

UNIQUE CONTINUATION FOR THE SCHRÖDINGER EQUATION WITH GRADIENT TERM

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ABSTRACT. We obtain a unique continuation result for the differential inequality $|(i\partial_t + \Delta)u| \leq |Vu| + |W \cdot \nabla u|$ by establishing L^2 Carleman estimates. Here, V is a scalar function and W is a vector function, which may be time-dependent or time-independent. As a consequence, we give a similar result for the magnetic Schrödinger equation.

1. INTRODUCTION

Given a partial differential equation or inequality in \mathbb{R}^n , we say that it has the unique continuation property from a nonempty open subset $\Omega \subset \mathbb{R}^n$ if its solution cannot vanish in Ω without being identically zero. Historically, such property was studied in connection with the uniqueness of the Cauchy problem. The major method for studying the property is based on so-called Carleman estimates which are weighted a priori estimates for the solution. The original idea goes back to Carleman [2], who first introduced it to obtain the property for the differential inequality $|\Delta u| \leq |V(x)u|$ with $V \in L^\infty(\mathbb{R}^2)$ concerning the stationary Schrödinger equation. Since then, the method has played a central role in almost all subsequent developments either for unbounded potentials V or fractional equations (see [3, 4, 11, 13, 20, 21] and the references therein). For the equation involving gradient term,

$$(1.1) \quad |\Delta u| \leq |V(x)u| + |W(x) \cdot \nabla u|,$$

see [16, 23, 24] and the references therein.

Now it can be asked whether the property is shared by the differential inequality

$$|i\partial_t u + \Delta u| \leq |Vu|$$

concerning the (time-dependent) Schrödinger equation which describes how the wave function u of a nonrelativistic quantum mechanical system with a potential V evolves over time. It would be an interesting problem to prove the property for this equation since such property can be viewed as one of the nonlocalization properties of the wave function which are an important issue in quantum mechanics. The unique continuation for this time-dependent case has been studied for decades from a half-space Ω in \mathbb{R}^{n+1} when the potential V is time-dependent ([14, 15, 17])

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or time-independent ([19, 22]). These results were based on Carleman estimates of the form

$$(1.2) \quad \|e^{\beta\phi(x,t)} f\|_{\mathcal{B}} \leq C \|e^{\beta\phi(x,t)} (i\partial_t + \Delta) f\|_{\mathcal{B}'},$$

where β is a real parameter, ϕ is a suitable weight function, and $\mathcal{B}, \mathcal{B}'$ are suitable Banach spaces of functions on \mathbb{R}^{n+1} .

The new point in this paper is that we allow the gradient term ∇u in the differential inequality

$$(1.3) \quad |i\partial_t u + \Delta u| \leq |Vu| + |W \cdot \nabla u|$$

in the spirit of (1.1). Here, V is a scalar function and W is a vector function, which may be time-dependent or time-independent. From the physical point of view, a motivation behind this form also comes from the following magnetic Schrödinger equation

$$(1.4) \quad i\partial_t u + \Delta_{\vec{A}} u = Vu$$

which describes the behavior of quantum mechanical systems in the presence of a magnetic field. Here, $\vec{A} = (A_1, \dots, A_n)$ is a magnetic vector potential and $\Delta_{\vec{A}}$ denotes the magnetic Laplacian defined by

$$\begin{aligned} \Delta_{\vec{A}} &= \sum_j (\partial_j + iA_j)^2 \\ &= \Delta + 2i\vec{A} \cdot \nabla + i\operatorname{div}\vec{A} - |\vec{A}|^2. \end{aligned}$$

Replacing V and W in (1.3) with $-i\operatorname{div}\vec{A} + |\vec{A}|^2 + V$ and $-2i\vec{A}$, respectively, we can reduce the unique continuation problem for the magnetic case to the one for the form (1.3).

To obtain unique continuation for (1.3), we find suitable weights ϕ and Banach spaces $\mathcal{B}, \mathcal{B}'$ to allow the gradient term in the left-hand side of (1.2), as follows (see Proposition 2.1 for details):

$$\beta \|e^{\beta\phi(x,t)} f\|_{\mathcal{B}} + \beta^{1/2} \|e^{\beta\phi(x,t)} |\nabla f|\|_{\mathcal{B}} \leq C \|e^{\beta\phi(x,t)} (i\partial_t + \Delta) f\|_{\mathcal{B}'},$$

where $\phi(x, t) = ct + |x|^2$ and $\mathcal{B} = \mathcal{B}' = L^2_{x,t}$. Making use of this estimate, we obtain the following unique continuation theorem which says that if the solution of (1.3) is supported on the inside of a paraboloid in \mathbb{R}^{n+1} , then it must vanish on all of \mathbb{R}^{n+1} .

