PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 6, June 2018, Pages 2711–2722
http://dx.doi.org/10.1090/proc/13940
Article electronically published on February 16, 2018

NEW FAMILIES IN THE HOMOTOPY OF THE MOTIVIC SPHERE SPECTRUM

MICHAEL J. ANDREWS

(Communicated by Michael A. Mandell)

Dedicated to the memory of Amelia Perry.

Abstract. Using iterates of the Adams self map $v_1^4 : \Sigma^8 S/2 \longrightarrow S/2$ one can construct infinite families of elements in the stable homotopy groups of spheres, the v_1-periodic elements of order 2. In this paper we work motivically over $\mathbb{C}$ and construct a non-nilpotent self map $w_1^4 : \Sigma^{20,12} S/\eta \longrightarrow S/\eta$. We then construct some infinite families of elements in the homotopy of the motivic sphere spectrum, w_1-periodic elements killed by η.

1. Introduction

Calculating stable homotopy groups of spheres has been an active field of research for decades now. Their structure has been better understood as a consequence of chromatic homotopy theory. On the other hand, motivic homotopy theory is a relatively new field of research. The stable motivic homotopy category contains the spectra of topology and the schemes of algebraic geometry. It is an enrichment of the stable homotopy category. Even though the homotopy of the motivic sphere spectrum is strictly more complicated than that of the classical sphere spectrum, the motivic perspective provides insight into the classical problem. This is beautifully demonstrated in [9–11], where Dan Isaksen uses motivic calculations to correct and improve upon the classical calculations that came before. This paper observes that chromatic ideas can also be used in the stable motivic homotopy category and, moreover, that the story is likely to be a richer one.

The chromatic approach to computing the homotopy of a finite 2-local complex X is recursive.

(1) Find a non-nilpotent self map $f : X \longrightarrow \Sigma^{-d} X$ and compute $f^{-1}\pi_*(X)$.
(2) Attack the problem of computing the f-torsion elements in $\pi_*(X)$ by replacing X with X/f and going back to step 1.

Before [4] and [7] it was not known that one could always construct the requisite self maps. However, in [1] Adams constructed a non-nilpotent map $v_1^4 : S/2 \to \Sigma^{-8} S/2$, and this gave the first hint that the above procedure is, in fact, implementable. One might say that Adams' work gave birth to chromatic homotopy theory.

Received by the editors June 29, 2017, and, in revised form, August 20, 2017.
2010 *Mathematics Subject Classification.* Primary 55Q45, 55Q51, 55T15, 14F42.
Key words and phrases. Non-nilpotent self map, motivic self map, motivic Adams-Novikov spectral sequence.

The power of Adams' self map is that it gives rise to infinite families in the stable homotopy groups of spheres. Let's recall how one obtains such families. We have elements of order 2,

$$\eta \in \pi_1(S^0),\ \eta^2 \in \pi_2(S^0),\ \eta^3 \in \pi_3(S^0),\ 8\sigma \in \pi_7(S^0),\ \epsilon \in \pi_8(S^0),\ \text{and } \epsilon\eta \in \pi_9(S^0),$$

which lift to elements of $\pi_*(S/2)$. By composing with the maps

$$S/2 \xrightarrow{(v_1^4)^n} \Sigma^{-8n} S/2 \xrightarrow{\text{pinch}} S^{1-8n}$$

we obtain families of elements in the homotopy groups of spheres, the v_1-periodic elements of $\pi_*(S^0)$ of order 2.

The nilpotence theorem [4] says that non-nilpotent self maps are detected by the Brown-Peterson spectrum BP in the sense that they induce non-zero homomorphisms in BP-homology. Motivically over $\mathbb{C}$ at the prime 2, the motivic Brown-Peterson spectrum does not enjoy the same property: there is an element $\eta \in \pi_{1,1}(S^{0,0})$ which is non-nilpotent and yet $BP_{*,*}(\eta) = 0$. Although, from the classical perspective, this non-nilpotent map is unexpected, we can still follow the algorithm with f taken to be η, and it suggests that we try to compute $\eta^{-1}\pi_{*,*}(S^{0,0})$. This computation was carried out in [2], and the description is very simple:

$$\eta^{-1}\pi_{*,*}(S^{0,0}) = \mathbb{F}_2[\eta^{\pm 1}, \sigma, \mu_9]/(\eta\sigma^2).$$

Here, $\eta \in \pi_{1,1}(S^{0,0})$ and $\sigma \in \pi_{7,4}(S^{0,0})$ are motivic Hopf invariant one elements, classes that exist over any ground field before η is inverted [5, 17]. As long as we work over $\mathbb{C}$ with the 2-completed motivic sphere, μ_9 exists before η is inverted: it can be described by the Toda bracket $\langle 8\sigma, 2, \eta\rangle \in \pi_{9,5}(S^{0,0})$.

The algorithm then suggests that we try to find a non-nilpotent self map of S/η. The main result of this paper is that such a self map exists (Proposition 3.5 and Theorem 3.6),

$$w_1^4 : S/\eta \longrightarrow \Sigma^{-20,-12} S/\eta,$$

and we use this map to construct six infinite families in the homotopy groups of the 2-completed motivic sphere spectrum over $\mathbb{C}$ (Theorem 3.14). The construction of the infinite families is parallel to the story we recalled above. We have elements killed by η (see [11, p. 95, Table 1] and [10, p. 4]):

$$\nu \in \pi_{3,2}(S^{0,0}),\ \nu^2 \in \pi_{6,4}(S^{0,0}),\ \nu^3 \in \pi_{9,6}(S^{0,0}),\ \eta^2\eta_4 \in \pi_{18,11}(S^{0,0}),$$
$$\overline{\sigma} \in \pi_{19,11}(S^{0,0}), \quad \text{and} \quad \overline{\sigma}\nu \in \pi_{22,13}(S^{0,0}),$$

