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EQUIVARIANT HILBERT SERIES OF MONOMIAL ORBITS

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ABSTRACT. The equivariant Hilbert series of an ideal generated by an orbit of a monomial under the action of the monoid $\operatorname{Inc}(\mathbb{N})$ of strictly increasing functions is determined. This is used to find the dimension and degree of such an ideal. The result also suggests that the description of the denominator of an equivariant Hilbert series of an arbitrary $\operatorname{Inc}(\mathbb{N})$-invariant ideal as given by Nagel and Römer is rather efficient.

1. Introduction

For a polynomial ring over a field K in finitely many variables, Hilbert showed that its ideals are finitely generated and the vector space dimensions of graded components of its homogeneous ideals eventually grow polynomially. Equivalently, their Hilbert series are rational. Recently, analogs of these results have been established for certain ideals in polynomial rings in infinitely many variables.

To describe this more precisely, fix a positive integer $c \geq 1$ and consider a polynomial ring $K[X] = K[X_{[c]\times\mathbb{N}}] = K[x_{i,j} \mid 1 \leq i \leq c,\ 1 \leq j]$. Let $\operatorname{Inc}(\mathbb{N})$ be the monoid of strictly increasing functions on the set $\mathbb{N}$ of positive integers

$$\operatorname{Inc}(\mathbb{N}) = \{\pi : \mathbb{N} \to \mathbb{N} \mid \pi(i) < \pi(i+1) \text{ for all } i \geq 1\}.$$

Setting $\pi \cdot x_{j,k} = x_{j,\pi(k)}$ induces an action of $\operatorname{Inc}(\mathbb{N})$ on $K[X]$. In [1] and [4] it is shown that any $\operatorname{Inc}(\mathbb{N})$-invariant ideal I of $K[X]$ is generated by finitely many orbits. This and related results are of great interest, for example, in algebraic statistics, the study of tensors, or in representation theory (see, e.g., [2–4, 8, 9]).

If I is a homogeneous ideal, in [7] an equivariant Hilbert series of $K[X]/I$ has been defined as a formal power series in two variables

$$H_{K[X]/I}(s,t) = \sum_{n\geq 0,\, j\geq 0} \dim_K [K[X_n]/I_n]_j \cdot s^n t^j,$$

where $K[X_n] = K[X_{[c]\times[n]}] = K[x_{i,j} \mid 1 \leq i \leq c,\ 1 \leq j \leq n]$ and $I_n = I \cap K[X_n]$. Note that any ideal of $K[X]$ that is invariant under the action of $\operatorname{Sym}(\infty)$ (induced by moving the column indices of the variables) is also $\operatorname{Inc}(\mathbb{N})$-invariant. For any homogeneous $\operatorname{Inc}(\mathbb{N})$-invariant ideal of $K[X]$, it has been shown in [7] that its

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equivariant Hilbert series is rational of the form

$$H_{K[X]/I}(s,t) = \frac{g(s,t)}{(1-t)^a \cdot \prod_{j=1}^{b} \left[(1-t)^{c_j} - s \cdot f_j(t)\right]}, \tag{1}$$

where a, b, c_j are non-negative integers with $c_j \le c$, $g(s,t) \in \mathbb{Z}[s,t]$, and each $f_j(t)$ is a polynomial in $\mathbb{Z}[t]$ satisfying $f_j(1) > 0$.

This form has been used in [7] to show in particular that the dimension of $K[X_n]/I_n$ eventually grows linearly in n and that the limit $\lim\limits_{n\to\infty} \sqrt[n]{\deg I_n}$ exists and is a positive integer. However, the equivariant Hilbert series is explicitly known for only a few ideals. Furthermore, a different argument for the rationality of the Hilbert series $H_{K[X]/I}(s,t)$ has been given more recently in [6], but without a more precise description of the rational function. The authors wonder about a good description of its denominator. In order to begin addressing these issues we consider any ideal I that is generated by the $\operatorname{Inc}(\mathbb{N})$-orbit of some monomial of $K[X]$. For ease of notation, let us focus on the case $c = 1$ in this introduction and write x_j for $x_{1,j}$. Let I be the ideal generated by the orbit of a monomial $x_{\mu_1}^{a_1} x_{\mu_2}^{a_2} \cdots x_{\mu_r}^{a_r}$, where $\mu_1 < \cdots < \mu_r$ and $r, a_1, \ldots, a_r \in \mathbb{N}$, which we write as $I = \langle \operatorname{Inc}(\mathbb{N}) \cdot x_{\mu_1}^{a_1} x_{\mu_2}^{a_2} \cdots x_{\mu_r}^{a_r} \rangle$. For example, if $I = \langle \operatorname{Inc}(\mathbb{N}) \cdot x_3^2 x_5^4 x_8 \rangle$, then one gets

$$I_n = \begin{cases} \langle x_{i_1}^2 x_{i_2}^4 x_{i_3} \mid 3 \le i_1, i_2 - i_1 \ge 2, i_3 - i_2 \ge 3,\ i_3 \le n \rangle & \text{if } n \ge 8, \\ 0 & \text{if } 0 \le n < 8. \end{cases}$$

As a special case of our main result (see Theorem 3.3), one gets for such ideals:

Theorem 1.1. *If $I = \langle \operatorname{Inc}(\mathbb{N}) \cdot x_{\mu_1}^{a_1} x_{\mu_2}^{a_2} \cdots x_{\mu_r}^{a_r} \rangle$, then*

$$H_{K[X]/I}(s,t) = \frac{g(s,t)}{(1-t)^{\mu_r - 1} \prod\limits_{j=1}^{r} \left[1 - s \cdot (1 + t + \cdots + t^{a_j - 1})\right]},$$

where $g(s,t) \in \mathbb{Z}[s,t]$ is a polynomial that is not divisible by any of the indicated irreducible factors of the denominator.

