. However, all the reasoning depends on the crucial change of variables. The change of variables works only in the parabolic-elliptic case (moreover, it is also very sensitive to the fact that the Jäger-Luckhaus-type simplification is studied, the usual Keller-Segel-type system was carried in the recent note [6]). The fully parabolic case was an open problem for several years. We answer this case in the present paper. Let us notice that also in the case of nonlocal diffusions in 1d, at least in the parabolic-elliptic case, critical diffusion does not exist; see [5].

To this end, we construct the following new functional associated to (1.1) (it is worth noticing that it holds only in dimension 1) satisfying

$$\frac{d}{dt}\mathcal{F}(u(t)) + \mathcal{D}(u(t), v(t)) = \int_0^1 \frac{ua(u)(v + \partial_t v)^2}{4},$$

where

$$\mathcal{F}(u(t)) := \frac{1}{2}\int_0^1 \frac{(a(u))^2}{u}|\partial_x u|^2 - \int_0^1 u \int_1^u a(r)\, dr, \tag{1.2}$$

$$\mathcal{D}(u(t), v(t)) := \int_0^1 ua(u)\left|\partial_x\left(\frac{a(u)}{u}\partial_x u\right) - \partial_x^2 v + \frac{(v + \partial_t v)}{2}\right|^2. \tag{1.3}$$

This functional is essentially the biggest novelty of our paper. It gives much richer information than the usual Lyapunov functional known to exist in Keller-Segel systems (that will be introduced later as L); see [4, 7]. Our functional is related to the Lyapunov functional introduced in [9], however the connection is delicate and, compared to [9], we needed a completely new method in order to arrive at (1.2). Actually, the Lyapunov functional in [9] should rather be seen as an inspiration for our formula. We shall discuss the connections between them in more detail at the beginning of Section 3.

Thanks to the known facts concerning the usual Lyapunov functional related to (1.1) we notice that the growth of $\mathcal{F}$ along the trajectories can be controlled. Then we obtain the following main result answering the open question concerning global existence in the critical quasilinear fully parabolic 1d Keller-Segel.

Theorem 1.1. *Let $a(u) = \frac{1}{1+u}$ and both $u_0, v_0 \geq 0$. Then the problem* (1.1) *has a unique classical positive solution, which exists globally in time.*

Let us also mention that the global existence result is extended also to the nonlinear diffusions a which are merely only nonintegrable at infinity continuous positive functions. It complements the result on finite-time blowup contained in [4] for nonlinear diffusions which are integrable at infinity. The above mentioned extension is a content of Theorem 5.1 in Section 5.

2. Preliminaries

The next lemma contains a crucial identity. It was shown in [6, Lemma 2.1] of the accompanying paper. As noticed in [6, Remark 2.2], the equality below holds only in dimension 1. For the reader's convenience we attach a short sketch of its proof.

Lemma 2.1. *Let $\phi \in C^3(0,1)$. Then the following identity holds:*

$$\phi \partial_x \mathcal{M}(\phi) = \partial_x \left(\phi a(\phi) \partial_x \left(\frac{a(\phi)}{\phi} \partial_x \phi \right) \right),$$

where

$$\mathcal{M}(\phi) := \frac{a(\phi)a'(\phi)}{\phi} |\partial_x \phi|^2 - \frac{(a(\phi))^2}{2\phi^2} |\partial_x \phi|^2 + \frac{(a(\phi))^2}{\phi} \partial_x^2 \phi.$$

Proof. A left hand side of the main identity equals

$$\begin{aligned}
&(a'(\phi))^2 |\partial_x \phi|^2 \partial_x \phi + a(\phi) a''(\phi) |\partial_x \phi|^2 \partial_x \phi \\
&\quad - \frac{2a(\phi)a'(\phi)}{\phi} |\partial_x \phi|^2 \partial_x \phi + a(\phi)a'(\phi) \partial_x (|\partial_x \phi|^2) \\
&\quad + \frac{(a(\phi))^2}{\phi^2} |\partial_x \phi|^2 \partial_x \phi - \frac{(a(\phi))^2}{2\phi} \partial_x \left(|\partial_x \phi|^2 \right) \\
&\quad - \frac{(a(\phi))^2}{\phi} \partial_x^2 \phi \partial_x \phi + 2a(\phi)a'(\phi) \partial_x^2 \phi \partial_x \phi + (a(\phi))^2 \partial_x^3 \phi.
\end{aligned}$$