which lift to elements of $\pi_{*,*}(S/\eta)$. By composing with the maps

$$S/\eta \xrightarrow{(w_1^4)^n} \Sigma^{-20n,-12n} S/\eta \xrightarrow{\text{pinch}} S^{2-20n,1-12n}$$

we obtain families of elements in the homotopy groups of the motivic sphere spectrum:

$$g^n(\nu) \in \pi_{3+20n,2+12n}(S^{0,0}),\ g^n(\nu^2) \in \pi_{6+20n,4+12n}(S^{0,0}),$$
$$g^n(\nu^3) \in \pi_{9+20n,6+12n}(S^{0,0}),\ g^n(\eta^2\eta_4) \in \pi_{18+20n,11+12n}(S^{0,0}),$$
$$g^n(\overline{\sigma}) \in \pi_{19+20n,11+12n}(S^{0,0}),\ g^n(\overline{\sigma}\nu) \in \pi_{22+20n,13+12n}(S^{0,0}).$$

These are w_1-periodic elements of $\pi_{*,*}(S^{0,0})$ killed by η. Here, we have used the letter g in honor of the element with the same name in the May spectral sequence (see [11, p. 97, Table 4]). The d_4-differential that g supports inspired the construction of w_1^4.

We emphasize here that it is not automatic that these composites are non-trivial and we have to detect this somehow. This should not be unfamiliar. The driving force behind [16] was to detect the non-triviality of the classical γ-family. Our tool for detection is the motivic Adams-Novikov spectral sequence. The homotopy classes above are detected by permanent cycles which, if non-zero, cannot be boundaries for degree reasons. We are then left with showing that these elements are non-zero on the E_2-page and we do this by mapping to the classical Adams E_2-page.

One could try to continue with the algorithm and attempt to compute $w_1^{-1}\pi_{*,*}(S/\eta)$. It is likely that such a calculation would be at least as difficult as Mahowald's classical computation of $v_1^{-1}\pi_*(S/2)$, [12, 13, 15]. We could also try to find a self map of $S/(\eta, w_1^4)$. It is likely that finding a minimal self map would be at least as difficult as Behrens, Hill, Hopkins, and Mahowald's work in [3].

At this point we explain where our intuition about these self maps comes from. In [2] we make use of an algebraic Novikov spectral for computing the E_2-page of the classical Adams-Novikov spectral sequence. Its E_2-page is $H^*(P; Q)$. Here $P = \mathbb{F}_2[\xi_1^2, \xi_2^2, \ldots]$ is the Hopf subalgebra of squares in the dual Steenrod algebra A, and Q is the associated graded of BP_* under a filtration. In some sense, Q contains the classical chromatic story. We speculate that a new chromatic motivic story is contained in P. In [2] we inverted the element $h_0 = \{[\xi_1^2]\}$ to compute the α_1-localized Adams-Novikov E_2-page. Using the close relationship between the classical Adams-Novikov spectral sequence and its motivic analog [8] this enabled our calculation of $\eta^{-1}\pi_{*,*}(S^{0,0})$. Since η corresponds to ξ_1^2, Haynes Miller suggested that there may be other non-nilpotent self maps corresponding to $\xi_2^2, \xi_3^2, \ldots \in P$. Informally, we call a self map corresponding to ξ_{n+1}^2, w_n. w_1 would have motivic degree $(5, 3)$, and w_2 would have motivic degree $(13, 7)$. It is not luck that Adams found v_1^4 and we find w_1^4. Since we have a self map v_2^{32} [3], one might guess that we also have a self map w_2^{32}.

The results of this paper have already led to interesting questions. Adams showed that his map v_1^4 is non-nilpotent by proving that it induces an isomorphism on K-theory. In [4] and [7] the story was developed further, and the definition of a v_n-self map was given in terms of the Morava K-theories. Due to the delay in publication of this paper, we are already aware of the work of Bogdan Gheorghe [6], in which he gives the appropriate definitions of w_n-periodicity using $K(w_n)$-theories.

The construction of the self map w_1^4 is not difficult; it is similar to the construction of Adams' self map. Adams' map has the property that the composite

$$S^7 \longrightarrow \Sigma^7 S/2 \xrightarrow{v_1^4} \Sigma^{-1} S/2 \xrightarrow{\text{pinch}} S^0$$

is 8σ. We construct w_1^4 so that $\eta^2\eta_4$ can be factored as

$$S^{18,11} \longrightarrow \Sigma^{18,11} S/\eta \xrightarrow{w_1^4} \Sigma^{-2,-1} S/\eta \xrightarrow{\text{pinch}} S^{0,0}.$$

2. The motivic Adams-Novikov spectral sequence and the detection maps

Working motivically requires specifying a ground field; for us, this is always taken to be $\mathbb{C}$. Throughout the paper all spectra are assumed to be 2-complete. Our main calculational tool is the motivic Adams-Novikov spectral sequence (MANSS). We need it for the spectra $S^{0,0}$, S/η, and $\mathrm{End}(S/\eta)$, that is, the motivic sphere spectrum, the cofiber of $\eta : S^{1,1} \longrightarrow S^{0,0}$, and the endomorphism spectrum of S/η.