We also determine the numerator polynomial $g(s,t)$ (see Theorem 2.4). For instance, if $I = \langle \operatorname{Inc}(\mathbb{N}) \cdot x_3^2 x_5^4 \rangle$ one gets (see, e.g., Example 2.5)

$$\begin{aligned} &H_{K[X]/I}(s,t) \\ &\quad = \frac{(1-t)^4 + s(1-t)^3(-1+t^2+t^4) + s^2t^6(1-t)^2 + s^3t^6(1-t) + s^4t^6}{(1-t)^4 \cdot [1 - s(1+t)] \cdot [1 - s(1+t+t^2+t^3)]}. \end{aligned} \tag{2}$$

The Hilbert series in the case of an arbitrary mononomial when $c \ge 1$ is qualitatively of the same form as in the case where $c = 1$ (see Theorem 3.3).

Let us compare the above result with the form of the equivariant Hilbert series of an arbitrary $\operatorname{Inc}(\mathbb{N})$-invariant ideal as given in equation (1). Example 7.3 in [7] shows that there is no a priori bound on the degree of the polynomials f_j appearing in the denominator and that they can have negative coefficients. Theorem 1.1 establishes that the number of irreducible factors in the denominator can be arbitrarily large. Thus, the description of the denominator in equation (1) seems rather efficient.

It is instructive to compare our results with the case of a noetherian graded hypersurface ring $A = K[y_1, \ldots, y_m]/\langle f \rangle$. It is a Cohen-Macaulay ring of dimension $m - 1$, and its multiplicity (degree) is $\deg f$. This information can be read off from its Hilbert series, which is $H_A(t) = \frac{1+t+\cdots+t^{\deg f-1}}{(1-t)^{m-1}}$. If I is generated by the Inc($\mathbb{N}$)-orbit of a monomial, then $\dim K[X_n]/I_n = n(c-1)+\mu_r-1$, and so the growth is dominated by $c-1$. However, the degrees of the ideals I_n eventually grow exponentially in n, and if $c=1$ (so $I = \langle \mathrm{Inc}(\mathbb{N}) \cdot x_{\mu_1}^{a_1} x_{\mu_2}^{a_2} \cdots x_{\mu_r}^{a_r} \rangle$) the growth rate is dominated by

$$\lim_{n\to\infty} \sqrt[n]{\deg I_n} = \max\{a_1,\ldots,a_r\},$$

which is *not* the degree of the orbit generator if $r \geq 2$. Again, there is a similar formula in the general case $c \geq 1$ (see Corollary 3.8). Notice that even though each $K[X_n]/I_n$ is Cohen-Macaulay the numerator polynomial of the Hilbert series of $K[X]/I$ in reduced form can have negative coefficients, as is the case in formula (2). However, the polynomials f_j appearing in the irreducible factors of the denominator have only non-negative coefficients (see also Remark 3.9).

The proofs of rationality of an equivariant Hilbert series in [7] and [6] both lead to an algorithm for computing it. However, here we develop a different method that makes the computations efficient. This is first carried out in Section 2 if $c=1$. We discuss this simpler case separately in order to stress the ideas and to simplify notation. The general case is treated in Section 3. In some sense we are able to reduce it to the case where $c=1$.

2. A special case

In this section we consider the special case where $c=1$; that is, the ring $K[X]$ has only one row of variables. Thus, we simplify notation and let $K[X] = K[x_j \mid j \in \mathbb{N}]$. Any monomial in $K[X]$ can be written as $x_{\mu_1}^{a_1} x_{\mu_2}^{a_2} \cdots x_{\mu_r}^{a_r}$, where $\mu_1 < \cdots < \mu_r$ and $r, a_1, \ldots, a_r \in \mathbb{N}$. The Inc($\mathbb{N}$)-invariant ideal I of $K[X]$ generated by the orbit of this monomial is

$$I = \langle \mathrm{Inc}(\mathbb{N}) \cdot x_{\mu_1}^{a_1} x_{\mu_2}^{a_2} \cdots x_{\mu_r}^{a_r} \rangle.$$

Set $\underline{\mu} = (\mu_1, \ldots, \mu_r)$.

Denote the set of non-negative integers by $\mathbb{N}_0$. So, for $n \in \mathbb{N}_0$, one has $K[X_n] = K[x_j \mid 1 \leq j \leq n]$. In particular, $K[X_0] = K$. Since Inc($\mathbb{N}$) acts on $K[X]$ by $\pi \cdot x_j = x_{\pi(j)}$, we get the following explicit description of the ideal $I_n = I \cap K[X_n]$:

$$I_n = \begin{cases} \langle x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_r}^{a_r} \mid \mu_1 \leq i_1,\ i_r \leq n, \text{ and } i_{j+1} - i_j \geq \mu_{j+1} - \mu_j \text{ for each } j \rangle & \text{if } n \geq \mu_r, \\ 0 & \text{if } 0 \leq n < \mu_r. \end{cases}$$

Similarly, if $r \geq 2$, we also consider the ideal

$$J = \langle \mathrm{Inc} \cdot x_{\mu_1}^{a_1} x_{\mu_2}^{a_2} \cdots x_{\mu_{r-1}}^{a_{r-1}} \rangle \subset K[X]$$

and $J_n = J \cap K[X_n]$ for $n \in \mathbb{N}_0$. The above description of the ideals I_n immediately gives the following simple but very useful observation.

Lemma 2.1. *If $n \geq 1$, then*

$$I_n = \langle I_{n-1} \rangle_{K[X_n]} + x_n^{a_r} \langle J_{n-\delta_r} \rangle_{K[X_n]},$$

where $\delta_r := \mu_r - \mu_{r-1} \geq 1$ and J_n is defined as the zero ideal if $n < 0$.

Recall that the *Hilbert series* of a proper homogeneous ideal $\mathfrak{a}$ of $K[X_n]$ is defined as the formal power series

$$H_{K[X_n]/\mathfrak{a}}(t) = \sum_{j \geq 0} \dim_K [K[X_n]/\mathfrak{a}]_j \cdot t^j.$$

Hilbert showed that it is a rational function of the form $H_{K[X_n]/\mathfrak{a}}(t) = \frac{f(t)}{(1-t)^d}$, where $f(t) \in \mathbb{Z}[t]$ and $d \in \mathbb{N}_0$. We say that $H_{K[X_n]/\mathfrak{a}}(t)$ is in *reduced* form if the numerator and denominator are relatively prime or, equivalently, if $f(1) \neq 0$. In this case d is the Krull dimension of $K[X_n]/\mathfrak{a}$, and $f(1) \geq 1$ is the *degree* of $\mathfrak{a}$ or *multiplicity* of $K[X_n]/\mathfrak{a}$. In particular, the zero ideal has degree one.