We see that the first, second and fourth term of the above formula sum up to $\partial_x \left(a(\phi)a'(\phi)|\partial_x \phi|^2 \right)$, while the last two terms give $\partial_x \left((a(\phi))^2 \partial_x^2 \phi \right)$. Hence

$$\begin{aligned}
\phi \partial_x \mathcal{M}(\phi) &= \partial_x \left(a(\phi)a'(\phi)|\partial_x \phi|^2 \right) + \partial_x \left((a(\phi))^2 \partial_x^2 \phi \right) \\
&\quad - \frac{2a(\phi)a'(\phi)}{\phi} |\partial_x \phi|^2 \partial_x \phi + \frac{(a(\phi))^2}{\phi^2} |\partial_x \phi|^2 \partial_x \phi - \frac{(a(\phi))^2}{\phi} \partial_x (|\partial_x \phi|^2) \\
&= \partial_x \left(a(\phi)a'(\phi)|\partial_x \phi|^2 + (a(\phi))^2 \partial_x^2 \phi - \frac{(a(\phi))^2}{\phi} |\partial_x \phi|^2 \right) \\
&= \partial_x \left(\phi a(\phi) \partial_x \left(\frac{a(\phi)}{\phi} \partial_x \phi \right) \right).
\end{aligned}$$

□

Next, we present several well-known facts. The following inequality is obtained in [2, 4, 14].

Lemma 2.2. *For $w \in H^1(0,1)$ and any $\delta > 0$ there exists $C_\delta > 0$ such that*

$$\|w\|_{L^4(0,1)}^4 \leq \delta \|w\|_{H^1(0,1)}^2 \left(\int_0^1 |w \log w| \right)^2 + C_\delta \|w\|_{L^1(0,1)}.$$

The following local existence result is known; see [4, 7].

Lemma 2.3. *For $a \in C^1(0,\infty) \cap C[0,\infty)$ and nonnegative $(u_0, v_0) \in (W^{1,\infty}(0,1))^2$ there exist $T_{\max} \leq \infty$ (depending only on $\|u_0\|_{L^\infty}$ and $\|v_0\|_{W^{1,\infty}}$) and exactly one pair (u,v) of positive functions*

$$(u,v) \in C([0,T_{\max}) \times [0,1]; \mathbb{R}^2) \cap C^{1,2}((0,T_{\max}) \times [0,1]; \mathbb{R}^2)$$

that solves (1.1) in the classical sense. Also, the solution (u,v) satisfies the mass conservation

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx \quad \text{for all } t \in (0, T_{\max}).$$

In addition, if $T_{\max} < \infty$, then

$$\limsup_{t \nearrow T_{\max}} \left(\|u(t)\|_{L^\infty(0,1)} + \|v(t)\|_{W^{1,\infty}(0,1)} \right) = \infty.$$

By virtue of the conservation of the total mass $\|u\|_{L^1(0,1)}$, we can get the following regularity estimates by the semigroup theory.

Lemma 2.4. *There exists some constant $M = M(\|u_0\|_{L^1(0,1)}, p, \|v_0\|_{L^p(0,1)}) > 0$ such that*

$$\sup_{t \in [0,T_{\max})} \|v\|_{L^p(0,1)} \leq M,$$

where $p \in [1,\infty)$.

Finally, let us recall some facts concerning the well-known Lyapunov functional. In the presentation we refer to [7, Lemma 4, Lemma 5]. The following functional $L(u,v) := \int_0^1 b(u) - \int_0^1 uv + 1/2\,\|v\|_{H^1(0,1)}^2$ satisfies

$$\frac{d}{dt} L(u(t), v(t)) = -\int_0^1 v_t^2 - \int_0^1 u|(b'(u) - v)_x|^2,$$

where $b \in C^2(0,\infty)$ is such that $b''(r) = \frac{a(r)}{r}$ for $r > 0$ and $b(1) = b'(1) = 0$.

Moreover, due to the inequality

$$\int_0^1 uv \leq \|v\|_{L^\infty(0,1)} \|u\|_{L^1(0,1)} \leq 1/2 \left(\int_0^1 u_0 \right)^2 + 1/2\,\|v\|_{H^1(0,1)}^2,$$

we see that L is bounded from below and so in particular there exists $C > 0$ such that for any $t < T_{\max}$

$$\int_0^t \int_0^1 (v_t(x,s))^2\,dxds \leq C. \tag{2.1}$$

3. New Lyapunov-like functional

In this section we construct a functional associated to the problem (1.1). It does not decrease along the trajectories, so it is not a classical Lyapunov functional. However, along the trajectories, we control its growth thanks to the information delivered by the Lyapunov functional L. Then we are able to derive the required estimates.