The motivic Adams-Novikov spectral sequence [8] is a convergent spectral sequence of the form

$$H^{s,t,w}(BP_*BP; BP_*(X)) \overset{s}{\Longrightarrow} \pi_{t-s,w}(X).$$

Here, BP is the motivic Brown-Peterson spectrum. We write BP_* instead of $BP_{*,*}$ in order to save space, and $H^*(BP_*BP; BP_*(X))$ is the cohomology of the Hopf algebroid BP_*BP with coefficients in $BP_*(X)$, which is the same thing as $\mathrm{Cotor}_{BP_*BP}(BP_*, BP_*(X))$. We recall that differentials in the MANSS interact with the s and t gradings as in the classical case, and they preserve the weight w, an additional feature in the motivic setting.

The $w = t/2$ slice of the motivic Adams-Novikov E_2-page for the sphere spectrum $S^{0,0}$ is plotted in the range $15 < t-s < 24$ in Figure 1. We can deduce the E_2-pages for S/η and $\mathrm{End}(S/\eta)$, up to extensions, in a smaller range using the cofibration sequences of (2.1) below.

Figure 1 is due to Ravenel [19]. Ignoring the naming of elements, his chart agrees with Isaksen's charts [10, 11] in the plotted range. We have chosen only to label the two elements which we will need to consider. We have labelled the first element $\beta_{4/3}$ as Ravenel does. We have labelled the second element z_{19} in accordance with Isaksen, [11, p. 132, Table 7].

Notation 2.1. We use i for "include" and c for "collapse" throughout this paper:

$$S^{0,0} \xrightarrow{\quad i \quad} S/\eta \xrightarrow{\quad c \quad} S^{2,1},$$
$$\Sigma^{-2,-1}S/\eta \xrightarrow{i=c^*} \mathrm{End}(S/\eta) \xrightarrow{c=i^*} S/\eta.$$

We never have to worry about differentials in our computations. For the most part, this is due to the existence of the following vanishing line and the corollary that follows.

Lemma 2.2. *When $X = S^{0,0}$, S/η, or $\mathrm{End}(S/\eta)$ we have $H^{s,t,w}(BP_*BP; BP_*(X)) \neq 0$ only when t is even and $w \leq t/2$.*

Proof. The result is true for $X = S^{0,0}$ by [8, (36)]. We have an element $\alpha_1 \in H^{1,2,1}(BP_*BP)$, and so multiplication by α_1 gives a map $H^*(BP_*BP) \longrightarrow \Sigma^{-1,-2,-1}H^*(BP_*BP)$.

The first cofibration sequence of (2.1) gives a short exact sequence:

$$0 \longrightarrow \mathrm{coker}\,\alpha_1 \longrightarrow H^*(BP_*BP; BP_*(S/\eta)) \longrightarrow \Sigma^{0,2,1}\mathrm{ker}\,\alpha_1 \longrightarrow 0.$$

Since $(\mathrm{coker}\,\alpha_1)^{s,t,w} \neq 0$ only when t is even and $w \leq t/2$, and the same is true for $\Sigma^{0,2,1}\mathrm{ker}\,\alpha_1$, the result holds when $X = S/\eta$. Similarly, we can use the second cofibration of the (2.1) sequence to show the result for $\mathrm{End}(S/\eta)$. $\square$

Corollary 2.3. *A non-zero element of $H^{s,2w,w}(BP_*BP; BP_*(X))$, where $X = S^{0,0}$, S/η, or $\mathrm{End}(S/\eta)$, cannot be the target of a differential in the MANSS.*

Proof. The differentials with the given group as the target can be enumerated:

$$d_{2r+1} : E_{2r+1}^{s-2r-1,2(w-r),w} \longrightarrow E_{2r+1}^{s,2w,w},\ r > 0.$$

But $E_{2r+1}^{s-2r-1,2(w-r),w} = 0$ since $w > w - r$. □

In this paper, we show many elements of the motivic Adams-Novikov E_2-page to be non-zero by mapping them to the classical Adams E_2-page. To define the so-called *detection map* we need to recall the structure of the Hopf algebroid (BP_*BP, BP_*) and the dual Steenrod algebra $(A, \mathbb{F}_2)$.

Notation 2.4. Recall that $BP_* = \mathbb{Z}_2[\tau, v_1, v_2, v_3 \ldots]$. Here $\mathbb{Z}_2$ denotes the 2-adics, τ has bigrading $(0, -1)$, and v_n has bigrading $(2^{n+1} - 2, 2^n - 1)$. $BP_*BP = BP_*[t_1, t_2, t_3, \ldots]$, where $|t_n| = |v_n|$ and there are structure maps making the pair (BP_*BP, BP_*) into a Hopf algebroid.

The dual Steenrod algebra is given as an algebra by $\mathbb{F}_2[\zeta_1, \zeta_2, \zeta_3, \ldots]$ where $|\zeta_n| = 2^n - 1$. Here ζ_n is the Hopf conjugate of the Milnor generator ξ_n, and the diagonal is given by the Milnor diagonal. We write $H^*(A; M)$ for $\mathrm{Cotor}_A(\mathbb{F}_2, M)$ when M is an A-comodule.

We can then define a map of Hopf algebroids.

Definition 2.5. Define $(BP_*BP, BP_*) \longrightarrow (A, \mathbb{F}_2)$ by demanding that τ, v_n, and t_n are mapped to 0, 0, and ζ_n, respectively. If we choose only to remember the weight of elements in (BP_*BP, BP_*), then this map preserves degree.

We also need maps between various homology groups, compatible with the map just defined. We note that as BP_*BP-comodules, $BP_*(S/\eta) = BP_*\langle 1, t_1\rangle$ and $BP_*(\mathrm{End}(S/\eta)) = BP_*(\Sigma^{-2,-1}S/\eta) \otimes^{\Delta}_{BP_*} BP_*(S/\eta)$.

Notation 2.6. Write $S/2$ for the classical mod 2 Moore spectrum and $H_*(-)$ for mod 2 homology.