Corollary 2.2.

(a) *If $n \geq \mu_r$, then $A_n := K[X_n]/I_n$ is a Cohen-Macaulay ring of dimension $\mu_r - 1$.*

(b) *Setting $B_n := K[X_n]/J_n$, one gets for the Hilbert series if $n \geq \delta_r$:*

$$H_{A_n}(t) = (1 + t + \cdots + t^{a_r - 1}) H_{A_{n-1}}(t) + \frac{t^{a_r}}{(1-t)^{\delta_r}} H_{B_{n-\delta_r}}(t).$$

Proof. Consider multiplication by $x_n^{a_r}$ on A_n. Lemma 2.1 shows that, for $n \geq 1$, it induces a short exact sequence

$$0 \to (K[X_n]/\langle J_{n-\delta_r}\rangle_{K[X_n]})(-a_r) \to A_n \to K[X_n]/\langle I_{n-1}, x_n^{a_r}\rangle_{K[X_n]} \to 0. \tag{3}$$

Since the generators of the ideal $J_{n-\delta_r}$ are in $K[X_{n-\delta_r}]$, we get

$$K[X_n]/\langle J_{n-\delta_r}\rangle_{K[X_n]} \cong \begin{cases} K[X_n] & \text{if } 0 \leq n < \delta_r, \\ B_{n-\delta_r}[x_{n-\delta_r+1}, \ldots, x_n] & \text{if } n \geq \delta_r. \end{cases}$$

Observe also that $K[X_n]/\langle I_{n-1}, x_n^{a_r}\rangle_{K[X_n]} \cong A_{n-1} \otimes_K K[x_n]/(x_n^{a_r})$, which implies that

$$\begin{aligned} H_{K[X_n]/\langle I_{n-1}, x_n^{a_r}\rangle_{K[X_n]}}(t) &= H_{A_{n-1}}(t) \cdot H_{K[x_n]/(x_n^{a_r})}(t) \\ &= H_{A_{n-1}}(t) \cdot (1 + t + \cdots + t^{a_r - 1}). \end{aligned}$$

Now, sequence (3) gives claim (b).

For proving (a), we use induction on $r \geq 1$. Let $r = 1$. If $n \geq \mu_1$, then note that $A_n = K[X_n]/\langle x_{\mu_1}^{a_1}, x_{\mu_1+1}^{a_1}, \ldots, x_n^{a_1}\rangle$, which is a Cohen-Macaulay ring of dimension $\mu_1 - 1$. If $r \geq 2$ and $n \geq \mu_r$, then the induction hypothesis gives that $K[X_n]/\langle J_{n-\delta_r}\rangle_{K[X_n]}$ is a Cohen-Macaulay ring with

$$\dim K[X_n]/\langle J_{n-\delta_r}\rangle_{K[X_n]} = \dim B_{n-\delta_r} + \delta_r = \mu_{r-1} - 1 + \delta_r = \mu_r - 1.$$

The above Hilbert series computation also yields $\dim K[X_n]/\langle I_{n-1}, x_n^{a_r}\rangle_{K[X_n]} = \dim A_{n-1}$. Furthermore, $x_n^{a_r}$ is not a zerodivisor of $K[X_n]/\langle I_{n-1}\rangle_{K[X_n]}$, and so $K[X_n]/\langle I_{n-1}, x_n^{a_r}\rangle_{K[X_n]}$ also is a Cohen-Macaulay ring.

Thus, claim (a) follows from sequence (3). ☐

Remark 2.3. In terms of Gorenstein liaison theory, Lemma 2.1 says that I_n is a basic double link of $\langle J_{n-\delta_r}\rangle_{K[X_n]}$ on $\langle I_{n-1}\rangle_{K[X_n]}$. The name stems from the fact that I_n can be obtained from $\langle J_{n-\delta_r}\rangle_{K[X_n]}$ by two Gorenstein links if $K[X_n]/\langle J_{n-\delta_r}\rangle_{K[X_n]}$ is generically Gorenstein (see [5, Proposition 5.10]).

We are ready to establish the main result of this section.

Theorem 2.4. *The equivariant Hilbert series of $A = K[X]/I$ is*

$$H_A(s,t) = \frac{g_{r,\underline{a},\underline{\mu}}(s,t)}{(1-t)^{\mu_r-1}\prod_{i=1}^{r}\Big[1-s(1+t+\cdots+t^{a_i-1})\Big]},$$

where $g_{r,\underline{a},\underline{\mu}}(s,t) \in \mathbb{Z}[s,t]$ is the polynomial with

$$g_{r,\underline{a},\underline{\mu}}(s,t)\cdot(1-t-s) = (1-t)^{\mu_r-r}\prod_{i=1}^{r}(1-t-s+st^{a_i}) - s^{\mu_r}t^{\sum_{i=1}^{r}a_i}.$$

Moreover, the above right-hand side is in reduced form; that is, the given numerator and denominator are relatively prime.

Proof. Denote the right-hand side in the definition of $g_{r,\underline{a},\underline{\mu}}(s,t)$ by $\widetilde{g}_{r,\underline{a},\underline{\mu}}(s,t)$, that is,

$$\widetilde{g}_{r,\underline{a},\underline{\mu}}(s,t) = (1-t)^{\mu_r-r}\prod_{i=1}^{r}(1-t-s+st^{a_i}) - s^{\mu_r}t^{\sum_{i=1}^{r}a_i}.$$

We first show by induction on $r \geq 1$ that

$$H_A(s,t) = \frac{\widetilde{g}_{r,\underline{a},\underline{\mu}}(s,t)}{(1-t)^{\mu_r-1}(1-t-s)\prod_{i=1}^{r}\Big[1-s(1+t+\cdots+t^{a_i-1})\Big]}. \tag{4}$$

