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MAXIMAL ORTHOPLECTIC FUSION FRAMES FROM MUTUALLY UNBIASED BASES AND BLOCK DESIGNS

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ABSTRACT. The construction of optimal line packings in real or complex Euclidean spaces has been shown to be a tantalizingly difficult task, because it includes the problem of finding maximal sets of equiangular lines. In the regime where equiangular lines are not possible, some optimal packings are known, for example, those achieving the orthoplex bound related to maximal sets of mutually unbiased bases. In this paper, we investigate the packing of subspaces instead of lines and determine the implications of maximality in this context. We leverage the existence of real or complex maximal mutually unbiased bases with a combinatorial design strategy in order to find optimal subspace packings that achieve the orthoplex bound. We also show that maximal sets of mutually unbiased bases convert between coordinate projections associated with certain balanced incomplete block designs and Grassmannian 2-designs. Examples of maximal orthoplectic fusion frames already appeared in the works by Shor, Sloane, and by Zauner. They are realized in dimensions that are a power of four in the real case or a power of two in the complex case.

1. INTRODUCTION

The problem of finding the best packings of lines, one-dimensional subspaces of a real or complex Euclidean space, is easy to state. Despite its simple geometric formulation, it has given rise to a surprisingly diverse literature over many years, ranging from relatively elementary, low dimensional examples [17] to more sophisticated constructions [13], some involving combinatorial [24,27,35] or group-theoretic aspects [10,38] and results on bounds on the relationship between the number of lines and achievable angles [21,28,36]. Maximal sets of equiangular lines are known to be optimal packings, but the number of lines that can be realized is hard to determine in general [16]. Special regard has been given to the construction of complex examples, motivated by applications in quantum information theory [39]. Numerical searches indicate that they exist in many cases [31,34], but a rigorous proof of their existence is restricted to low dimensions; see [31] and references therein.

Next to lines, packings of higher-dimensional subspaces have also been investigated [11,13]. In this case, even less seems to be known about general construction principles that realize tight bounds [3,4,6,18]. More recently, these packing problems have been studied in the context of frame theory. Apart from geometric optimality criteria, frame design aims at tightness, which implies that the projections

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onto the subspaces sum to a multiple of the identity. The case of higher-dimensional subspaces corresponds to fusion frames. If the number of subspaces is not too large, then in close similarity to line packings, equidistant fusion frames present optimal solutions [22]. Examples of such constructions follow similar strategies as in the frame case [2, 5, 11, 19]. For a larger number of subspaces, such equiangular arrangements cannot be realized, and one needs to find an alternative bound for the characterization of optimal packings, for example the orthoplex bound for lines or subspaces [13]. In an earlier paper, we constructed optimal line packings when the number of lines goes slightly beyond the threshold beyond which equiangular lines are impossible to realize [7]. In this paper, we study the orthoplex bound for subspace packings and investigate cases in which the bound is achieved while the number of subspaces is maximal.

The main results are as follows. In order to maximize the number of subspaces while achieving the orthoplex bound, the dimension of the subspaces is necessarily half of the dimension of the ambient space, and the chordal distance between subspaces assumes only two values. Because of the relation with the orthoplex bound, we call these subspace packings maximal orthoplectic fusion frames. The family of examples we describe here has already appeared in the literature, either as optimal real subspace packings [32], whose discovery is ascribed to a "remarkable coincidence", or among the more general family of quantum 2-designs [39] in complex Hilbert spaces of prime power dimensions. In the complex case, it was observed that the projections are affine, Grassmannian designs [39] (see also [29]), where the construction is attributed to Rötteler. In the present paper, we examine rigidity properties in the construction of maximal orthoplectic fusion frames obtained from the theory of packings and designs [4, 5, 29, 39]. We treat the real and complex case on the same footing, involving a new construction principle. To this end, we leverage earlier constructions of orthoplex-bound achieving optimal line packings associated with mutually unbiased bases introduced by Schwinger [30]. Maximal sets of mutually unbiased bases are known to exist in the complex case in prime power dimensions [1, 9, 15, 37] and in the real case if the dimension is a power of four [12, 23]; see also [8]. We obtain maximal orthoplectic fusion frames by augmenting these maximal mutually unbiased bases with block designs, subsets of the index set that satisfy certain combinatorial conditions. The designs we construct for our purposes are known as balanced incomplete block designs and at the same time associated with optimal constant-weight binary codes [20]; see also [33]. The resulting families of subspaces are constructed in any real Hilbert space whose dimension is a power of four or in any complex Hilbert space whose dimension is a power of two.

With hindsight, one can interpret the results presented here as another instance of maximal packings having design properties (see the general bounds on the size of codes in metric spaces in the paper by Levenshteĭn [25]) which apply to binary codes and block designs as well as to spherical or even Grassmannian codes and designs; see also [3, 4]. The rigidity in the design properties helps narrow down the construction of maximal packings by additional structural requirements. In our case, these requirements appear in combinatorial design elements and in the emerging Grassmannian packings.