We note that as A-comodules, $H_*(S/2) = \mathbb{F}_2\langle 1, \zeta_1\rangle$ and $H_*(\mathrm{End}(S/2)) = H_*(\Sigma^{-1}S/2) \otimes^{\Delta}_{\mathbb{F}_2} H_*(S/2)$.

Definition 2.7. Define $BP_*(S/\eta) \to H_*(S/2)$ and $BP_*(\mathrm{End}(S/\eta)) \to H_*(\mathrm{End}(S/\eta))$ by demanding that τ, v_n, 1, and t_1 are mapped to 0, 0, 1, and ζ_1, respectively.

We are now ready to define our detection maps.

Definition 2.8 (The detection maps)**.** The maps of Definition 2.5 and Definition 2.7 induce maps

$$d : H^*(BP_*BP) \longrightarrow H^*(A), \quad d : H^*(BP_*BP; BP_*(S/\eta)) \longrightarrow H^*(A; H_*(S/2)),$$
$$d : H^*(BP_*BP; BP_*(\mathrm{End}(S/\eta))) \longrightarrow H^*(A; H_*(\mathrm{End}(S/2))).$$

We label each map by d for "detection".

3. The self map, the homotopy classes, and the main results

In this section we state our main results, which are Proposition 3.5, Theorem 3.6 and Theorem 3.14. We define the homotopy classes which appear in Theorem 3.14. Doing so requires defining a number of auxilary homotopy classes. In section 4 we prove Theorem 3.14 by working at the algebraic level with the elements detecting

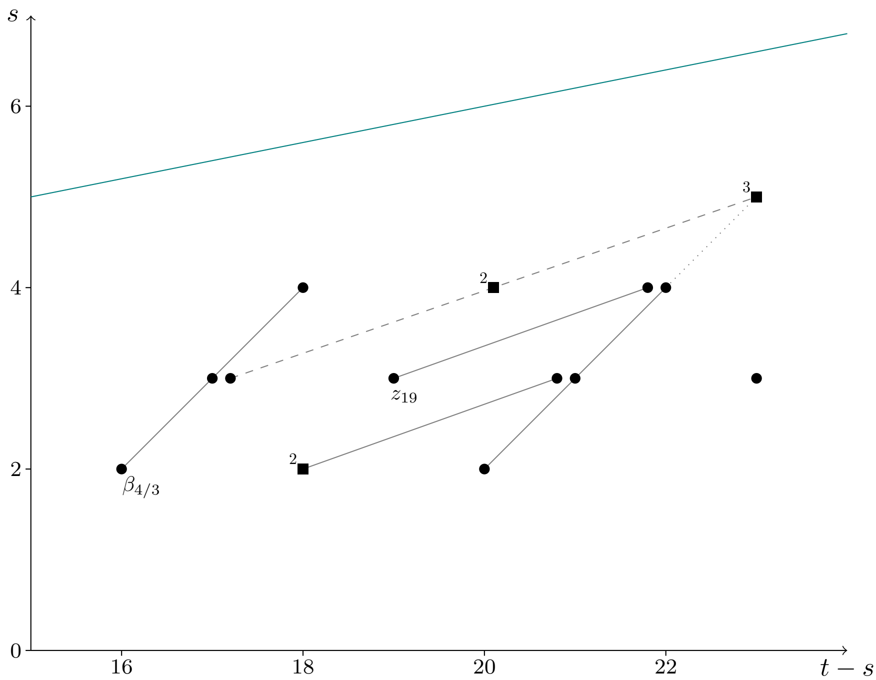

FIGURE 1. $H^{s,t,t/2}(BP_*BP)$ in the range $15 < t - s < 24$, minus the algebra generated by the α's. Round nodes indicate copies of $\mathbb{Z}/2$. Square nodes labelled with an n indicate copies of $\mathbb{Z}/2^n$. Lines indicate multiplication by α_1 and $\alpha_{2/2}$. Dashed lines indicate hitting twice a generator. Dotted lines indicate hitting four times a generator. Above the teal line, there are only α_1-free elements.

these classes. For this reason, we keep track of all the elements detecting our homotopy classes.

The first elements that one encounters in homotopy theory are the Hopf invariant one elements.

Definition 3.1. We write $\eta \in \pi_{1,1}(S^{0,0})$, $\nu \in \pi_{3,2}(S^{0,0})$, and $\sigma \in \pi_{7,4}(S^{0,0})$ for the motivic Hopf invariant one elements, [5]. These elements are detected by α_1, $\alpha_{2/2}$, and $\alpha_{4/4}$, respectively.

Mahowald discovered the η_j-family, [14]. These are classes which are detected by h_1h_j in the Adams spectral sequence, and, thus, they are defined up to higher Adams filtration. We need the motivic analog of η_4, but we are more precise, defining it without any indeterminacy. First, we have to address the class in the MANSS that detects it, and we record one of its properties.