Let $r=1$. One has $A_n = K[X_n]$ if $n < \mu_1$. If $n \geq \mu_1$, then we get

$$A_n = K[X_n]/(x_{\mu_1}^{a_1}, x_{\mu_1+1}^{a_1}, \dots, x_n^{a_1}) \cong K[X_{\mu_1-1}] \otimes_K \big(K[z]/\langle z^{a_1}\rangle\big)^{\otimes(n-\mu_1+1)}.$$

Thus we obtain for the equivariant Hilbert series

$$\begin{aligned}
H_A(s,t) &= \sum_{n=0}^{\mu_1-1}\frac{1}{(1-t)^n}s^n + \sum_{n\geq\mu_1}\frac{1}{(1-t)^{\mu_1-1}}(1+t+\cdots+t^{a_1-1})^{n-\mu_1+1}\cdot s^n \\
&= \sum_{n=0}^{\mu_1-2}\Big(\frac{s}{1-t}\Big)^n + \Big(\frac{s}{1-t}\Big)^{\mu_1-1}\sum_{n\geq\mu_1-1}\Big[s(1+t+\cdots+t^{a_1-1})\Big]^{n-\mu_1+1} \\
&= \frac{1-\Big(\frac{s}{1-t}\Big)^{\mu_1-1}}{1-\frac{s}{1-t}} + \Big(\frac{s}{1-t}\Big)^{\mu_1-1}\frac{1}{1-s(1+t+\cdots+t^{a_1-1})} \\
&= \frac{\big[(1-t)^{\mu_1-1}-s^{\mu_1-1}\big]\cdot\big[1-t-s(1-t^{a_1})\big]+s^{\mu_1-1}\big[1-t-s\big]}{(1-t)^{\mu_1-1}(1-t-s)\Big[1-s(1+t+\cdots+t^{a_1-1})\Big]} \\
&= \frac{(1-t)^{\mu_1-1}(1-t-s+st^{a_1})-s^{\mu_1}t^{a_1}}{(1-t)^{\mu_1-1}(1-t-s)\Big[1-s(1+t+\cdots+t^{a_1-1})\Big]},
\end{aligned}$$

as desired.

Let $r \geq 2$. Using Corollary 2.2(b), we get

$$\begin{aligned}
H_A(s,t) - 1 &= \sum_{n\geq 1} H_{A_n}(t) s^n \\
&= \sum_{n=1}^{\delta_r - 1} t^{a_r} \cdot H_{K[X_n]}(t) \cdot s^n + \sum_{n \geq \delta_r} \frac{t^{a_r}}{(1-t)^{\delta_r}} H_{B_{n-\delta_r}}(t) \cdot s^n \\
&\quad + \sum_{n\geq 1} [1 + t + \cdots + t^{a_r - 1}] \cdot H_{A_{n-1}}(t) \cdot s^n \\
&= t^{a_r} \cdot \frac{s}{1-t} \cdot \frac{1 - \left(\frac{s}{1-t}\right)^{\delta_r - 1}}{1 - \frac{s}{1-t}} + \frac{t^{a_r}}{(1-t)^{\delta_r}} s^{\delta_r} H_B(s,t) \\
&\quad + [1 + t + \cdots + t^{a_r - 1}] \cdot s \cdot H_A(s,t).
\end{aligned}$$

Solving for the equivariant Hilbert series of A, a straightforward computation gives the following recursive formula:

$$H_A(s,t) \quad = \quad \frac{1 + \frac{t^{a_r} s \left[(1-t)^{\delta_r - 1} - s^{\delta_r - 1}\right]}{(1-t)^{\delta_r - 1}(1-s-t)} + \frac{t^{a_r} s^{\delta_r}}{(1-t)^{\delta_r}} H_B(s,t)}{1 - s[1 + t + \cdots + t^{a_r - 1}]}.$$

Applying the induction hypothesis to B and noting $\mu_r = \mu_{r-1} + \delta_r$, we get

$$\begin{aligned}
H_A(s,t) \cdot [1 - s \cdot (1 + t + \cdots + t^{a_r - 1})] &= 1 + \frac{t^{a_r} s (1-t)^{\mu_{r-1}} \left[(1-t)^{\delta_r - 1} - s^{\delta_r - 1}\right]}{(1-t-s)(1-t)^{\mu_r - 1}} \\
+ \frac{t^{a_r} s^{\delta_r} (1-t)^{\mu_{r-1} - r + 1} \prod_{i=1}^{r-1} [1 - t - s + s t^{a_i}] - s^{\mu_r} t^{\sum_{i=1}^{r} a_i}}{(1-t)^{\mu_r - 1}(1-t-s) \prod_{i=1}^{r-1} \left[1 - s(1 + t + \cdots + t^{a_i - 1})\right]}&.
\end{aligned}$$

Using $(1-t) \cdot [1 - s(1 + t + \cdots + t^{a_i - 1})] = [1 - t - s + s t^{a_i}]$, this gives

$$\begin{aligned}
&H_A(s,t) \cdots (1-t)^{\mu_r - 1}(1-s-t) \prod_{i=1}^{r} [1 - s \cdot (1 + t + \cdots + t^{a_i - 1})] \\
&= H_A(s,t) \cdot (1-t)^{\mu_r - r}(1-s-t)[1 - s \cdot (1 + t + \cdots + t^{a_r - 1})] \prod_{i=1}^{r-1} [1 - t - s + s t^{a_i}] \\
&= -s^{\mu_r} t^{\sum_{i=1}^{r} a_i} + \prod_{i=1}^{r-1} [1 - t - s + s t^{a_i}] \\
&\quad \cdot \ \Big\{(1-t)^{\mu_r - r}(1-s-t) + t^{a_r} s (1-t)^{\mu_{r-1} - r + 1} \left[(1-t)^{\delta_r - 1} - s^{\delta_r - 1}\right] \\
&\quad + t^{a_r} s^{\delta_r} (1-t)^{\mu_{r-1} - r + 1}\Big\} \\
&= -s^{\mu_r} t^{\sum_{i=1}^{r} a_i} + \prod_{i=1}^{r-1} [1 - t - s + s t^{a_i}] \cdot (1-t)^{\mu_r - r} \left\{1 - t - s + s t^{a_r}\right\}.
\end{aligned}$$

Now equation (4) follows.