The functional $\mathcal{F}$ we introduce seems to be the main novelty of our paper. It is a new functional in the context of chemotaxis, yielding quite strong estimates. Unfortunately, it seems to us, that it works only in the one-dimensional setting. Let us discuss its connection to the Lyapunov functional which appears in [9]. In [9] the so-called parabolic-elliptic Jäger-Luckhaus-type modification of the 1d Keller-Segel system is studied. A main tool in [9] is a change of variables. It simplifies chemotaxis equations considered there to a single parabolic equation and it is proved that this parabolic equation has a Lyapunov functional. The information coming from the obtained Lyapunov functional allows the authors to establish global existence. However, the change of variables is very sensitive to the fact that the version of the Keller-Segel studied in [9] is a parabolic-elliptic Jäger-Luckhaus-type modification. It cannot be prosecuted in our fully parabolic case. However, when translating back the Lyapunov functional in [9] to the original variables, we arrive at the functional which reminds us in its form of our functional (1.2). Still, there are essential differences, mainly stemming from the fact that (1.2) is not even a Lyapunov functional. Moreover, our derivation of (1.2) requires completely new computations. As a reader sees below, we make use of the crucial Lemma 2.1 which allows us to treat a test function $\mathcal{M}(u)$ in a tricky way. As we see in Lemma 3.1, proceeding with integration by parts in the derivation of (1.2) depends heavily on our observation in Lemma 2.1.

We proceed in several steps. Based on Lemma 2.1 we obtain Lemma 3.1. The detailed proof follows the lines of [6, Lemma 3.1]. Below we give a sketch of it.

Lemma 3.1. *Let* (u, v) *be a solution of* (1.1) *in* $(0,T)\times(0,1)$*. Then the following identity holds:*

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^1 \frac{(a(u))^2}{u}|\partial_x u|^2\right) + \int_0^1 ua(u)\left|\partial_x\left(\frac{a(u)}{u}\partial_x u\right)\right|^2 = \int_0^1 ua(u)\partial_x^2 v\cdot\partial_x\left(\frac{a(u)}{u}\partial_x u\right).$$

Proof. First, we multiply the first equation of (1.1) by $\mathcal{M}(u)$. Integration by parts yields

$$\int_0^1 u_t\mathcal{M}(u) = -\int_0^1\left(\frac{a(u)\partial_x u}{u} - \partial_x v\right)\cdot u\partial_x\mathcal{M}(u).$$

Utilizing the claim of Lemma 2.1, we arrive at (the full details can be found in [6, (3.1)])

$$\int_0^1 u_t\mathcal{M}(u) = \int_0^1 ua(u)\left|\partial_x\left(\frac{a(u)}{u}\partial_x u\right)\right|^2 - \int_0^1 ua(u)\partial_x^2 v\cdot\partial_x\left(\frac{a(u)\partial_x u}{u}\right).$$

On the other hand, one checks that

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^1\frac{(a(u))^2}{u}|\partial_x u|^2\right) = -\int_0^1 u_t\mathcal{M}(u).$$

Indeed,

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\int_0^1 \frac{(a(u))^2}{u}|\partial_x u|^2\right)\\
&\quad= \int_0^1 \frac{2a(u)a'(u)u-(a(u))^2}{2u^2}|\partial_x u|^2\partial_t u + \int_0^1 \frac{(a(u))^2}{u}\partial_x u \partial_x \partial_t u\\
&\quad= \int_0^1 \frac{2a(u)a'(u)u-(a(u))^2}{2u^2}|\partial_x u|^2\partial_t u - \int_0^1 \partial_x\left(\frac{(a(u))^2}{u}\partial_x u\right)\partial_t u\\
&\quad= -\int_0^1 \frac{2a(u)a'(u)u-(a(u))^2}{2u^2}|\partial_x u|^2\partial_t u - \int_0^1 \frac{(a(u))^2}{u}\partial_x^2 u \partial_t u,
\end{aligned}$$

and the claim follows. □

Next, integration by parts gives

$$\begin{aligned}
-\int_0^1 \partial_x\left(\frac{a(u)}{u}\partial_x u\right)u^2 a(u) &= \int_0^1 \partial_x(u^2 a(u))\frac{a(u)\partial_x u}{u}\\
&= \int_0^1 a(u)\partial_x u\cdot a(u)\partial_x u + \int_0^1 \partial_x(ua(u))a(u)\partial_x u. \qquad (3.1)
\end{aligned}$$