This paper is organized as follows. After the introduction, we fix notation and recall known distance and cardinality bounds on fusion frames in Section 2.

We relate the orthoplex bound for fusion frames with the notion of mutual unbiasedness in Section 3 and study implications for the structure of maximal orthoplectic fusion frames as packings and as Grassmannian designs. Finally, Section 4 presents the construction of a family of maximal orthoplectic fusion frames.

2. Distance bounds and Grassmannian fusion frames

Definition 2.1. Let $l, m, n \in \mathbb{N}$ and let $\mathbb{F}$ denote the field $\mathbb{R}$ or $\mathbb{C}$. An $(\mathbf{n}, \mathbf{l}, \mathbf{m})$-**fusion frame** is a set $\mathcal{F} = \{P_j\}_{j=1}^n$, where each P_j is an orthogonal projection onto an l-dimensional subspace of $\mathbb{F}^m$, such that there exist positive numbers A and B with $0 < A \leq B$ for which the chain of inequalities $A\|x\|^2 \leq \sum_{j=1}^n \|P_j x\|^2 \leq B\|x\|^2$ holds for every $x \in \mathbb{F}^m$. If we can choose $A = B$, then $\mathcal{F}$ is **tight.** If there is $C \geq 0$ such that $\mathrm{tr}(P_i P_j) = C$ for each pair $i \neq j$ in the index set $[\![n]\!] \equiv \{1, 2, \ldots, n\}$, then $\mathcal{F}$ is called **equiangular**.

By the polarization identity, the tightness property is equivalent to the fusion frame resolving the identity I_m on $\mathbb{F}^m$ according to

$$\frac{1}{A}\sum_{j=1}^{n} P_j = I_m \,.$$

More general types of fusion frames are obtained by relaxing the condition that all subspaces have the same dimension and by scaling the projections with non-negative weight factors.

For any two projections P and P' onto l-dimensional subspaces of $\mathbb{F}^m$, the *chordal distance* is defined by $d_c(P, P') = \frac{1}{\sqrt{2}}\|P - P'\| = (l - \mathrm{tr}(PP'))^{1/2}$. In order to characterize optimal packings with respect to d_c, we use an embedding that maps the projections to vectors in a higher-dimensional Hilbert space. We denote the dimension of this space as

$$d_{\mathbb{F}}(m) = \begin{cases} \frac{(m+2)(m-1)}{2}, & \mathbb{F} = \mathbb{R}, \\ m^2 - 1, & \mathbb{F} = \mathbb{C}. \end{cases}$$

Theorem 2.2 ([13])**.** *If $\mathcal{F} = \{P_j\}_{j=1}^n$ is an (n, l, m)-fusion frame, then letting*

$$V_j = \sqrt{\frac{m}{l(m-l)}}\left(P_j - \frac{l}{m} I_m\right)$$

defines a set of unit-norm vectors $\{V_j\}_{j=1}^n$ in the $d_{\mathbb{F}}(m)$-dimensional real Euclidean space of symmetric/Hermitian $m \times m$ matrices with vanishing trace, equipped with the Hilbert-Schmidt norm, such that the inner products satisfy

$$\mathrm{tr}\,(P_j P_k) = \frac{l^2}{m} + \frac{l(m-l)}{m}\,\mathrm{tr}(V_j V_k),$$

for every $j, k \in [\![n]\!]$. Furthermore, if $\mathcal{F}$ forms a tight fusion frame for $\mathbb{F}^m$, then $\sum_{j=1}^n V_j = 0$.

We use Rankin's distance bound for vectors on the sphere [28] in the formulation used by Conway, Hardin, and Sloane [13].

Theorem 2.3. *Let d be a positive integer. Any n vectors $\{v_1, v_2, \ldots, v_n\}$ on the unit sphere in $\mathbb{R}^d$ have a minimum Euclidean distance* $\min_{j,k \in [\![n]\!], j \neq k} \|v_j - v_k\| \leq \sqrt{\frac{2n}{n-1}}$, *and if equality is achieved, then* $n \le d+1$ *and the vectors form a simplex. Additionally, if* $n > d+1$*, then the minimum Euclidean distance is* $\min_{j,k\in[\![n]\!],j\neq k}\|v_j - v_k\| \le \sqrt{2}$*, and if equality holds in this case, then* $n \le 2d$*. Moreover, if* $n = 2d$*, then equality holds if and only if the vectors form an orthoplex, the union of an orthonormal basis with the negatives of its basis vectors.*

In terms of the inner products between n unit vectors in $\mathbb{R}^d$, the Rankin bound is $\max_{j,k\in[\![n]\!],j\neq k}\langle v_j, v_k\rangle \ge -\frac{1}{n-1}$, and if $n > d+1$, then it improves to the estimate $\max_{j,k\in[\![n]\!],j\neq k}\langle v_j, v_k\rangle \ge 0$.