It remains to show that $\widetilde{g}_{r,\underline{a},\underline{\mu}}(s,t)$ is divisible by $(1-t-s)$ in $\mathbb{Z}[s,t]$, but not by any of the polynomials $[1-s(1+t+\cdots+t^{a_i-1})]$. The first claim follows because

$$\begin{aligned}\widetilde{g}_{r,\underline{a},\underline{\mu}}(1-t,t) &= (1-t)^{\mu_r-r}\prod_{i=1}^{r}[(1-t)-(1-t)+(1-t)t^{a_i}] - (1-t)^{\mu_r}t^{\sum_{j=1}^{r} a_j}\\ &= (1-t)^{\mu_r-r}(1-t)^r t^{\sum_{i=1}^{r} a_i} - (1-t)^{\mu_r}t^{\sum_{j=1}^{r} a_j} = 0.\end{aligned}$$

In order to show the other claims we compute

$$\widetilde{g}_{r,\underline{a},\underline{\mu}}\left(\frac{1}{1+t+\cdots+t^{a_i-1}},t\right) = \widetilde{g}_{r,\underline{a},\underline{\mu}}\left(\frac{1-t}{1-t^{a_i}},t\right) = -\left(\frac{1-t}{1-t^{a_i}}\right)^{\mu_r}\cdot t^{\sum_{i=1}^{r} a_i}.$$

Since this is not the zero polynomial, it follows that $[1-s(1+t+\cdots+t^{a_i})]$ does not divide $\widetilde{g}_{r,\underline{a},\underline{\mu}}(s,t)$, as desired. □

We give the numerator polynomial in the reduced form of the Hilbert series for small r.

Example 2.5. For $r=1,2,3$, one gets

$$\begin{aligned}g_{1,\underline{a},\underline{\mu}}(s,t) &= (1-t)^{\mu_1-1} + t^{a_1}\sum_{j=0}^{\mu_1-2}(1-t)^j s^{\mu_1-1-j},\\ g_{2,\underline{a},\underline{\mu}}(s,t) &= (1-t)^{\mu_2-1} + s(1-t)^{\mu_2-2}(-1+t^{a_1}+t^{a_2})\\ &\quad + t^{a_1+a_2}\cdot\sum_{j=0}^{\mu_2-3}(1-t)^j s^{\mu_2-1-j},\\ g_{3,\underline{a},\underline{\mu}}(s,t) &= (1-t)^{\mu_3-1} + s(1-t)^{\mu_3-2}(-2+t^{a_1}+t^{a_2}+t^{a_3})\\ &\quad + s^2(1-t)^{\mu_3-3}(1-t^{a_1}-t^{a_2}-t^{a_3}+t^{a_1+a_2}+t^{a_1+a_3}+t^{a_2+a_3})\\ &\quad + t^{a_1+a_2+a_3}\cdot\sum_{j=0}^{\mu_3-4}(1-t)^j s^{\mu_3-1-j}.\end{aligned}$$

Here we use the convention that a sum $\sum_{j=0}^{e}$ is defined to be zero if $e<0$.

We can also use our methods to determine the degree of each ideal I_n.

Proposition 2.6. *If $n \geq \mu_r - 1$, then $\deg I_n$ is the coefficient of $t^{n-\mu_r+1}$ in the power series $\prod_{i=1}^{r}\frac{1}{1-a_i t}$. In other words,*

$$\prod_{i=1}^{r}\frac{1}{1-a_i t} = \sum_{n\geq\mu_r-1}\deg I_n\cdot t^{n-\mu_r+1}.$$

Proof. One can deduce this from Theorem 2.4. However, there is an easier, more direct approach.

Since $I_{\mu_r-1}=0$ by definition, we get $\deg I_{\mu_r-1}=1$ for each $r\geq 1$, as claimed. To determine $\deg I_n$ for larger n, we use induction on $r\geq 1$. If $r=1$, then $I_n = \langle x_{\mu_1}^{a_1}, x_{\mu_r+1}^{a_1},\ldots,x_n^{a_1}\rangle$, and so $\deg I_n = a_1^{n-\mu_1+1}$. Now the geometric series gives the claim, that is, $\sum\limits_{n\geq\mu_1-1}\deg I_n t^{n-\mu_1+1} = \sum\limits_{n\geq 0} a_1^n t^n = \frac{1}{1-a_1 t}$.

Let $r\geq 2$. If $n\geq\delta_r$, then Lemma 2.1 gives

$$\deg I_n = a_r\deg I_{n-1} + \deg J_{n-\delta_r}. \tag{5}$$

By induction on r, one has

$$\prod_{i=1}^{r-1} \frac{1}{1-a_i t} = \sum_{n \geq \mu_{r-1}-1} \deg J_n t^{n-\mu_{r-1}+1} = \sum_{n-\delta_r \geq \mu_{r-1}-1} \deg J_{n-\delta_r} t^{n-\mu_r+1}.$$

Hence we obtain

$$\begin{aligned}\prod_{i=1}^{r} \frac{1}{1-a_i t} &= \Big(\sum_{n-\delta_r \geq \mu_{r-1}-1} \deg J_{n-\delta_r} \cdot t^{n-\mu_r+1} \Big) \cdot \Big(\sum_{k \geq 0} a_r^k t^k \Big) \\ &= \sum_{n \geq \mu_r - 1} \left[\sum_{i=0}^{n-\mu_r+1} a_r^{n-\mu_r+1-i} \cdot \deg J_{\mu_{r-1}-1+i} \right] t^{n-\mu_r+1}.\end{aligned}$$

This implies our assertion because

$$\sum_{i=0}^{n-\mu_r+1} a_r^{n-\mu_r+1-i} \cdot \deg J_{\mu_{r-1}-1+i} = \deg I_n.$$

Indeed, if $n = \mu_r - 1$, then this formula is true since $a_r^0 \cdot \deg J_{\mu_{r-1}-1} = 1 = \deg I_{\mu_r - 1}$. Let $n \geq \mu_r$. Using equation (5), one has

$$\begin{aligned}\sum_{i=0}^{n-\mu_r+1} a_r^{n-\mu_r+1-i} \cdot \deg J_{\mu_{r-1}-1+i} &= \deg J_{n-\mu_r+\mu_{r-1}} + a_r \sum_{i=0}^{n-\mu_r} a_r^{n-\mu_r-i} \deg J_{\mu_{r-1}-1+i} \\ &= \deg J_{n-\delta_r} + a_r \deg I_{n-1} = \deg I_n,\end{aligned}$$

as desired. □

One can use the last result to explicitly compute $\deg I_n$. This is easiest if $a_1, \ldots, a_r$ are pairwise distinct.