Theorem 1.1. *Let $V \in L^\infty$ and let $|W| \in L^\infty$. Suppose that $u \in H_t^1 \cap H_x^2$ is a solution of (1.3) which vanishes in the outside of a paraboloid given by*

$$(1.5) \quad \{(x, t) \in \mathbb{R}^{n+1} : c(t - t_0) + |x - x_0|^2 > 0\},$$

where $c \in \mathbb{R} \setminus \{0\}$ and $(x_0, t_0) \in \mathbb{R}^{n+1}$. Then u is identically zero in \mathbb{R}^{n+1} . Here, H_t^1 denotes the space of functions whose derivatives up to order 1, with respect to the time variable t , belong to L^2 . H_x^2 is similarly defined as H_t^1 .

Remark 1.2. The assumption $u \in H_t^1 \cap H_x^2$ can be relaxed to

$$(1.6) \quad u, \partial_t u, \Delta u \in L_t^2(\mathbb{R}; L_x^2(|x - x_0|^2 \leq c(t_0 - t)))$$

because we are assuming $u = 0$ on the set $\{(x, t) \in \mathbb{R}^{n+1} : c(t - t_0) + |x - x_0|^2 > 0\}$. Since $u(x, t)$ vanishes at infinity for each t , it follows from integration by parts that ∇u also satisfies (1.6).

Regarding the condition (1.6), we give a remark that the solutions $u(x, t) = e^{it\Delta}u_0(x)$ to the free Schrödinger equation with the initial data $u_0 \in C_0^\infty(\mathbb{R}^n)$, $n > 2$, satisfy (1.6). Indeed, we consider the case $c < 0$ without loss of generality. Then we may consider $t \geq t_0$ only. Since $e^{it\Delta}u_0(x) \in L_x^2$ for each t , it is enough to show that

$$(1.7) \quad \int_M^\infty \int_{|x-x_0|^2 \leq c(t_0-t)} |e^{it\Delta}u_0(x)|^2 dx dt < \infty$$

for a sufficiently large $M > t_0$. But, using the following well-known decay

$$\sup_{x \in \mathbb{R}^n} |e^{it\Delta}u_0(x)| \lesssim |t|^{-n/2} \|u_0\|_{L^1(\mathbb{R}^n)},$$

the integral in (1.7) is bounded by

$$\begin{aligned} C \|u_0\|_{L^1(\mathbb{R}^n)}^2 \int_M^\infty (t - t_0)^{n/2} t^{-n} dt &\leq C \|u_0\|_{L^1(\mathbb{R}^n)}^2 \int_M^\infty t^{-n/2} dt \\ &\leq C \|u_0\|_{L^1(\mathbb{R}^n)}^2 \end{aligned}$$

if $n > 2$. Since $\partial_t e^{it\Delta}u_0 = e^{it\Delta}i\Delta u_0$ and $\Delta e^{it\Delta}u_0 = e^{it\Delta}\Delta u_0$, the condition (1.6) for these follows from the same argument with $\Delta u_0 \in L^1$.

As mentioned above, the theorem directly implies the following result for the magnetic case (1.4).

Corollary 1.3. *Let $V \in L^\infty$, let $|\vec{A}| \in L^\infty$, and let $\operatorname{div} \vec{A} \in L^\infty$. If $u \in H_t^1 \cap H_x^2$ is a solution of (1.4) which vanishes in the outside of a paraboloid given by (1.5), then u is identically zero in \mathbb{R}^{n+1} .*

There are related results for Schrödinger equations which describe the behavior of the solutions at two different times which ensure $u \equiv 0$. Since the Schrödinger equation is time reversible, it seems natural to consider this type of unique continuation from the behavior at two different time moments. Such results were first obtained by various authors ([6, 8, 12, 18, 25]) for Schrödinger equations of the form

$$(1.8) \quad (i\partial_t + \Delta)u = Vu + F(u, \bar{u}),$$

where $V(x, t)$ is a time-dependent potential and F is a nonlinear term.