Using the embedding from Theorem 2.2, we reformulate the Rankin bound for the Hilbert-Schmidt inner products of the projections of a fusion frame. This results in a bound that has already been derived in an alternative way before [22, 26] and in an improved bound for a larger number of subspaces, as noted in [29].

Corollary 2.4. *If* $\mathcal{F} = \{P_j\}_{j=1}^n$ *is an* (n, l, m)*-fusion frame, then*

$$\max_{j,k\in[\![n]\!],j\neq k} \operatorname{tr}(P_j P_k) \ge \frac{nl^2 - ml}{m(n-1)},$$

and if equality is achieved, then the fusion frame is equiangular and $n \le d_{\mathbb{F}}(m) + 1$*. If* $n > d_{\mathbb{F}}(m) + 1$*, then*

$$\max_{j,k\in[\![n]\!],j\neq k} \operatorname{tr}(P_j P_k) \ge \frac{l^2}{m},$$

and if equality is achieved, then $n \le 2d_{\mathbb{F}}(m)$*. Moreover, if* $n = 2d_{\mathbb{F}}(m)$*, then equality in this bound implies that for each* $j \in [\![m]\!]$*,* $\operatorname{tr}(P_j P_k) = \frac{2(2l-m)}{m}$ *for exactly one* $k \in [\![m]\!] \setminus \{j\}$ *and* $\operatorname{tr}(P_j P_k) = \frac{l^2}{m}$ *for all other* $k \in [\![m]\!] \setminus \{j\}$*.*

Definition 2.5. Let $\mathcal{F} = \{P_j\}_{j=1}^n$ be an (n, l, m)-fusion frame. If $\mathcal{F}$ is a solution to the subspace packing problem, that is, it minimizes $\max_{j\neq k} \operatorname{tr}(P_j P_k)$ among all (n, l, m)-fusion frames, then it is called a **Grassmannian** fusion frame. If $\mathcal{F}$ is Grassmannian, then it is called **orthoplex-bound achieving** if $n > d_{\mathbb{F}}(m) + 1$ and $\max\limits_{j,k\in[\![n]\!],j\neq k} \operatorname{tr}(P_j P_k) = l^2/m$. If $\mathcal{F}$ is orthoplex-bound achieving and $n = 2d_{\mathbb{F}}(m)$, then $\mathcal{F}$ is referred to as a **maximal orthoplectic fusion frame**.

We wish to construct maximal orthoplectic fusion frames, which means the projections must embed exhaustively into the vertices of an orthoplex in $\mathbb{R}^{d_{\mathbb{F}}(m)}$. The feasability of this depends on whether the embedding admits antipodal points in the higher-dimensional Euclidean sphere, which in turn depends on the relationship between l and m.

Proposition 2.6. *If* $\mathcal{F} = \{P_j\}_{j=1}^n$ *is an* (n, l, m)*-fusion frame and* $\mathcal{V} = \{V_j\}_{j=1}^n \subset \mathbb{R}^{d_{\mathbb{F}}(m)}$ *are the embedded vectors obtained from Theorem 2.2, then*

$$\min_{j,k\in[\![n]\!],j\neq k} \langle V_j, V_k\rangle \ge -\frac{l}{m-l}$$

for every $j, k \in [\![n]\!]$*. In particular, the embedding admits antipodal points only if* $l \ge \frac{m}{2}$*.*

Proof. This follows immediately from the non-negativity of the inner product between projections, $0 \le \min\limits_{j,k\in[\![n]\!],j\neq k} \operatorname{tr}(P_j P_k) = \frac{l^2}{m} + \frac{l(m-l)}{m} \min\limits_{j,k\in[\![n]\!],j\neq k}\langle V_j, V_k\rangle$. □

Partitioning a maximal orthoplectic fusion frame into orthogonal pairs of projections shows, together with the preceding proposition, that $m = 2l$.

Corollary 2.7. *If $\mathcal{F} = \{P_j\}_{j=1}^n$ is a maximal orthoplectic (n, l, m)-fusion frame, then $m = 2l$.*

Maximal orthoplectic fusion frames enjoy another property that has been studied in the literature: they are part of a family of *Grassmannian* 2*-designs*, as shown by Zauner [39]. For our purposes, this is important because it imposes more rigidity on their construction. We follow Zauner's convention for the definition of these designs.