Corollary 2.7. *If $a_1, \ldots, a_r$ are pairwise distinct, then $\deg I_n = \sum_{i=1}^{r} \frac{a_i^{n-\mu_r+r}}{\prod_{j \neq i} (a_i - a_j)}$, provided $n \geq \mu_r - 1$.*

Proof. Using partial fractions, one can write

$$\prod_{i=1}^{r} \frac{1}{1-a_i t} = \frac{C_1}{1-a_1 t} + \cdots + \frac{C_r}{1-a_r t},$$

where

$$C_i = \prod_{j \neq i} \frac{1}{1 - \frac{a_j}{a_i}} = \frac{a_i^{r-1}}{\prod_{j \neq i} (a_i - a_j)}.$$

Hence

$$\prod_{i=1}^{r} \frac{1}{1-a_i t} = \sum_{i=1}^{r} \frac{a_i^{r-1}}{\prod_{j \neq i} (a_i - a_j)} \cdot \frac{1}{1-a_i t} = \sum_{i=1}^{r} \Big[\frac{a_i^{r-1}}{\prod_{j \neq i} (a_i - a_j)} \cdot \sum_{k \geq 0} a_i^k t^k \Big].$$

Now we conclude by Proposition 2.6. □

3. The general case

We extend the results of the previous section. We use the notation established in the introduction. So we fix an integer $c \geq 1$ and consider the polynomial rings $K[X_n] = K[x_{i,j} \mid 1 \leq i \leq c,\ 1 \leq j \leq n]$ and $K[X] = K[x_{i,j} \mid 1 \leq i \leq c,\ 1 \leq j]$. Any monomial of positive degree in $K[X]$ can be written as

$$x^{\underline{a}} = \prod_{i=1}^{c} \prod_{j=1}^{s} x_{i,j}^{a_{i,j}},$$

where $\underline{a} = (a_{i,j})$ is a $c \times s$ non-zero matrix whose entries are non-negative integers. Denote the indices of the non-zero columns of $\underline{a}$ by $\mu_1, \ldots, \mu_r$, where $\mu_1 < \mu_2 < \cdots < \mu_r$. We may assume that the last column of $\underline{a}$ is not zero, that is, $\mu_r = s$ and $\underline{a} \in \mathbb{N}_0^{c\times\mu_r}$. Thus, we can rewrite $x^{\underline{a}}$ more explicitly as

$$x^{\underline{a}} = (x_{1,\mu_1}^{a_{1,\mu_1}} \cdots x_{1,\mu_r}^{a_{1,\mu_r}}) \cdot (x_{2,\mu_1}^{a_{2,\mu_1}} \cdots x_{2,\mu_r}^{a_{2,\mu_r}}) \cdots (x_{c,\mu_1}^{a_{c,\mu_1}} \cdots x_{c,\mu_r}^{a_{c,\mu_r}}).$$

Put $\underline{\mu} = (\mu_1, \ldots, \mu_r)$.

In order to determine the equivariant Hilbert series of $K[X]/I$, where $I = \langle \operatorname{Inc}(\mathbb{N}) \cdot x^{\underline{a}} \rangle$, we also consider the ideal

$$J = \langle \operatorname{Inc} \cdot \prod_{i=1}^{c} \prod_{j=1}^{\mu_{r-1}} x_{i,j}^{a_{i,j}} \rangle$$

if $r \geq 2$. Thus, we get for $I_n = I \cap K[X]$ and $J_n = J \cap K[X]$ that $I_n = 0$ if $n < \mu_r$ and that $J_n = 0$ if $n < \mu_{r-1}$. Moreover, there is again a useful relation among these ideals.

Lemma 3.1. *If $n \geq 1$, then*

$$I_n := \langle I_{n-1} \rangle_{K[X_n]} + \prod_{i=1}^{c} x_{i,n}^{a_{i,\mu_r}} \langle J_{n-\delta_r} \rangle_{K[X_n]},$$

where $\delta_r := \mu_r - \mu_{r-1} \geq 1$.

It follows that I_n is a basic double link of $J_{n-\delta_r}$ on I_{n-1} because of the following consequence. We use the notation $A_n = K[X_n]/I_n$, $B_n = K[X_n]/J_n$, and $b_j = \sum_{i=1}^{c} a_{i,\mu_j}$ for $j = 1, \ldots, r$. Thus, b_j is the total degree of the divisor of $x^{\underline{a}}$ whose factors are the variables appearing in column μ_j.

Corollary 3.2.

(a) *If $n \geq \mu_r$, then A_n is a Cohen-Macaulay ring of dimension $n(c-1)+\mu_r-1$.*
(b) *If $n \geq \delta_r$, then one has for the Hilbert series:*

$$H_{A_n}(t) = \frac{1-t^{b_r}}{(1-t)^c} H_{A_{n-1}}(t) + \frac{t^{b_r}}{(1-t)^{c\delta_r}} H_{B_{n-\delta_r}}(t).$$

Proof. Multiplication by $\prod_{i=1}^{c} x_{i,n}^{a_{i,\mu_r}}$ on A_n induces the exact sequence

$$0 \to K[X_n]/\langle J_{n-\delta_r} \rangle_{K[X_n]}(-b_r) \to A_n \to K[X_n]/\langle I_{n-1}, \prod_{i=1}^{c} x_{i,n}^{a_{i,\mu_r}} \rangle_{K[X_n]} \to 0.$$

Furthermore, we have

$$K[X_n]/\langle J_{n-\delta_r}\rangle_{K[X_n]} \cong \begin{cases} K[X_n] & \text{if } 0 \le n < \delta_r, \\ B_{n-\delta_r}\big[x_{i,j} \mid 1 \le i \le c, n-\delta_r < j \le n\big] & \text{if } n \ge \delta_r. \end{cases}$$

Now the claims follow as in the proof of Corollary 2.2. ☐

Our main result is the promised extension of Theorem 2.4.