Definition 3.2 ([19])**.** We have the following short exact sequences of BP_*BP-comodules:

$$0 \longrightarrow BP_* \xrightarrow{2} BP_* \longrightarrow BP_*/2 \longrightarrow 0,$$
$$0 \longrightarrow BP_*/2 \xrightarrow{v_1^3} BP_*/2 \longrightarrow BP_*/(2, v_1^3) \longrightarrow 0.$$

$\beta_{4/3}$ is the image of v_2^4 under the composite

$$H^{0,24,12}(BP_*BP; BP_*/(2, v_1^3)) \xrightarrow{\delta_1} H^{1,18,9}(BP_*BP; BP_*/2)$$

$$\xrightarrow{\delta_0} H^{2,18,9}(BP_*BP; BP_*).$$

Lemma 3.3 ([19]). *We have $\alpha_1^3\beta_{4/3} = 0$.*

Proof. From [2], we know that α_1-free elements of $H^*(BP_*BP; BP_*/2)$ are also v_1-free. Since $\delta_1(v_2^4)$ is v_1-torsion, it is α_1-torsion, and so $\beta_{4/3}$ is α_1-torsion. Because $\alpha_{8/5}$ is α_1-free, this means we cannot have $\alpha_1^3\beta_{4/3} = \alpha_1^4\alpha_{8/5}$, and so we must have $\alpha_1^3\beta_{4/3} = 0$. □

Definition 3.4. In the motivic Adams-Novikov spectral sequence $\beta_{4/3}$ detects a unique homotopy class. We call this homotopy class $\eta_4 \in \pi_{16,9}(S^{0,0})$.

The main result of this paper is that we have a non-nilpotent self map $w_1^4 : \Sigma^{20,12}S/\eta \longrightarrow S/\eta$. The next proposition gives a map; non-nilpotence is left for Theorem 3.6. We will use this map to construct the infinite families of Theorem 3.14.

Proposition 3.5. *There is an element $x \in H^{4,24,12}(BP_*BP; BP_*(\mathrm{End}(S/\eta)))$ which maps to $\alpha_1^2\beta_{4/3}$ under the collapse maps of* (2.1)*:*

$$H^{4,24,12}(BP_*BP; BP_*(\mathrm{End}(S/\eta))) \longrightarrow H^{4,24,12}(BP_*BP; BP_*(S/\eta))$$

$$\longrightarrow H^{4,22,11}(BP_*BP; BP_*(S^{0,0}))$$

$$x \longmapsto y \longmapsto \alpha_1^2\beta_{4/3}.$$

x is a permanent cycle in the MANSS for $\mathrm{End}(S/\eta)$ and so detects a map $w_1^4 : \Sigma^{20,12}S/\eta \longrightarrow S/\eta$. Moreover, $\eta^2\eta_4$ is the composite

$$S^{18,11} \xrightarrow{i} \Sigma^{18,11}S/\eta \xrightarrow{w_1^4} \Sigma^{-2,-1}S/\eta \xrightarrow{c} S^{0,0}.$$

Proof. Since $\alpha_1^3\beta_{4/3} = 0$, there exists y mapping to $\alpha_1^2\beta_{4/3}$. We see that $\alpha_1 y = 0$, and so there exists an x mapping to y.

All the target groups of the differentials emanating from

$$H^{4,24,12}(BP_*BP; BP_*(\mathrm{End}(S/\eta)))$$

in the motivic Adams-Novikov spectral sequence are zero, and so x is a permanent cycle. Because x maps to $\alpha_1^2\beta_{4/3}$ and there are no elements of higher Novikov filtration in that stem and weight, we obtain the factorization of $\eta^2\eta_4$ in the proposition statement. □

One of our main results is the following theorem. We will postpone the proof until section 4.

Theorem 3.6. *$w_1^4 : \Sigma^{20,12}S/\eta \longrightarrow S/\eta$ is non-nilpotent.*

We will define one of the homotopy classes that we need by a Toda bracket. First, we must recall some relations in homotopy.

Lemma 3.7 (Isaksen)**.** *We have the following relations:* $\eta\nu = 0$, $\nu\sigma = 0$, *and* $\eta\sigma^2 = 0$.

Proof. In [5] it is proved that $\eta\nu = 0$ and $\nu\sigma = 0$. Moreover, $\eta\sigma^2 = 0$ holds classically [20], and this immediately implies the motivic version because there are no "exotic" classes in the 15-stem with weight 9, [11]. □

Definition 3.8. We define $\overline{\sigma} \in \pi_{19,11}(S^{0,0})$ by the Toda bracket $\langle \nu, \sigma, \eta\sigma \rangle$. There is no indeterminacy in this Toda bracket, and the class is non-zero, [11].

To see the element which detects $\overline{\sigma}$ in the motivic Adams-Novikov spectral sequence we make note of the following property.

Lemma 3.9 (Isaksen)**.** *We have the following relation:* $\eta\overline{\sigma} = 0$.

Proof. $\langle \eta, \nu, \sigma \rangle$ is defined and seen to be 0. Thus $\eta\overline{\sigma} = \eta\langle \nu, \sigma, \eta\sigma \rangle = \langle \eta, \nu, \sigma \rangle \eta\sigma = 0$. □

Corollary 3.10. $\overline{\sigma}$ *is the unique element detected by* z_{19}.

To define the infinite families of Theorem 3.14 we need to lift some homotopy classes in $\pi_{*,*}(S^{0,0})$ to $\pi_{*,*}(S/\eta)$. We also need to keep track of the elements detecting these classes. That is the purpose of the next two definitions.

Definition 3.11. We write $\widetilde{\nu} \in \pi_{5,3}(S/\eta)$ for a fixed choice of lift of $\nu \in \pi_{3,2}(S^{0,0})$ under the map $c : S/\eta \longrightarrow S^{2,1}$. This element has Novikov filtration one; we write $\widetilde{\alpha}_{2/2} \in H^{1,6,3}(BP_*BP; BP_*(S/\eta))$ for the element which detects it. $\widetilde{\alpha}_{2/2}$ lifts $\alpha_{2/2} \in H^{1,4,2}(BP_*BP)$.

Definition 3.12. We fix a lift $\widetilde{z}_{19} \in H^{3,24,12}(BP_*BP; BP_*(S/\eta))$ of $z_{19} \in H^{3,22,11}(BP_*BP)$ under the map $c : S/\eta \longrightarrow S^{2,1}$. This is a permanent cycle and detects a homotopy class which we call $\widetilde{\overline{\sigma}} \in \pi_{21,12}(S/\eta)$. $\widetilde{\overline{\sigma}}$ lifts $\overline{\sigma} \in \pi_{19,11}(S^{0,0})$.