Definition 2.8. An (n, l, m)-fusion frame $\mathcal{F} = \{P_j\}_{j=1}^n$ is called a **Grassmannian t-design** if

$$\sum_{j=1}^{n} P_j^{\otimes t} = \sum_{j=1}^{n} (UP_jU^*)^{\otimes t}$$

for any orthogonal matrix or unitary U in the real or complex case, respectively, where $P_j^{\otimes t}$ is the t-fold Kronecker/tensor product of P_j with itself and U^* is the adjoint of U or the transpose of U in the real case.

Equivalently, the right-hand side of the defining identity can be averaged with respect to the Haar measure μ on the group $\mathcal{U}$ of orthogonal or unitary $m \times m$ matrices. This formulation implies a simple characterization of the design property based on the *t-coherence tensor*

$$K_{t,l,m} = \int_{\mathcal{U}} (UPU^*)^{\otimes t} d\mu(U)\,,$$

where P is any rank-l orthogonal projection matrix. Because of the analogy with bounds for constant-weight codes [33], Zauner calls the following estimate a generalized Sidelnikov inequality.

Theorem 2.9 (Zauner, Theorem 2.5 of [39])**.** *Let $\mathcal{F} = \{P_j\}_{j=1}^n$ be an (n, l, m)-fusion frame. Then*

$$\frac{1}{n^2} \sum_{i,j=1}^{n} (\mathrm{tr}(P_iP_j))^t \geq \mathrm{tr}(K_{t,l,m}^2),$$

and equality holds if and only if $\mathcal{F}$ is a Grassmannian t-design.

Proof. Let $C = \sum_{j=1}^n P_j^{\otimes t} - nK_{t,l,m}$. Then we have $\mathrm{tr}(C^2) = \sum_{i,j=1}^n (\mathrm{tr}(P_iP_j))^t - n^2\,\mathrm{tr}(K_{t,l,m}^2) \geq 0$, and cases of equality are characterized by $C = 0$, which is the Grassmannian t-design property. □

Zauner also computes the value for the Hilbert-Schmidt norm $K_{2,l,m}$ that provides the lower bound.

Lemma 2.1 (Zauner, Lemma 2.7 of [39])**.** *Given positive integers l and m with $l \leq m$, the squared Hilbert-Schmidt norm of the 2-coherence tensor is*

$$\mathrm{tr}(K_{2,l,m}^2) = \frac{l^4}{m^2} + \frac{l^2(m-l)^2}{d_{\mathbb{F}}(m)m^2}.$$

Next, we verify that maximal orthoplectic fusion frames are Grassmannian 2-designs.

Theorem 2.10. *Given a maximal orthoplectic $(2d_{\mathbb{F}}(m), m/2, m)$-fusion frame $\mathcal{F} = \{P_j\}_{j=1}^{2d_{\mathbb{F}}(m)}$, equality holds in the generalized Sidelnikov inequality in Theorem* 2.9 *and the fusion frame is a Grassmannian* 2*-design.*

Proof. We first compute the value of the lower bound from Theorem 2.9:

$$\operatorname{tr}(K_{2,m/2,m}^2) = \frac{m^2(d_{\mathbb{F}}(m)+1)}{16 d_{\mathbb{F}}(m)} .$$

Using Corollary 2.4, we compute the average squared inner product for the projection matrices:

$$\frac{1}{4d_{\mathbb{F}}(m)^2} \sum_{j,j'=1}^{2d_{\mathbb{F}}(m)} (\operatorname{tr}(P_j P_{j'}))^2 = \frac{1}{2d_{\mathbb{F}}(m)} \left(l^2 + \frac{l(2l-m)}{m} + (2d_{\mathbb{F}}(m)-2)\frac{l^4}{m^2} \right) .$$

This last expression is verified to be equal to $\operatorname{tr}(K_{2,m/2,m}^2)$ by $l = m/2$. As stated in Theorem 2.9, this characterizes Grassmannian 2-designs. $\square$

3. Mutually unbiased bases and fusion frames

Definition 3.1. If $\mathcal{B} = \{b_j\}_{j=1}^m$ and $\mathcal{B}' = \{b'_j\}_{j=1}^m$ are a pair of orthonormal bases for $\mathbb{F}^m$, then they are **mutually unbiased** if $|\langle b_j, b'_{j'}\rangle|^2 = \frac{1}{m}$ for every $j, j' \in [\![m]\!]$. A collection of orthonormal bases $\{\mathcal{B}_k\}_{k\in K}$ is called a set of **mutually unbiased bases** if the pair $\mathcal{B}_k$ and $\mathcal{B}_{k'}$ is mutually unbiased for every $k \neq k'$.

The number of mutually unbiased bases is bounded in terms of m.