For the $1 - D$ cubic Schrödinger equations, i.e., $V \equiv 0$, $F = \pm |u|^2 u$, $n = 1$ in (1.8), Zhang [25] showed that if $u = 0$ in the same semiline at two times t_0, t_1 , then $u \equiv 0$ in $\mathbb{R} \times [t_0, t_1]$. His proof is based on the inverse scattering theory. In the work of Kenig–Ponce–Vega [12], this result was completely extended to higher dimensions under the assumption that $u = 0$ in the complement of a cone with an opening $< \pi$ at two times. Key steps in their proof were energy estimates for the Fourier transform of the solution and the use of Isakov’s results [10] on local unique continuation. The size of the set on which the solution vanishes at two times was improved by Ionescu and Kenig [8] to the case of semispaces, i.e., cones with an opening $= \pi$. Latter, Escauriaza–Kenig–Ponce–Vega [6]¹ showed that it suffices to assume that the solution decays sufficiently fast at two times to have unique continuation results.

¹This approach was motivated by a deep relationship between the unique continuation and uncertainty principles for the Fourier transform. As a consequence, several remarkable results were later obtained in both cases, linear Schrödinger equations with variable coefficients and nonlinear ones (see, for example, [1] and the references therein). See also a good survey paper [7] explaining several results on the subject.

These results were extended in [5, 9] to the case where the nonlinear term F in (1.8) involves the gradient terms $\nabla u, \nabla \bar{u}$. More precisely, it was shown in [9] that if the solution vanishes in the complement of a ball at two times t_0, t_1 , then $u \equiv 0$ in $\mathbb{R}^n \times [t_0, t_1]$. In the spirit of [6], this support condition at two times was improved in [5] to the assumption that the solution decay sufficiently fast. By applying these works to each time interval $[n, n + 1]$, $n \in \mathbb{Z}$, particularly for the linear equation

$$(1.9) \quad i\partial_t u + \Delta u = Vu + W \cdot \nabla u,$$

one can show that if the solution to (1.9) vanishes in the outside of a paraboloid given by (1.5), then $u \equiv 0$ under the assumption that

$$(1.10) \quad V \in B_x^{2,\infty} L_t^\infty(\mathbb{R}^{n+1}), \quad |W| \in B_x^{1,\infty} L_t^\infty(\mathbb{R}^{n+1}),$$

where $B^{p,q}$, $1 \leq p, q \leq \infty$, are Banach spaces with the properties that $B^{p,p} = L^p$, $1 \leq p \leq \infty$, and $B^{p_1,q_1} \hookrightarrow B^{p_2,q_2}$ if $q_1 \geq q_2$ and $p_1 \leq p_2$. (See [5, 9] for details.) But in this paper, the assumption (1.10) is improved because

$$B_x^{p,\infty} L_t^\infty(\mathbb{R}^{n+1}) \hookrightarrow B_x^{\infty,\infty} L_t^\infty(\mathbb{R}^{n+1}) = L^\infty(\mathbb{R}^{n+1})$$

for $p = 1, 2$.

2. L^2 CARLEMAN ESTIMATE

In this section we obtain the following L^2 Carleman estimate which is a key ingredient for the proof of Theorem 1.1 in the next section.

Proposition 2.1. *Let $\beta > 0$ and let $c \in \mathbb{R}$. Then we have for $f \in C_0^\infty(\mathbb{R}^{n+1})$*

$$(2.1) \quad \beta \|e^{\beta(ct+|x|^2)} f\|_{L_{x,t}^2} + \beta^{\frac{1}{2}} \|e^{\beta(ct+|x|^2)} |\nabla f|\|_{L_{x,t}^2} \leq \|e^{\beta(ct+|x|^2)} (i\partial_t + \Delta) f\|_{L_{x,t}^2}.$$

Proof. To show (2.1), we first set $f = e^{-\beta(ct+|x|^2)} g$ and note that

$$\begin{aligned} \|e^{\beta(ct+|x|^2)} |\nabla f|\|_{L_{x,t}^2} &= \|e^{\beta(ct+|x|^2)} |e^{-\beta(ct+|x|^2)} (-2\beta g x + \nabla g)|\|_{L_{x,t}^2} \\ &\leq 2\beta \| |x|g \|_{L_{x,t}^2} + \| |\nabla g| \|_{L_{x,t}^2}. \end{aligned}$$