Owing to (3.1) and the following straightforward equality:

$$\int_0^1 u\partial_x v\cdot a(u)\partial_x u + \int_0^1 \partial_x(ua(u))u\partial_x v = \int_0^1 \partial_x\left(u^2 a(u)\right)\cdot\partial_x v,$$

testing the first equation of (1.1) by $\int_1^u a(r)\,dr + ua(u)$ and integrating over $(0,1)$, we arrive at

Lemma 3.2. *Let (u,v) be a solution of (1.1) in $(0,T)\times(0,1)$. Then the following identity holds:*

$$\frac{d}{dt}\left(\int_0^1 u\int_1^u a(r)\,dr\right) = \int_0^1 \partial_x\left(\frac{a(u)}{u}\partial_x u\right)u^2 a(u) + \int_0^1 \partial_x\left(u^2 a(u)\right)\cdot\partial_x v.$$

Finally, we introduce a crucial observation. For $\mathcal{F}$ and $\mathcal{D}$ given in (1.2) and (1.3) respectively, the following formula holds.

Lemma 3.3. *Let (u,v) be a solution of (1.1) in $(0,T)\times(0,1)$. The following identity is satisfied:*

$$\frac{d}{dt}\mathcal{F}(u(t)) + \mathcal{D}(u(t),v(t)) = \int_0^1 \frac{ua(u)(v+\partial_t v)^2}{4}.$$

Proof. Multiplying the second equation in (1.1) by $ua(u)\partial_x^2 v$ and integrating over $(0,1)$ we have

$$\begin{aligned}
\int_0^1 ua(u)\partial_t v\partial_x^2 v &= \int_0^1 ua(u)|\partial_x^2 v|^2 - \int_0^1 v\cdot ua(u)\partial_x^2 v + \int_0^1 u^2 a(u)\partial_x^2 v\\
&= \int_0^1 ua(u)|\partial_x^2 v|^2 - \int_0^1 v\cdot ua(u)\partial_x^2 v - \int_0^1 \partial_x\left(u^2 a(u)\right)\cdot\partial_x v. \qquad (3.2)
\end{aligned}$$

Combining Lemma 3.2 and (3.2) we get

$$\frac{d}{dt}\left(-\int_0^1 u\int_1^u a(r)\,dr\right)+\int_0^1 ua(u)|\partial_x^2 v|^2-\int_0^1 v\cdot ua(u)\partial_x^2 v-\int_0^1 ua(u)\partial_t v\partial_x^2 v$$
$$=-\int_0^1 \partial_x\left(\frac{a(u)}{u}\partial_x u\right)\cdot u^2 a(u),$$

and then using the second equation of (1.1) we see that

$$\frac{d}{dt}\left(-\int_0^1 u\int_1^u a(r)\,dr\right)+\int_0^1 ua(u)|\partial_x^2 v|^2-\int_0^1 v\cdot ua(u)\partial_x^2 v-\int_0^1 ua(u)\partial_t v\partial_x^2 v$$
$$=-\int_0^1 ua(u)\partial_x\left(\frac{a(u)}{u}\partial_x u\right)(\partial_t v-\partial_x^2 v+v).$$

Thus it follows from Lemma 3.1 that

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^1\frac{(a(u))^2}{u}|\partial_x u|^2-\int_0^1 u\int_1^u a(r)\,dr\right)+\int_0^1 ua(u)\left|\partial_x\left(\frac{a(u)}{u}\partial_x u\right)-\partial_x^2 v\right|^2$$
$$+\int_0^1 ua(u)(v+\partial_t v)\cdot\left(\partial_x\left(\frac{a(u)}{u}\partial_x u\right)-\partial_x^2 v\right)=0,$$

that is,

$$\begin{aligned}\frac{d}{dt}&\left(\frac{1}{2}\int_0^1\frac{(a(u))^2}{u}|\partial_x u|^2-\int_0^1 u\int_1^u a(r)\,dr\right)\\&+\int_0^1 ua(u)\left|\partial_x\left(\frac{a(u)}{u}\partial_x u\right)-\partial_x^2 v+\frac{v+\partial_t v}{2}\right|^2\\&\quad=\int_0^1\frac{ua(u)(v+\partial_t v)^2}{4},\end{aligned}$$

which is the desired inequality. □

From now on, we focus on the critical case $a(u)=1/(1+u)$. Then $\mathcal{F}$ defined in (1.2) takes the form

$$\mathcal{F}=\frac{1}{2}\int_0^1\frac{|\partial_x u|^2}{u(1+u)^2}-\int_0^1 u\log(1+u). \tag{3.3}$$

As mentioned before we can control the growth of $\mathcal{F}$.