We are ready to construct the homotopy classes of interest. We let Φ_n be the following composite:

$$S/\eta \xrightarrow{(w_1^4)^n} \Sigma^{-20n,-12n} S/\eta \xrightarrow{\quad c \quad} S^{2-20n,1-12n}.$$

Recall the inclusion $i : S^{0,0} \longrightarrow S/\eta$ and that $\pi_{*,*}(S/\eta)$ is a $\pi_{*,*}(S^{0,0})$-module.

Definition 3.13. For $n \geq 0$, we define

$$g^n(\nu) \in \pi_{3+20n,2+12n}(S^{0,0}),\ g^n(\nu^2) \in \pi_{6+20n,4+12n}(S^{0,0}),$$
$$g^n(\nu^3) \in \pi_{9+20n,6+12n}(S^{0,0}),\ g^n(\eta^2\eta_4) \in \pi_{18+20n,11+12n}(S^{0,0}),$$
$$g^n(\overline{\sigma}) \in \pi_{19+20n,11+12n}(S^{0,0}),\ g^n(\overline{\sigma}\nu) \in \pi_{22+20n,13+12n}(S^{0,0})$$

by

$$g^n(\nu) = (\Phi_n)_*(\widetilde{\nu}),\ g^n(\nu^2) = (\Phi_n)_*(\widetilde{\nu}\nu),\ g^n(\nu^3) = (\Phi_n)_*(\widetilde{\nu}\nu^2),$$
$$g^n(\eta^2\eta_4) = (\Phi_{n+1})_*(i),\ g^n(\overline{\sigma}) = (\Phi_n)_*(\widetilde{\overline{\sigma}}),\ g^n(\overline{\sigma}\nu) = (\Phi_n)_*(\widetilde{\overline{\sigma}}\nu).$$

One of our main results is the following theorem. We will postpone the proof until section 4.

Theorem 3.14. *The homotopy classes* $g^n(\nu)$, $g^n(\nu^2)$, $g^n(\nu^3)$, $g^n(\eta^2\eta_4)$, $g^n(\overline{\sigma})$, $g^n(\overline{\sigma}\nu)$ *are non-zero; i.e., they are* w_1*-periodic.*

Proving the theorem comes down to algebra, and we can make the analogous construction algebraically. We have a map $\mathrm{End}(S/\eta) \wedge S/\eta = \mathrm{Hom}(S/\eta, S/\eta) \wedge \mathrm{Hom}(S^0, S/\eta) \longrightarrow \mathrm{Hom}(S^0, S/\eta) = S/\eta$ given by composition. Let φ be the composite

$$H^*(BP_*BP; BP_*(\mathrm{End}(S/\eta))) \otimes^{\Delta}_{BP_*} H^*(BP_*BP; BP_*(S/\eta))$$
$$\xrightarrow{\text{composition}} H^*(BP_*BP; BP_*(S/\eta))$$
$$\xrightarrow{c} H^*(BP_*BP; BP_*(S^{0,0}))$$

and recall that $H^*(BP_*BP; BP_*(S/\eta))$ is an $H^*(BP_*BP)$-module. Recall, also, the element x of Proposition 3.5.

Definition 3.15. For $n \geq 0$, we define

$$g^n(\alpha_{2/2}) = \varphi(x^n \otimes \widetilde{\alpha}_{2/2}),\ g^n(\alpha_{2/2}^2) = \varphi(x^n \otimes \widetilde{\alpha}_{2/2}\alpha_{2/2}),\ g^n(\alpha_{2/2}^3) = \varphi(x^n \otimes \widetilde{\alpha}_{2/2}\alpha_{2/2}^2),$$
$$g^n(\alpha_1^2\beta_{4/3}) = \varphi(x^{n+1} \otimes 1),\ g^n(z_{19}) = \varphi(x^n \otimes \widetilde{z}_{19}),\ g^n(z_{19}\alpha_{2/2}) = \varphi(x^n \otimes \widetilde{z}_{19}\alpha_{2/2}).$$

The construction of these elements together with Moss's convergence theorem [18] tells us that we have the following result.

Lemma 3.16. *The elements*

$$g^n(\alpha_{2/2}),\ g^n(\alpha_{2/2}^2),\ g^n(\alpha_{2/2}^3),\ g^n(\alpha_1^2\beta_{4/3}),\ g^n(z_{19}),\ \text{and } g^n(z_{19}\alpha_{2/2})$$

detect $g^n(\nu)$, $g^n(\nu^2)$, $g^n(\nu^3)$, $g^n(\eta^2\eta_4)$, $g^n(\overline{\sigma})$, *and* $g^n(\overline{\sigma}\nu)$, *respectively.*

4. Proof of main results

In this section we prove Theorem 3.6 and Theorem 3.14. This comes down to analyzing the effect of the detection map (2.8) on the class x of Proposition 3.5 and the effect of the detection map on the classes of Definition 3.15.

In order to prove Theorem 3.6 we need the following two lemmas.

Lemma 4.1. *Let d denote the detection map of Definition* 2.8. *We have* $d(\alpha_1) = h_0$, $d(\alpha_{2/2}) = h_1$, *and* $d(\alpha_{4/4}) = h_2$.

Proof. One can compute directly with cocycle representatives in $\Omega^*(BP_*BP)$. For instance, $\alpha_{4/4}$ is represented by $5[t_1^4] - 2[t_1t_2] + 9v_1[t_1^3] - v_1[t_2] + 7v_1^2[t_1^2] + 2v_1^3[t_1] - v_2[t_1]$. ☐

Lemma 4.2. *Under the detection map we have* $d(\beta_{4/3}) = h_0h_3$.