Theorem 3.2 (Delsarte, Goethals, and Seidel [14])**.** *Let $\mathbb{F}^m$ be a real or complex Hilbert space and let $\{\mathcal{B}_k\}_{k\in K}$ be a set of mutually unbiased bases for $\mathbb{F}^m$, where $\mathcal{B}_k = \left\{b_j^{(k)}\right\}$ for each $k \in K$, and let $r = |K|$. If $\mathbb{F} = \mathbb{R}$, then $r \leq m/2+1$, and if $\mathbb{F} = \mathbb{C}$, then $r \leq m+1$. If equality is achieved in either case, then the real span of the corresponding projection operators, $\left\{b_j^{(k)} \otimes \left(b_j^{(k)}\right)^* : j \in [\![m]\!], k \in K\right\}$, is that of all symmetric or Hermitian operators on $\mathbb{F}^m$, respectively.*

For the real case, it is known that, for most values of m, the maximal number of mutually unbiased bases is less than or equal to 3 [23]. However, if m is a power of 4, then examples exist that achieve the bound in Theorem 3.2 [12]. In the complex case, the bound is achieved if m is a prime power [37].

Theorem 3.3 (Cameron and Seidel [12], Wootters and Fields [37])**.** *If m is a prime power, then a family of $m+1$ mutually unbiased bases for $\mathbb{C}^m$ exists. If m is a power of* 4*, then a family of $m/2+1$ mutually unbiased bases exists for $\mathbb{R}^m$.*

Henceforth, we abbreviate $k_{\mathbb{R}}(m) = m/2+1$ and $k_{\mathbb{C}}(m) = m+1$. The rank one orthogonal projections corresponding to maximal sets of mutually unbiased bases give rise to Grassmannian 2-designs.

Proposition 3.4. *If $\{\mathcal{B}_k\}_{k\in K}$ is a set of mutually unbiased bases for $\mathbb{F}^m$ with $|K| = k_{\mathbb{F}}(m)$, then the family of rank one projections $\left\{b_j^{(k)} \otimes \left(b_j^{(k)}\right)^* : k \in K, j \in [\![m]\!]\right\}$ forms a Grassmannian* 2*-design.*

Proof. We only need to compare both sides of the inequality from Theorem 2.9. To evaluate the left-hand side, we observe that the Hilbert-Schmidt inner product is expressed in terms of the basis vectors as $\left(\operatorname{tr}\left(P_j^{(k)}P_{j'}^{(k')}\right)\right)^2 = \left|\langle b_j^{(k)}, b_{j'}^{(k')}\rangle\right|^4$. Given any fixed basis vector $b_j^{(k)}$, the fourth power of the absolute value of its inner product with the other vectors in the set are 0, which occurs $m-1$ times; $1/m^2$, which occurs $(k_{\mathbb{F}}(m)-1)m$ times; and 1, which occurs once. Averaging these gives

$$\frac{1}{m^2 k_{\mathbb{F}}(m)^2}\sum_{j,j'=1}^{m}\sum_{k,k'=1}^{k_{\mathbb{F}}(m)}\left(\operatorname{tr}\left(P_j^{(k)}P_{j'}^{(k')}\right)\right)^2 = \frac{k_{\mathbb{F}}(m)+m-1}{m^2 k_{\mathbb{F}}(m)}.$$

Comparing with the value of $\operatorname{tr}(K_{2,l,m}^2)$ in the special case $l=1$ and using $d_{\mathbb{F}}(m) = (m-1)k_{\mathbb{F}}(m)$ show that equality holds in the inequality in Theorem 2.9. $\square$

The version of the orthoplex bound for projections motivates the notion of mutual unbiasedness for fusion frames.

Definition 3.5. If $\mathcal{F} = \{P_j\}_{j=1}^n$ is an (n,l,m)-fusion frame and $\mathcal{F}' = \{P_j'\}_{j=1}^{n'}$ is an (n',l,m)-fusion frame, then $\mathcal{F}$ and $\mathcal{F}'$ are **mutually unbiased** if $\operatorname{tr}(P_j P_{j'}') = \frac{l^2}{m}$ for every $j \in [\![n]\!]$ and $j' \in [\![n']\!]$. A collection of fusion frames $\{\mathcal{F}_k\}_{k\in K}$ for $\mathbb{F}^m$, where each $\mathcal{F}_k$ consists of projections onto l-dimensional subspaces, is a set of **mutually unbiased fusion frames** if the pair $\mathcal{F}_k$ and $\mathcal{F}_{k'}$ is mutually unbiased for every $k \neq k'$.

Given a subset of a fixed orthonormal basis, the orthogonal projection onto the span is given by the sum of the corresponding rank one projections. Projections formed in this way are called *coordinate projections.*

Definition 3.6. Given an orthonormal basis $\mathcal{B} = \{b_j\}_{j=1}^m$ for $\mathbb{F}^m$ and a subset $\mathcal{J} \subset [\![m]\!]$, the **$\mathcal{J}$-coordinate projection** with respect to $\mathcal{B}$ is $P_{\mathcal{J}} = \sum_{j\in\mathcal{J}} b_j \otimes b_j^*$.