Proposition 3.4. *Let (u,v) be a solution of (1.1) in $(0,T)\times(0,1)$ with nonlinear diffusion $a(u)=1/(1+u)$. Then there exists a constant $C>0$ such that for any $t<T_{\max}$*

$$\int_0^t\int_0^1\frac{ua(u)(v+v_t)^2}{4}\le C(t+1).$$

Proof. In view of the choice of a, $ua(u)\le 1$, moreover by Lemma 2.4 and (2.1) both v and v_t belong to $L^2(0,t;L^2(0,1))$ for any $t<T_{\max}$. □

4. Proof of the main theorem

By Lemma 3.3 and Proposition 3.4 we obtain the upper bound of the functional $\mathcal{F}(u)$. To derive the required estimates, we first establish the lower bound of the functional.

Proposition 4.1. *Let* (u, v) *be a solution of* (1.1) *with* $a(u) = 1/(1+u)$ *in* $(0, T) \times (0, 1)$ *and* $u_0 \geq 0$, $\int_0^1 u_0 = M > 0$. *The following estimates hold:*

$$\mathcal{F}(u) \geq \frac{1}{4} \int_0^1 \frac{|\partial_x u|^2}{u(1+u)^2} - C(M) \tag{4.1}$$

with some $C = C(M) > 0$, *and*

$$\int_0^1 u \log(1+u) \leq C(M) + 1/4 \left(\int_0^1 \frac{|\partial_x u|^2}{u(1+u)^2} \right)^{\frac{1}{2}}. \tag{4.2}$$

Proof.

$$\begin{aligned}
-\int_0^1 u(x) \log(1+u(x))\, dx &\geq -M \left\| \log(1+u) \right\|_{L^\infty(0,1)} \\
&\geq -M \left\| \log(1+u) \right\|_{W^{1,1}(0,1)} \\
&= -M \left\| \log(1+u) \right\|_{L^1(0,1)} - M \int_0^1 \left| \partial_x \left(\log(1+u) \right) \right| dx \\
&\geq -M^2 - M \int_0^1 \sqrt{u} \frac{|\partial_x u|}{(1+u)\sqrt{u}}\, dx \\
&\geq -M^2 - M^{3/2} \left(\int_0^1 \frac{|\partial_x u|^2}{u(1+u)^2}\, dx \right)^{1/2},
\end{aligned}$$

where in the last inequality we used the Cauchy–Schwarz inequality. From the above inequality we derive the lower bound for $\mathcal{F}(u)$

$$\begin{aligned}
\mathcal{F}(u) &\geq \frac{1}{4} \int_0^1 \frac{|\partial_x u|^2}{u(1+u)^2} + \frac{1}{4} \int_0^1 \frac{|\partial_x u|^2}{u(1+u)^2} - M^{\frac{3}{2}} \left(\int_0^1 \frac{|\partial_x u|^2}{u(1+u)^2} \right)^{\frac{1}{2}} \\
&\quad + M^3 - M^3 - M^2 \\
&= \frac{1}{4} \int_0^1 \frac{|\partial_x u|^2}{u(1+u)^2} + \frac{1}{4} \left(\left(\int_0^1 \frac{|\partial_x u|^2}{u(1+u)^2} \right)^{\frac{1}{2}} - 2M^{\frac{3}{2}} \right)^2 - M^3 - M^2,
\end{aligned}$$

which gives us in turn (4.1). Next, owing to the form of $\mathcal{F}$ in (3.3) and (4.1) we immediately see that

$$\int_0^1 u \log(1+u) \leq \left(\frac{1}{2} - \frac{1}{4} \right) \left(\int_0^1 \frac{|\partial_x u|^2}{u(1+u)^2} \right)^{\frac{1}{2}} + C(M),$$

which gives us (4.2). □

Below we obtain regularity estimates which depend on the time interval $T > 0$.