Theorem 3.3. *Setting* $\underline{b}=(b_1,\dots,b_r)$*, the equivariant Hilbert series of* $A=K[X]/I$ *is*

$$H_A(s,t) = \frac{g_{r,c,\underline{b},\underline{\mu}}(s,t)}{(1-t)^{c(\mu_r-r-1)+r}\prod_{j=1}^{r}\big[(1-t)^{c-1} - s(1+t+\cdots+t^{b_j-1})\big]},$$

where $g_{r,c,\underline{b},\underline{\mu}}(s,t) \in \mathbb{Z}[s,t]$ *is the polynomial with*

$$g_{r,c,\underline{b},\underline{\mu}}(s,t)\cdot\big[(1-t)^c - s\big] = (1-t)^{c(\mu_r-r)}\prod_{j=1}^{r}\big[(1-t)^c - s + st^{b_j}\big] - s^{\mu_r}t^{\sum_{j=1}^{r} b_j}.$$

Furthermore, the above rational function is in reduced form; that is, the given numerator and denominator polynomials are relatively prime. (*Notice that the exponent* $[c(\mu_r-r-1)+r]$ *of* $(1-t)$ *is negative if and only if* $r<c$ *and* $\mu_r=r$.)

Proof. We argue as in the proof of Theorem 2.4. Set

$$\widetilde{g}_{r,c,\underline{a},\underline{\mu}}(s,t) = (1-t)^{c(\mu_r-r)}\prod_{j=1}^{r}\big[(1-t)^c - s + st^{b_j}\big] - s^{\mu_r}t^{\sum_{j=1}^{r} b_j}.$$

Using induction on $r \ge 1$, one shows that

(6)
$$H_A(s,t) = \frac{\widetilde{g}_{r,c,\underline{a},\underline{\mu}}(s,t)}{(1-t)^{c(\mu_r-r-1)+r}\big[(1-t)^c - s\big]\prod_{j=1}^{r}\big[(1-t)^{c-1} - s(1+t+\cdots+t^{b_j-1})\big]}.$$

Indeed, let $r=1$. If $n \ge \mu_1$, then we get

$$A_n \cong K[X_{\mu_1-1}] \otimes \Big(K[z_1,\dots,z_c]/(z_1^{a_{1,\mu_1}}\cdots z_c^{a_{c,\mu_1}})\Big)^{\otimes n-\mu_1+1},$$

where $z_1,\dots,z_c$ are new variables. It follows that

$$H_A(s,t) = \sum_{n=0}^{\mu_1-1}\frac{1}{(1-t)^{nc}}s^n + \sum_{n\ge\mu_1}\frac{1}{(1-t)^{c(\mu_1-1)}}\Big(\frac{1+t+\cdots+t^{b_1-1}}{(1-t)^{c-1}}\Big)^{n-\mu_1+1}s^n.$$

Now a computation as in the proof of Theorem 2.4 gives the desired formula.

Let $r \geq 2$. Corollary 3.2 implies that

$$\begin{aligned}
H_A(s,t) - 1 &= \sum_{n \geq 1} H_{A_n}(t) s^n \\
&= \sum_{n=1}^{\delta_r - 1} t^{b_r} \cdot H_{K[X_n]}(t) \cdot s^n + \sum_{n \geq \delta_r} \frac{t^{b_r}}{(1-t)^{c\delta_r}} H_{B_{n-\delta_r}}(t) \cdot s^n \\
&\quad + \sum_{n \geq 1} \frac{1 + t + \cdots + t^{b_r - 1}}{(1-t)^{c-1}} \cdot H_{A_{n-1}}(t) \cdot s^n \\
&= t^{b_r} \cdot \frac{s}{(1-t)^c} \cdot \frac{1 - \left(\frac{s}{(1-t)^c}\right)^{\delta_r - 1}}{1 - \frac{s}{(1-t)^c}} + \frac{t^{b_r}}{(1-t)^{c\delta_r}} s^{\delta_r} H_B(s,t) \\
&\quad + \frac{s(1 + t + \cdots + t^{b_r - 1})}{(1-t)^{c-1}} \cdot H_A(s,t).
\end{aligned}$$

This gives

$$\begin{aligned}
&H_A(s,t) \cdot \frac{(1-t)^{c-1} - s(1 + t + \cdots + t^{b_r - 1})}{(1-t)^{c-1}} \\
&\qquad = 1 + t^{b_r} s \frac{(1-t)^{c(\delta_r - 1)} - s^{\delta_r - 1}}{(1-t)^{c(\delta_r - 1)}[(1-t)^c - s]} + \frac{t^{b_r} s^{\delta_r}}{(1-t)^{c\delta_r}} H_B(s,t).
\end{aligned}$$

Applying the induction hypothesis to B, a computation similar to the one in the proof of Theorem 2.4 establishes equation (6).

It remains to show that $\widetilde{g}_{r,c,\underline{a},\underline{\mu}}(s,t)$ is divisible by $((1-t)^c - s)$ in $\mathbb{Z}[s,t]$, but not by any of the polynomials $[(1-t)^{c-1} - s(1 + t + \cdots + t^{a_i - 1})]$. The first claim is true because

$$\begin{aligned}
\widetilde{g}_{r,c,\underline{a},\underline{\mu}}((1-t)^c, t) &= (1-t)^{c(\mu_r - r)} \prod_{i=1}^{r} [(1-t)^c - (1-t)^c + (1-t)^c t^{b_i}] \\
&\quad - (1-t)^{c\mu_r} t^{\sum_{j=1}^{r} b_j} \\
&= (1-t)^{c(\mu_r - r)} (1-t)^{rc} t^{\sum_{i=1}^{r} b_i} - (1-t)^{c\mu_r} t^{\sum_{j=1}^{r} b_j} = 0.
\end{aligned}$$

Substituting $s = \dfrac{(1-t)^{c-1}}{1 + t + \cdots + t^{b_r - 1}} = \dfrac{(1-t)^c}{1 - t^{b_r}}$, we get

$$\widetilde{g}_{r,c,\underline{a},\underline{\mu}}\left(\frac{(1-t)^{c-1}}{1 + t + \cdots + t^{b_1 - 1}}, t\right) = -\frac{(1-t)^{(c-1)\mu_r}}{(1 + t + \cdots + t^{b_r - 1})^{\mu_r}} \cdot t^{\sum_{j=1}^{r} b_j}.$$

Since this is not the zero polynomial the argument is complete now. $\square$

Again we give the numerator polynomial in the reduced form of the Hilbert series for small r, where we assume that $c(\mu_r - r - 1) + r \geq 0$.