Proof. We can compute directly with cocycle representatives in $\Omega^*(BP_*BP; BP_*/2)$ and $\Omega^*(BP_*BP)$. The differential on v_2^4 is $v_1^4[t_1^8] + v_1^8[t_1^4] \pmod 2$, and so $\delta_1(v_2^4)$ is represented by $v_1[t_1^8] + v_1^5[t_1^4]$. If we apply the differential to $v_1[t_1^8] + v_1^5[t_1^4]$, divide by 2, and evaluate mod $(2, v_1)$, we obtain $[t_1|t_1^8]$. Thus $d(\beta_{4/3})$ is represented by $[\zeta_1|\zeta_1^8]$, and we are done. ☐

Now we address Theorem 3.6 even though it will follow, independently, as a corollary of Theorem 3.14. The key is to recall how Adams' self map $v_1^4 : \Sigma^8 S/2 \longrightarrow S/2$ is detected in the classical Adams spectral sequence. We have the following cofibration sequences:

(4.3)
$$S^0 \xrightarrow{i} S/2 \xrightarrow{c} S^1,$$
$$\Sigma^{-1}S/2 \xrightarrow{i=c^*} \mathrm{End}(S/2) \xrightarrow{c=i^*} S/2.$$

Proposition 4.4. *There exists a unique non-zero element*

$$\overline{x} \in H^{4,12}(A; H_*(\mathrm{End}(S/2))).$$

It maps to $h_0^3 h_3$ *under the collapse maps of* (4.3) *and is a permanent cycle in the Adams spectral sequence for* $\mathrm{End}(S/2)$ *detecting* $v_1^4 : \Sigma^8 S/2 \longrightarrow S/2$*:*

$$\begin{array}{ccccc} H^{4,12}(A; H_*(\mathrm{End}(S/2))) & \xrightarrow{c=i^*} & H^{4,12}(A; H_*(S/2)) & \xrightarrow{c} & H^{4,11}(A; H_*(S^0)), \\ \overline{x} & \longmapsto & & \longrightarrow & h_0^3 h_3. \end{array}$$

Moreover, $\overline{x}$ *is non-nilpotent.*

We now prove Theorem 3.6 by proving the following corollary.

Corollary 4.5. *The element* $x \in H^{4,24,12}(BP_*BP; BP_*(\mathrm{End}(S/\eta)))$ *of Lemma* 3.5 *is non-nilpotent, and so* w_1^4 *is non-nilpotent.*

Proof. We consider the following diagram in which the horizontal maps are obtained by applying the appropriate two collapse maps ((2.1) and (4.3)) and the vertical maps are detection maps (2.8). It is straightforward to see that this diagram commutes:

$$\begin{array}{ccc} H^{4,24,12}(BP_*BP; BP_*(\mathrm{End}(S/\eta))) & \longrightarrow & H^{4,22,11}(BP_*BP) \\ \downarrow d & & \downarrow d \\ H^{4,12}(A; H_*(\mathrm{End}(S/2))) & \longrightarrow & H^{4,11}(A) \end{array}$$

Start with x. We chose x so that it maps right to $\alpha_1^2\beta_{4/3}$ and we know $d(\alpha_1^2\beta_{4/3}) = h_0^3 h_3$ by Lemmas 4.1 and 4.2. So $d(x)$ gives a lift of $h_0^3 h_3$, but, by Proposition 4.4, $\overline{x}$ is the unique such lift, so $d(x) = \overline{x}$. Since $\overline{x}$ is non-nilpotent, x is non-nilpotent. Moreover, Corollary 2.3 tells us that no power of x can ever be hit by a differential. We deduce that w_1^4 is non-nilpotent. □

In order to prove Theorem 3.14 we need the following lemma.

Lemma 4.6. *Under the detection map of Definition* 2.8 *we have* $d(z_{19}) = c_0$.

Proof. We note the Massey product $\langle h_1, h_2, h_0 h_2\rangle$ has zero indeterminacy and defines c_0, [11]. Since $\nu\sigma = 0$ and $\eta\sigma^2 = 0$, we have $\alpha_{2/2}\alpha_{4/4} = 0$ and $\alpha_1\alpha_{4/4}^2 = 0$. This means that $\langle \alpha_{2/2}, \alpha_{4/4}, \alpha_1\alpha_{4/4}\rangle$ is defined and its elements give a lift for c_0. The only elements in the correct trigrading to lift c_0 are linear combinations of $\alpha_1^2\alpha_9$ and z_{19}. Since α_9 maps to zero, z_{19} must map to c_0. □

We are now ready to prove Theorem 3.14.

Proof of Theorem 3.14. By Lemma 3.16 and Corollary 2.3 we see that it is enough to prove that each of the following elements is non-zero in $H^*(BP_*BP)$:

$$g^n(\alpha_{2/2}),\ g^n(\alpha_{2/2}^2),\ g^n(\alpha_{2/2}^3),\ g^n(\alpha_1^2\beta_{4/3}),\ g^n(z_{19}),\ \text{and } g^n(z_{19}\alpha_{2/2}). \tag{4.7}$$

We do this by mapping to $H^*(A)$ using the detection map $d : H^*(BP_*BP) \longrightarrow H^*(A)$ of Definition 2.8. In the case $n = 0$, they map to

$$h_1, h_1^2, h_1^3, h_0^3 h_3, c_0, c_0 h_1$$

by Lemmas 4.1, 4.2, and 4.6, and so we're done.