Hence it is enough to show that

$$(2.2) \quad \begin{aligned} \|e^{\beta(ct+|x|^2)} (i\partial_t + \Delta) e^{-\beta(ct+|x|^2)} g\|_{L_{x,t}^2} \\ \geq \beta \|g\|_{L_{x,t}^2} + 2\beta^{\frac{3}{2}} \| |x|g \|_{L_{x,t}^2} + \beta^{\frac{1}{2}} \| |\nabla g| \|_{L_{x,t}^2} \end{aligned}$$

for $g \in C_0^\infty(\mathbb{R}^{n+1})$.

By direct calculation, we see that

$$e^{\beta(ct+|x|^2)} (i\partial_t + \Delta) e^{-\beta(ct+|x|^2)} g = (i\partial_t + \Delta + 4\beta^2|x|^2 - 2n\beta - 4\beta x \cdot \nabla - i\beta c)g.$$

Let $A := i\partial_t + \Delta + 4\beta^2|x|^2$ and $B := 2n\beta + 4\beta x \cdot \nabla_x + i\beta c$ such that $A^* = A$ and $B^* = -B$. (Here A^* and B^* denote adjoint operators.) Then we get

$$\begin{aligned} \|e^{\beta(ct+|x|^2)} (i\partial_t + \Delta) e^{-\beta(ct+|x|^2)} g\|_{L_{x,t}^2}^2 &= \langle (A - B)g, (A - B)g \rangle_{L_{x,t}^2} \\ &= \langle Ag, Ag \rangle_{L_{x,t}^2} + \langle Bg, Bg \rangle_{L_{x,t}^2} + \langle (BA - AB)g, g \rangle_{L_{x,t}^2} \\ &\geq \langle (BA - AB)g, g \rangle_{L_{x,t}^2}. \end{aligned}$$

Since the only terms in A and B which are not commutative each other are $4\beta x \cdot \nabla$ and $(\Delta + 4\beta^2|x|^2)$, we see that

$$\begin{aligned} BA - AB &= (4\beta x \cdot \nabla)(\Delta + 4\beta^2|x|^2) - (\Delta + 4\beta^2|x|^2)(4\beta x \cdot \nabla) \\ &= 32\beta^3|x|^2 - 8\beta\Delta. \end{aligned}$$

Hence it follows that

$$\begin{aligned} \langle (BA - AB)g, g \rangle_{L^2_{x,t}} &= 32\beta^3 \langle |x|^2 g, g \rangle_{L^2_{x,t}} + 8\beta \langle -\Delta g, g \rangle_{L^2_{x,t}} \\ (2.3) \qquad \qquad \qquad &= 32\beta^3 \| |x|g \|^2_{L^2_{x,t}} + 8\beta \| |\nabla g| \|^2_{L^2_{x,t}}. \end{aligned}$$

Next we notice that

$$-\sum_{i=1}^n \frac{1}{2} x_i \frac{\partial}{\partial x_i} |g(x, t)|^2 = -\text{Re}(\nabla g \cdot \overline{xg}).$$

By integrating this over \mathbb{R}^{n+1} and then integrating by parts on the left-hand side, we see that

$$\begin{aligned} \frac{n}{2} \|g\|^2_{L^2_{x,t}} &= -\text{Re} \left(\int_{\mathbb{R}} \int_{\mathbb{R}^n} \nabla g \cdot \overline{xg} dx dt \right) \\ &\leq \left| \int_{\mathbb{R}} \int_{\mathbb{R}^n} \nabla g \cdot \overline{xg} dx dt \right| \\ &\leq \| |\nabla g| \|_{L^2_{x,t}} \| |x|g \|_{L^2_{x,t}}. \end{aligned}$$

Hence, by combining this inequality² and (2.3), we conclude that

$$\langle (BA - AB)g, g \rangle_{L^2_{x,t}} \geq (31\beta^3) \| |x|g \|^2_{L^2_{x,t}} + (7\beta) \| |\nabla g| \|^2_{L^2_{x,t}} + n\beta^2 \|g\|^2_{L^2_{x,t}},$$

since

$$2\beta^2 \| |\nabla g| \|_{L^2_{x,t}} \| |x|g \|_{L^2_{x,t}} \leq \beta^3 \| |\nabla g| \|^2_{L^2_{x,t}} + \beta \| |x|g \|^2_{L^2_{x,t}}.$$