Example 3.4. For $r = 1, 2, 3$, one has

$$\begin{aligned}
g_{1,c,\underline{a},\underline{\mu}} &= (1-t)^{c(\mu_1-1)} + t^{b_1} \cdot \sum_{j=0}^{\mu_1-2} (1-t)^{cj} s^{\mu_1-1-j}, \\
g_{2,c,\underline{a},\underline{\mu}}(s,t) &= (1-t)^{c(\mu_2-1)} + s(1-t)^{c(\mu_2-2)}(-1+t^{b_1}+t^{b_2}) \\
&\quad + t^{b_1+b_2} \cdot \sum_{j=0}^{\mu_2-3} (1-t)^{cj} s^{\mu_2-1-j}, \\
g_{3,c,\underline{a},\underline{\mu}}(s,t) &= (1-t)^{c(\mu_3-1)} + s(1-t)^{c(\mu_3-2)}(-2+t^{b_1}+t^{b_2}+t^{b_3}) \\
&\quad + s^2(1-t)^{c(\mu_3-3)}(1-t^{b_1}-t^{b_2}-t^{b_3}+t^{b_1+b_2}+t^{b_1+b_3}+t^{b_2+b_3}) \\
&\quad + t^{b_1+b_2+b_3} \cdot \sum_{j=0}^{\mu_3-4} (1-t)^{cj} s^{\mu_3-1-j}.
\end{aligned}$$

Notice that these polynomials simplify if the μ_i's are as small as possible, that is, $\mu_i = i$. For example, then one gets $g_{1,c,\underline{a},\underline{\mu}} = 1$ and

$$g_{2,c,\underline{a},\underline{\mu}}(s,t) = (1-t)^c + s(-1+t^{b_1}+t^{b_2}).$$

Remark 3.5. Observe the similarity of the formulas in Theorems 2.4 and 3.3. Indeed, Theorem 3.3 is formally obtained from Theorem 2.4 by replacing each a_j by the total column degree b_j and $(1-t)$ by $(1-t)^c$.

Now we determine the degree of I_n.

Proposition 3.6. *If $n \geq \mu_r - 1$, then $\deg I_n$ is the coefficient of $t^{n-\mu_r+1}$ in the power series $\prod_{j=1}^{r} \frac{1}{1-b_j t}$. That is,*

$$\prod_{j=1}^{r} \frac{1}{1-b_j t} = \sum_{n \geq \mu_r - 1} \deg I_n \cdot t^{n-\mu_r+1}.$$

Proof. If $r \geq 2$ and $n \geq \delta_r$, Lemma 3.1 gives

$$\deg I_n = b_r \deg I_{n-1} + \deg J_{n-\delta_r}.$$

Now we conclude as in the proof of Proposition 2.6. □

Analogously to Corollary 2.7 this gives the following explicit formula.

Corollary 3.7. *If $b_1, \ldots, b_r$ are pairwise distinct, then* $\deg I_n = \sum_{i=1}^{r} \frac{b_i^{n-\mu_r+r}}{\prod_{j \neq i}(b_i - b_j)}$, *provided $n \geq \mu_r - 1$.*

For any $\mathrm{Inc}(\mathbb{N})$-invariant ideal I of $K[X]$, it is shown in [7, Theorem 7.9] that the two limits $\lim_{n\to\infty} \frac{\dim K[X_n]/I_n}{n}$ and $\lim_{n\to\infty} \sqrt[n]{\deg I_n}$ exist and are non-negative integers, where $I_n = I \cap K[X_n]$. Following [7, Remark 7.14], we refer to these integers as the *dimension* of $K[X]/I$ and the *degree* of I, respectively. If I is generated by the orbit of a monomial, we obtain the following values.

Corollary 3.8. *For $I = \langle \mathrm{Inc}(\mathbb{N}) \cdot \prod_{i=1}^{c} \prod_{j=1}^{\mu_r} x_{i,j}^{a_{i,j}} \rangle$, one has*

(a) $\dim K[X]/I = c - 1$,
(b) $\deg I = \max\{b_1, \ldots, b_r\}$.

Proof. (a) is a consequence of Corollary 3.2.

(b) follows by using partial fractions as in [7, Lemma A.3]. We leave the details to the interested reader. □

We conclude with some comments about non-negativity of the coefficients of the polynomials appearing in an equivariant Hilbert series.

Remark 3.9. If A is a graded Cohen-Macaulay quotient of a noetherian polynomial ring, then it is well-known that the numerator polynomial in its reduced Hilbert series has non-negative coefficients only. We have seen above that in the case of an $\mathrm{Inc}(\mathbb{N})$-invariant ideal I of $K[X]$ the condition that all rings $K[X_n]/I_n$ are Cohen-Macaulay is not sufficient to guarantee that the numerator polynomial $g(s,t)$ in a reduced Hilbert series of $K[X]/I$ as in equation (1) has non-negative coefficients only (see, e.g., Example 3.4). However, the coefficients in the polynomials $f_j(t)$ appearing in the denominator of the Hilbert series all have non-negative coefficients if I is generated by the orbit of a monomial. This suggests the following question.

Question 3.10. Assume I is an $\mathrm{Inc}(\mathbb{N})$-invariant ideal of $K[X]$ such that each ring $K[X_n]/I_n$ is Cohen-Macaulay. Is it then true that the coefficients of the polynomials $f_j(t)$ appearing in the reduced form of the Hilbert series (1) are all non-negative?

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