Given a pair of mutually unbiased bases, one can select coordinate projections from the respective bases to form mutually unbiased fusion frames.

Proposition 3.7. *If $\mathcal{B} = \{b_j\}_{j=1}^m$ and $\mathcal{B}' = \{b'_j\}_{j=1}^m$ are a pair of mutually unbiased bases for $\mathbb{F}^m$ and $\mathcal{J}, \mathcal{J}' \subset [\![m]\!]$ with $l = |\mathcal{J}| = |\mathcal{J}'|$, then $\operatorname{tr}(P_{\mathcal{J}} P'_{\mathcal{J}'}) = \frac{l^2}{m}$, where $P_{\mathcal{J}}$ is the $\mathcal{J}$-coordinate projection with respect to $\mathcal{B}$ and $P'_{\mathcal{J}'}$ is the $\mathcal{J}'$-coordinate projection with respect to $\mathcal{B}'$. Moreover, if $\mathcal{F} = \{P_{\mathcal{J}}\}_{\mathcal{J}\in\mathbb{S}}$ is a set of coordinate projections with respect to $\mathcal{B}$, $\mathcal{F}' = \{P'_{\mathcal{J}'}\}_{\mathcal{J}'\in\mathbb{S}'}$ is a set of coordinate projections with respect to $\mathcal{B}'$, $\mathcal{F}$ is an $(|\mathbb{S}|, l, m)$-fusion frame, and $\mathcal{F}'$ is an $(|\mathbb{S}'|, l, m)$-fusion frame, then $\mathcal{F} \cup \mathcal{F}'$ are mutually unbiased.*

Proof. We compute

$$\operatorname{tr}(P_{\mathcal{J}}P'_{\mathcal{J}'}) = \sum_{j\in\mathcal{J},j'\in\mathcal{J}'} \operatorname{tr}\left(b_j \otimes (b_j)^* b'_{j'} \otimes (b'_{j'})^*\right) = \sum_{j\in\mathcal{J},j'\in\mathcal{J}'} |\langle b_j, b'_{j'}\rangle|^2 = \frac{l^2}{m}.$$

The claim about mutual unbiasedness follows directly from this computation. $\square$

We repeat the embedding of fusion frames for the special case of coordinate projections. Tight fusion frames of coordinate projections have also been investigated as commutative quantum designs by Zauner [39]. We first focus on the structure of optimal packings of coordinate projections.

Theorem 3.8. *Let $\mathcal{B} = \{b_j\}_{j=1}^m$ be an orthonormal basis for $\mathbb{F}^m$ and let $\mathbb{S} = \{\mathcal{J}_j\}_{j=1}^n$ be a set of subsets of $[\![m]\!]$, each of size $|\mathcal{J}_j| = l$. If $\mathcal{F} = \{P_j\}_{j=1}^n$ is a family of projections for which P_j is the $\mathcal{J}_j$-coordinate projection with respect to $\mathcal{B}$, then the set of unit vectors $\mathcal{V} = \{V_j\}_{j=1}^n$ obtained as in Theorem 2.2 resides in a $m-1$-dimensional subspace of the real Euclidean space of symmetric matrices and*

$$\operatorname{tr}(P_j P_{j'}) = \frac{l^2}{m} + \frac{l(m-l)}{m}\langle V_j, V_{j'}\rangle,$$

for every $j, j' \in [\![n]\!]$.

Proof. By definition, $\mathcal{F}$ is a set of rank-l orthogonal projections that can be regarded as diagonal matrices when represented in the basis $\mathcal{B}$. The mapping $P_j \mapsto V_j := \sqrt{\frac{m}{l(m-l)}}(P_j - \frac{l}{m}I_m)$ embeds the projections into the real diagonal matrices with zero trace. This implies $\dim\big(\operatorname{span}\{V_j\}_{j=1}^n\big) \leq m-1$. The identity for the Hilbert-Schmidt inner products follows directly from the definition of $\{V_j\}_{j=1}^n$. $\square$

In this special case, the Rankin bound can be expressed in terms of the subsets indexing the coordinate projections, because $\operatorname{tr}(P_{\mathcal{J}}P_{\mathcal{J}'}) = |\mathcal{J} \cap \mathcal{J}'|$ for any $\mathcal{J}, \mathcal{J}' \subset [\![m]\!]$.

Corollary 3.9. *If $\mathbb{S}$ is a collection of n subsets of $[\![m]\!]$ for which $n > m$ and each $\mathcal{J} \in \mathbb{S}$ has size $|\mathcal{J}| = l$, then*

$$\max_{\mathcal{J},\mathcal{J}' \in \mathbb{S}, \mathcal{J} \neq \mathcal{J}'} |\mathcal{J} \cap \mathcal{J}'| \geq \frac{l^2}{m}\,.$$

Moreover, if $n = 2(m-1)$ and equality holds in this bound, then $\mathbb{S}$ can be partitioned into $m-1$ disjoint pairs of subsets of size $m/2$.