This implies (2.2) because of $\sqrt{3}(a^2 + b^2 + c^2)^{1/2} \geq |a| + |b| + |c|$. Indeed,

$$\begin{aligned} &\| e^{\beta(ct+|x|^2)} (i\partial_t + \Delta) e^{-\beta(ct+|x|^2)} g \|_{L^2_{x,t}} \\ &= \langle (A - B)g, (A - B)g \rangle_{L^2_{x,t}}^{1/2} \\ &\geq \left(31\beta^3 \| |x|g \|^2_{L^2_{x,t}} + 7\beta \| |\nabla g| \|^2_{L^2_{x,t}} + n\beta^2 \|g\|^2_{L^2_{x,t}} \right)^{1/2} \\ &\geq \frac{1}{\sqrt{3}} \left(\sqrt{31\beta^3} \| |x|g \|_{L^2_{x,t}} + \sqrt{7\beta} \| |\nabla g| \|_{L^2_{x,t}} + \sqrt{n\beta^2} \|g\|_{L^2_{x,t}} \right) \\ &\geq \beta \|g\|_{L^2_{x,t}} + 2\beta^{\frac{3}{2}} \| |x|g \|_{L^2_{x,t}} + \beta^{\frac{1}{2}} \| |\nabla g| \|_{L^2_{x,t}}. \end{aligned}$$

□

²If we consider L^2_x instead of $L^2_{x,t}$, this inequality is indeed the Heisenberg uncertainty principle in n dimensions ($\| |\nabla g| \|_{L^2_x} = 2\pi \| |\xi\hat{g}| \|_{L^2_\xi}$).

3. PROOF OF THEOREM 1.1

This section is devoted to proving Theorem 1.1 using Proposition 2.1.

By translation we may first assume that $(x_0, t_0) = (0, 0)$ so that the solution u vanishes in the paraboloid $\{(x, t) \in \mathbb{R}^{n+1} : ct + |x|^2 > 0\}$. Now, from induction it suffices to show that $u = 0$ in the following set:

$$S = \{(x, t) \in \mathbb{R}^{n+1} : -1 < ct + |x|^2 \leq 0\}.$$

To show this, we make use of the Carleman estimate in Proposition 2.1.

Let $\psi : \mathbb{R}^{n+1} \rightarrow [0, \infty)$ be a smooth function such that $\text{supp } \psi \subset B(0, 1)$ and

$$\int_{\mathbb{R}^{n+1}} \psi(x, t) dx dt = 1.$$

Also, let $\phi : \mathbb{R}^{n+1} \rightarrow [0, 1]$ be a smooth function such that $\phi = 1$ in $B(0, 1)$ and $\phi = 0$ in $\mathbb{R}^{n+1} \setminus B(0, 2)$. For $0 < \varepsilon < 1$ and $R \geq 1$, we set $\psi_\varepsilon(x, t) = \varepsilon^{-(n+1)}\psi(x/\varepsilon, t/\varepsilon)$ and $\phi_R(x, t) = \phi(x/R, t/R)$.

Now we apply the Carleman estimate (2.1) to the following C_0^∞ function:

$$v(x, t) = (u * \psi_\varepsilon)(x, t)\phi_R(x, t).$$

Then, we see that

$$\beta \|e^{\beta(ct+|x|^2)}v\|_{L^2_{x,t}} + \beta^{\frac{1}{2}} \|e^{\beta(ct+|x|^2)}|\nabla v|\|_{L^2_{x,t}} \leq \|e^{\beta(ct+|x|^2)}(i\partial_t + \Delta)v\|_{L^2_{x,t}}.$$

Note that

$$\nabla v = (\nabla u * \psi_\varepsilon)\phi_R + (u * \psi_\varepsilon)\nabla\phi_R$$

and

$$(i\partial_t + \Delta)v = ((i\partial_t + \Delta)u * \psi_\varepsilon)\phi_R + (u * \psi_\varepsilon)(i\partial_t + \Delta)\phi_R + 2(\nabla u * \psi_\varepsilon) \cdot \nabla\phi_R.$$