Proposition 4.2. *Let* (u, v) *be a solution of* (1.1) *with* $a(u) = 1/(1+u)$ *in* $(0, T) \times (0, 1)$. *Then there exists some constant* $C > 0$ *such that*

$$\int_0^1 u \log(1+u) + \int_0^1 \frac{|\partial_x u|^2}{u(1+u)^2} \leq C(1+T) \qquad \text{for all } t \in (0, T).$$

Proof. Due to Proposition 3.4 we have the existence of constant $C > 0$ such that

$$\int_0^t \int_0^1 \frac{ua(u)(v+v_t)^2}{4} \leq C(t+1),$$

it follows that for all $t \in (0,T)$,

$$\mathcal{F}(u(t)) \leq \mathcal{F}(u_0) + C(T+1).$$

Thus (4.1) implies that

$$\frac{1}{4}\int_0^1 \frac{|\partial_x u|^2}{u(1+u)^2} \leq \mathcal{F}(u(t)) + C(M) \leq \mathcal{F}(u_0) + C(M) + C(T+1)$$

with some $C(M) > 0$. Next, (4.2) gives the claim. □

Proof of Theorem 1.1. With the information in Proposition 4.2 as well as the inequality in Lemma 2.2 we can use [4, Lemma 3] to deduce that there exists some constant $C_T > 0$ such that

$$\int_0^1 (1+u)^3 \leq C_T \qquad \text{for all } t \in (0,T).$$

By the iterative argument (see [4, Proof of Theorem 1] or [14]) we have for any $p \in (1,\infty)$

$$\|1+u(t)\|_{L^p(0,1)} \leq C_T \qquad \text{for all } t \in (0,T)$$

with some $C_T > 0$. Finally by the standard regularity estimates for the quasilinear parabolic equation ([4, Proposition 3]) we can derive boundedness of u,

$$\|1+u(t)\|_{L^\infty(0,1)} \leq C_T \qquad \text{for all } t \in (0,T),$$

which implies global existence of solutions to (1.1). □

5. Appendix

In this section we extend the global existence result to the bounded positive nonlinearities a satisfying

$$\text{(5.1)} \quad a \in C([0,\infty)) \cap C^2(0,\infty), \;\; a \notin L^1(1,\infty),$$
$$\text{moreover there exists } C > 0 : ua(u) \leq C \;\text{ for any } u > 0.$$

The extension is worth investigating, since in view of [4, Theorem 2] any nonlinearity a integrable at infinity allows us to pick up an initial data so that solutions emanating from them blow up in finite time. Our result of this section, roughly speaking, shows that we have a sharp dichotomy, global existence for nonintegrable nonlinearities a and finite-time blowup for integrable ones.

Let us notice that in particular nonlinearities a of the form

$$a(u) = (1+u)^{-1}\left(\log(1+u)\right)^{-\alpha}, \; \alpha \in [0,1],$$

satisfy assumptions (5.1), so our result is applicable for such a's.

We have the following theorem.

Theorem 5.1. *Let* $a(u)$ *satisfy* (5.1) *and both* $u_0, v_0 \geq 0$ *satisfy assumptions in Lemma* 2.3. *Then the problem* (1.1) *has a unique classical positive solution, which exists globally in time.*

Proof. We know that a positive solution exists locally in time. In order to arrive at global existence, we need to show global bounds of L^∞ norm of u. We shall do it in three steps. The first step, being a crucial one, contains a bound from below on the functional $\mathcal{F}$. Consequently we obtain an estimate of $\int_0^1 u \int_1^u a(r)drdx$, which feeds a further procedure. In the second step we extend the Biler-Hebisch-Nadzieja inequality in Lemma 2.2 to the case of functions $\int_1^u a(r)dr$ (instead of special case log). The last step is a completely classical application of the first two steps yielding a bound of L^∞ norm of u as in [4, Lemma 3].