A more general bound of this type has been derived by Johnson in the context of *constant-weight codes* [20, inequality (14)]. The case of equality was also known to Levenshteĭn as a special case of a Delsarte code among the constant-weight codes of size $m(m-1)/l = 2(m-1)$ and minimum Hamming distance $2(l-1)(m-l)/(m-2) = l$; see [25, Table 8.4, p. 63].

In light of Corollary 3.9 and Proposition 3.7, we pursue the idea of generating tight, orthoplex-bound achieving fusion frames by using coordinate projections from *maximal sets of mutually unbiased bases*, which are sets of mutually unbiased bases that achieve the cardinality bound in Theorem 3.2. In order to construct an orthoplex-bound achieving fusion frame, we need $n > d_{\mathbb{F}}(m)+1$ subspaces. Thus, given a maximal set of mutually unbiased bases, we need a sufficient number of coordinate projections per basis with low Hilbert-Schmidt inner products. In order to bound the inner products between coordinate projections corresponding to a given orthonormal basis, the intersection of any two different index sets $\mathcal{J}$ and $\mathcal{J}'$ must have a small intersection, which we call a cohesiveness bound.

Definition 3.10. Let $\mathbb{S}$ be a collection of subsets of $[\![m]\!]$, each $\mathcal{J} \in \mathbb{S}$ of size l. We say that $\mathbb{S}$ is c-**cohesive** if there exists $c > 0$ such that $\max_{\substack{\mathcal{J},\mathcal{J}' \in \mathbb{S} \\ \mathcal{J} \neq \mathcal{J}'}} |\mathcal{J} \cap \mathcal{J}'| \leq c$. If $\mathbb{S}$ is l^2/m-cohesive and $|\mathbb{S}| = 2(m-1)$, then it is **maximally orthoplectic**.

Theorem 3.11. *Let $\mathbb{S}$ be an l^2/m-cohesive collection of subsets of $[\![m]\!]$, where each $\mathcal{J} \in \mathbb{S}$ is of size l, let $\{\mathcal{B}_k\}_{k\in K}$ be a set of mutually unbiased bases for $\mathbb{F}^m$, where*

$|K||\mathbb{S}| > d_{\mathbb{F}}(m)+1$ and $\mathcal{B}_k = \left\{b_j^{(k)}\right\}_{j=1}^m$ for each $k \in K$, and let $n = |K||\mathbb{S}|$. If the set $\mathcal{F} = \left\{P_{\mathcal{J}}^{(k)} : k \in K, \mathcal{J} \in \mathbb{S}\right\}$ forms an (n,l,m)-fusion frame, where each $P_{\mathcal{J}}^{(k)}$ denotes the $\mathcal{J}$-coordinate projection with respect to $\mathcal{B}_k$, then $\mathcal{F}$ is an orthoplex-bound achieving (n,l,m)-fusion frame. Moreover, if $([\![m]\!],\mathbb{S})$ is maximally orthoplectic and if $|K| = k_{\mathbb{F}}(m)$, then the set $\mathcal{F}$ is a tight, maximal orthoplectic fusion frame.

Proof. The cardinality requirement in the definition of orthoplex-bound achieving fusion frames is satisfied since $n > d_{\mathbb{F}}(m)+1$. Let $k, k' \in K$. If $k \neq k'$, then

$$\operatorname{tr}\left(P_{\mathcal{J}}^{(k)} P_{\mathcal{J}'}^{(k')}\right) = \frac{l^2}{m}$$

for every $\mathcal{J}, \mathcal{J}' \in \mathbb{S}$ by Proposition 3.7. If $k = k'$, then the fact that $\mathbb{S}$ is l^2/m-cohesive yields

$$\max_{\substack{\mathcal{J},\mathcal{J}' \in \mathbb{S}\\ \mathcal{J}\neq\mathcal{J}'}} \operatorname{tr}\left(P_{\mathcal{J}}^{(k)} P_{\mathcal{J}'}^{(k)}\right) = \max_{\substack{\mathcal{J},\mathcal{J}' \in \mathbb{S}\\ \mathcal{J}\neq\mathcal{J}'}} |\mathcal{J} \cap \mathcal{J}'| \leq \frac{l^2}{m},$$

which shows that $\mathcal{F}$ is an orthoplex-bound achieving fusion frame. Finally, if $\mathbb{S}$ is maximally orthoplectic and the set of mutually unbiased bases is maximal, then Corollary 3.9 shows that the coordinate projections belonging to each basis sum to a multiple of the identity, so the corresponding fusion frame is tight. Hence, the union of all the coordinate projections belonging to the mutually unbiased bases forms a set of $n = k_{\mathbb{F}}(m)\,(2m-2) = 2d_{\mathbb{F}}(m)$ orthogonal projections whose pairwise inner products are bounded by l^2/m, which shows that $\mathcal{F}$ is a maximal orthoplectic fusion frame. □