Denote by P the Adams periodicity operator $\langle h_0^3 h_3, h_0, -\rangle$. It is known [1] that the following elements are non-zero for all n:

$$P^n(h_1),\ P^n(h_1^2),\ P^n(h_1^3),\ P^n(h_0^3 h_3),\ P^n(c_0),\ P^n(c_0 h_1). \tag{4.8}$$

Thus, it is enough to prove that the elements of (4.7) are mapped under d to the elements of (4.8), respectively. Since the proofs are identical, we will only prove that $g^n(\alpha_{2/2})$ maps to $P^n(h_1)$. We proceed by induction on n. We take as the inductive hypothesis that $g^{n-1}(\alpha_{2/2})$ maps to $P^{n-1}(h_1)$. The definition of $g^n(\alpha_{2/2})$ (3.15) gives

$$g^n(\alpha_{2/2}) \in \langle \alpha_1^2 \beta_{4/3}, \alpha_1, g^{n-1}(\alpha_{2/2})\rangle.$$

Using Lemmas 4.1 and 4.2, we see that $g^n(\alpha_{2/2})$ maps to $P^n(h_1)$, which completes the inductive step. □

Acknowledgments

I would like to thank Haynes Miller for his bold suggestion that such self maps exist. I would like to thank Zhouli Xu for conversations we had at the University of Chicago after the existence of w_1^4 was proved. Upon showing him my result, he showed great enthusiasm, and within an hour of talking we had the six infinite families discussed in this paper. Thank you to Dan Isaksen for his beautiful spectral sequence charts [9–11] and for a Toda bracket argument he provided. More importantly, I am grateful to him for spotting a flawed argument in what I thought was a simpler proof of these results.

References

[1] J. F. Adams, *On the groups $J(X)$. IV*, Topology **5** (1966), 21–71, DOI 10.1016/0040-9383(66)90004-8. MR0198470

[2] M. J. Andrews and H. R. Miller, *Inverting the Hopf map in the Adams-Novikov spectral sequence*, http://math.ucla.edu/~mjandr/Inverting_the_Hopf_map.pdf, 2014. Accessed: 2017-08-18. To appear in Topology.

[3] M. Behrens, M. Hill, M. J. Hopkins, and M. Mahowald, *On the existence of a v_2^{32}-self map on $M(1,4)$ at the prime 2*, Homology Homotopy Appl. **10** (2008), no. 3, 45–84. MR2475617

[4] Ethan S. Devinatz, Michael J. Hopkins, and Jeffrey H. Smith, *Nilpotence and stable homotopy theory. I*, Ann. of Math. (2) **128** (1988), no. 2, 207–241, DOI 10.2307/1971440. MR960945

[5] Daniel Dugger and Daniel C. Isaksen, *Motivic Hopf elements and relations*, New York J. Math. **19** (2013), 823–871. MR3141814

[6] B. Gheorghe, *Exotic motivic periodicities*, preprint, 2017, privately communicated.

[7] Michael J. Hopkins and Jeffrey H. Smith, *Nilpotence and stable homotopy theory. II*, Ann. of Math. (2) **148** (1998), no. 1, 1–49, DOI 10.2307/120991. MR1652975

[8] Po Hu, Igor Kriz, and Kyle Ormsby, *Remarks on motivic homotopy theory over algebraically closed fields*, J. K-Theory **7** (2011), no. 1, 55–89, DOI 10.1017/is010001012jkt098. MR2774158

[9] D. C. Isaksen, *Classical and motivic Adams charts*, preprint, arXiv:1401.4983 (2014).

[10] ———, *Classical and motivic Adams-Novikov charts*, preprint, arXiv:1408.0248 (2014).

[11] ———, *Stable stems*, preprint, arXiv:1407.8418 (2014).

[12] Mark Mahowald, *The order of the image of the J-homomorphisms*, Bull. Amer. Math. Soc. **76** (1970), 1310–1313, DOI 10.1090/S0002-9904-1970-12656-8. MR0270369

[13] Mark Mahowald, *Description homotopy of the elements in the image of the J-homomorphism*, Manifolds—Tokyo 1973 (Proc. Internat. Conf., Tokyo, 1973), Univ. Tokyo Press, Tokyo, 1975, pp. 255–263. MR0370577

[14] Mark Mahowald, *A new infinite family in ${}_2\pi_*{}^s$*, Topology **16** (1977), no. 3, 249–256, DOI 10.1016/0040-9383(77)90005-2. MR0445498

[15] Mark Mahowald, *bo-resolutions*, Pacific J. Math. **92** (1981), no. 2, 365–383. MR618072

[16] Haynes R. Miller, Douglas C. Ravenel, and W. Stephen Wilson, *Periodic phenomena in the Adams-Novikov spectral sequence*, Ann. of Math. (2) **106** (1977), no. 3, 469–516, DOI 10.2307/1971064. MR0458423
[17] Fabien Morel, *An introduction to $\mathbb{A}^1$-homotopy theory*, Contemporary developments in algebraic K-theory, ICTP Lect. Notes, XV, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004, pp. 357–441. MR2175638
[18] R. Michael F. Moss, *Secondary compositions and the Adams spectral sequence*, Math. Z. **115** (1970), 283–310, DOI 10.1007/BF01129978. MR0266216
[19] Douglas C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, vol. 121, Academic Press, Inc., Orlando, FL, 1986. MR860042
[20] Hirosi Toda, *Composition methods in homotopy groups of spheres*, Annals of Mathematics Studies, No. 49, Princeton University Press, Princeton, N.J., 1962. MR0143217

Department of Mathematics, University of California Los Angeles, Box 951555, Los Angeles, California 90095-1555
Email address: mjandr@math.ucla.edu