Step 1. There exists a constant $C > 0$ such that the following estimate holds true:

$$\int_0^1 u \int_1^u a(r)drdx + \int_0^1 \frac{a^2(u)|\partial_x u|^2}{u} \le C(1+T) \text{ for any } t < T. \tag{5.2}$$

Indeed,

$$\begin{aligned}
-\int_0^1 u(x)\int_1^u a(r)dr\,dx &\ge -M\left\|\int_1^u a(r)dr\right\|_{L^\infty(0,1)}\\
&\ge -M\left\|\int_1^u a(r)dr\right\|_{W^{1,1}(0,1)}\\
&= -M\left\|\int_1^u a(r)dr\right\|_{L^1(0,1)} - M\int_0^1\left|\partial_x\left(\int_1^u a(r)dr\right)\right|dx\\
&\ge -CM^2 - M\int_0^1 \sqrt{u}\frac{a(u)|\partial_x u|}{\sqrt{u}}\,dx\\
&\ge -CM^2 - M^{3/2}\left(\int_0^1 \frac{a^2(u)|\partial_x u|^2}{u}\,dx\right)^{1/2},
\end{aligned}$$

where in the last inequality we used the Cauchy–Schwarz inequality. From the above inequality we derive the lower bound for $\mathcal{F}(u)$ such that

$$\begin{aligned}
\mathcal{F}(u) \ge{}& \frac{1}{4}\int_0^1 \frac{a^2(u)|\partial_x u|^2}{u} + \frac{1}{4}\int_0^1 \frac{a^2(u)|\partial_x u|^2}{u}\\
&- M^{\frac{3}{2}}\left(\int_0^1 \frac{a^2(u)|\partial_x u|^2}{u}\right)^{\frac{1}{2}} + M^3 - M^3 - CM^2\\
={}& \frac{1}{4}\int_0^1 \frac{a^2(u)|\partial_x u|^2}{u} + \frac{1}{4}\left(\left(\int_0^1 \frac{a^2(u)|\partial_x u|^2}{u}\right)^{\frac{1}{2}} - 2M^{\frac{3}{2}}\right)^2 - M^3 - CM^2.
\end{aligned}$$

Hence

$$\mathcal{F}(u) \ge \frac{1}{4}\int_0^1 \frac{a^2(u)|\partial_x u|^2}{u} - C(M). \tag{5.3}$$

We next argue as in the end of the proof of Proposition 4.1 to see that

$$\int_0^1 u \int_1^u a(r)dr \le \left(\frac{1}{2} - \frac{1}{4}\right)\left(\int_0^1 \frac{a^2(u)|\partial_x u|^2}{u}\right)^{\frac{1}{2}} + C(M). \tag{5.4}$$

Next, by (5.1), Proposition 3.4 is still valid and we have the existence of constant

$C > 0$ such that

$$\int_0^t \int_0^1 \frac{ua(u)(v+v_t)^2}{4} \leq C(t+1),$$

and it follows from Lemma 3.3 that for $t \in (0,T)$,

$$\mathcal{F}(u(t)) \leq \mathcal{F}(u_0) + C(T+1).$$

Thus the lower bound (5.3) on $\mathcal{F}$ implies that

$$\frac{1}{4}\int_0^1 \frac{a^2(u)|\partial_x u|^2}{u} \leq \mathcal{F}(u(t)) + C(M) \leq \mathcal{F}(u_0) + C(M) + C(T+1)$$

with some $C(M) > 0$. Next, (5.4) gives the claim of Step 1.

Step 2. The following inequality, being a generalization of the one in [2], can be obtained exactly as in [4, Proposition 6].

Lemma 5.2. *Let $F : \mathbb{R}_+ \to \mathbb{R}$ be a nonnegative function such that $F(x) \to \infty$ when $x \to \infty$. Moreover, assume that there exists $N > 0$ such that F is increasing for $x > N$. For $w \in H^1(0,1)$ and any $\delta > 0$ there exists $C_\delta > 0$ such that*

$$\|w\|^4_{L^4(0,1)} \leq \delta \|w\|^2_{H^1(0,1)} \left(\int_0^1 |w|F(w)\right)^2 + C_\delta \|w\|_{L^1(0,1)}.$$

Notice that due to the nonintegrability of a at infinity, $|\int_1^u a(r)dr|$ can be taken as F. Moreover, we know that boundedness of $\int_1^u a(r)dr$ implies boundedness of its modulus.

Step 3. In this step, using (5.2) and Lemma 5.2 we estimate the L^p norms of u. Again, we start by multiplying the first equation in (1.1) by $(1+u)^2$ in exactly the same way as in [4, Lemma 3.1]. Replacing the use of Proposition 6 there by our Lemma 5.2, we get the bound

$$\int_0^1 (1+u)^3 dx \leq C_T.$$

Next we increase the integrability with the bootstrap procedure the same way as in [4].

□

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