Since v is supported in $\{(x, t) \in \mathbb{R}^{n+1} : ct + |x|^2 \leq \varepsilon\}$ and we are assuming $u \in H^1_t \cap H^2_x$, by letting $R \rightarrow \infty$, we get

$$\begin{aligned} \beta \|e^{\beta(ct+|x|^2)}(u * \psi_\varepsilon)\|_{L^2_{x,t}} + \beta^{\frac{1}{2}} \|e^{\beta(ct+|x|^2)}|\nabla u * \psi_\varepsilon|\|_{L^2_{x,t}} \\ \leq \|e^{\beta(ct+|x|^2)}(i\partial_t + \Delta)(u * \psi_\varepsilon)\|_{L^2_{x,t}}. \end{aligned}$$

Again by letting $\varepsilon \rightarrow 0$,

$$\beta \|e^{\beta(ct+|x|^2)}u\|_{L^2_{x,t}} + \beta^{\frac{1}{2}} \|e^{\beta(ct+|x|^2)}|\nabla u|\|_{L^2_{x,t}} \leq \|e^{\beta(ct+|x|^2)}(i\partial_t + \Delta)u\|_{L^2_{x,t}}.$$

Hence it follows that

$$\begin{aligned} \beta \|e^{\beta(ct+|x|^2)}u\|_{L^2_{x,t}(S)} + \beta^{\frac{1}{2}} \|e^{\beta(ct+|x|^2)}|\nabla u|\|_{L^2_{x,t}(S)} \\ \leq \|e^{\beta(ct+|x|^2)}(i\partial_t + \Delta)u\|_{L^2_{x,t}(S)} + \|e^{\beta(ct+|x|^2)}(i\partial_t + \Delta)u\|_{L^2_{x,t}(\mathbb{R}^{n+1} \setminus S)}. \end{aligned}$$

By (1.3), we see that

$$\begin{aligned} \|e^{\beta(ct+|x|^2)}(i\partial_t + \Delta)u\|_{L^2_{x,t}(S)} \leq \|V\|_{L^\infty} \|e^{\beta(ct+|x|^2)}u\|_{L^2_{x,t}(S)} \\ + \| |W| \|_{L^\infty} \|e^{\beta(ct+|x|^2)}|\nabla u|\|_{L^2_{x,t}(S)}. \end{aligned}$$

Hence if we choose β large enough so that $\|V\|_{L^\infty} \leq \beta/2$ and $\|W\|_{L^\infty} \leq \beta^{1/2}/2$, we get

$$\begin{aligned} \beta \|e^{\beta(ct+|x|^2)}u\|_{L^2_{x,t}(S)} + \beta^{\frac{1}{2}} \|e^{\beta(ct+|x|^2)}|\nabla u|\|_{L^2_{x,t}(S)} \\ \leq 2 \|e^{\beta(ct+|x|^2)}(i\partial_t + \Delta)u\|_{L^2_{x,t}(\mathbb{R}^{n+1}\setminus S)}. \end{aligned}$$

Since u vanishes in $\{(x, t) \in \mathbb{R}^{n+1} : ct + |x|^2 > 0\}$ and $u \in H_t^1 \cap H_x^2$, we also see that

$$\begin{aligned} \|e^{\beta(ct+|x|^2)}(i\partial_t + \Delta)u\|_{L^2(\mathbb{R}^{n+1}\setminus S)} &= \|e^{\beta(ct+|x|^2)}(i\partial_t + \Delta)u\|_{L^2(\{(x,t):ct+|x|^2 \leq -1\})} \\ &\leq e^{-\beta} \|(i\partial_t + \Delta)u\|_{L^2(\{(x,t):ct+|x|^2 \leq -1\})} \\ &\leq Ce^{-\beta}. \end{aligned}$$

Hence, we get

$$\beta \|e^{\beta(ct+|x|^2+1)}u\|_{L^2(S)} + \beta^{\frac{1}{2}} \|e^{\beta(ct+|x|^2+1)}|\nabla u|\|_{L^2(S)} \leq 2C.$$

Since $ct + |x|^2 + 1 > 0$ for $(x, t) \in S$,

$$\beta \|u\|_{L^2(S)} + \beta^{\frac{1}{2}} \|\nabla u\|_{L^2(S)} \leq 2C.$$

By letting $\beta \rightarrow \infty$ we now conclude that $u = 0$ on S . This completes the proof.

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