Following Zauner's ideas, we repeat the study of design properties for the special case of a fusion frame formed by coordinate projections. To this end, we define the *diagonal coherence tensor*,

$$D_{t,l,m} = \frac{1}{\binom{m}{l}} \sum_{\mathcal{J} \in \mathbb{J}} D_{\mathcal{J}}^{\otimes t},$$

where $\mathbb{J}$ is the set of all subsets of $[\![m]\!]$ of size l, and, for each $\mathcal{J} \in \mathbb{J}$, $D_{\mathcal{J}}$ is the $\mathcal{J}$-coordinate projection with respect to the canonical basis. An elementary counting argument shows that $D_{1,l,m} = \frac{l}{m} I_m$ and

$$D_{2,l,m} = \frac{l}{m}\sum_{j=1}^m E_{j,j}\otimes E_{j,j} + \frac{l(l-1)}{m(m-1)} \sum_{\substack{j,j'=1\\ j\neq j'}}^m E_{j,j}\otimes E_{j',j'},$$

where $\{e_j\}_{j=1}^m$ denotes the canonical basis for $\mathbb{F}^m$ and $E_{j,j'} = e_j \otimes e_{j'}^*$ for each $j, j' \in [\![m]\!]$. By squaring the diagonal entries and summing, we compute

$$\operatorname{tr}(D_{2,l,m}^2) = m\frac{l^2}{m^2} + m(m-1)\frac{l^2(l-1)^2}{m^2(m-1)^2} = \frac{l^2}{m(m-1)}(l^2-2l+m)\,.$$

With this notation, the combinatorial notion of a *block t-design* is characterized conveniently.

Definition 3.12. A **t-(m,l,λ) block design** $\mathbb{S}$ is a collection of subsets of $[\![m]\!]$ called **blocks**, where each block $\mathcal{J} \in \mathbb{S}$ has cardinality l, such that every subset of $[\![m]\!]$ with cardinality t is contained in exactly λ blocks. When the parameters are not important or implied by the context, then $\mathbb{S}$ is also referred to as a **block t-design**. The special case of a block 2-design is also referred to as a **balanced incomplete block design**.

Theorem 3.13 (Zauner [39, Theorem 1.12])**.** *A collection $\mathbb{S}$ of subsets of $[\![m]\!]$ where each $\mathcal{J} \in \mathbb{S}$ has size l is a t-(m,l,λ) block design if and only if*

$$\frac{1}{|\mathbb{S}|}\sum_{\mathcal{J}\in\mathbb{S}} D_{\mathcal{J}}^{\otimes t} = D_{t,l,m},$$

with $\lambda = |\mathbb{S}| \operatorname{tr}\left(D_{t,l,m} \bigotimes_{s=1}^{t} E_{s,s}\right)$, where $\{e_j\}_{j=1}^m$ denotes the canonical basis for $\mathbb{F}^m$ and $E_{s,s} = e_s \otimes e_s^*$ for each $s \in [\![m]\!]$.

In the special case where $t = 1$ in Theorem 3.13, we obtain the correspondence between the block 1-design property of $\mathbb{S}$ and tightness of the corresponding fusion frame of coordinate projections.

Corollary 3.14. *If $\mathcal{B}$ is any orthonormal basis for $\mathbb{F}^m$, then a set of coordinate projections $\{P_{\mathcal{J}}\}_{\mathcal{J}\in\mathbb{S}}$ with respect to $\mathcal{B}$ is a tight fusion frame if and only if $\mathbb{S}$ is a 1-design.*

Given any positive integers l and m with $l \le m$, one can choose the set of all blocks of size l from $[\![m]\!]$ to form a tight fusion frame in this way.

Example 3.15. If $\mathbb{S} = \{\mathcal{J} : \mathcal{J} \subset [\![m]\!], |\mathcal{J}| = l\}$, the set of all blocks of size l, then $\mathbb{S}$ forms a t-(m,l,λ) block design. Given an orthonormal basis $\mathcal{B}$ for $\mathbb{F}^m$, the corresponding set of coordinate projections with respect to $\mathcal{B}$, $\mathcal{F} = \{P_{\mathcal{J}}\}_{\mathcal{J}\in\mathbb{S}}$ forms a tight (n,l,m)-fusion frame by Corollary 3.14, where $n = |\mathbb{S}| = \binom{m}{l}$.

With the same proof as in Theorem 2.9, we obtain an analogous characterization of block t-designs.

Corollary 3.16. *Given a collection $\mathbb{S}$ of subsets of $[\![m]\!]$, where each $\mathcal{J} \in \mathbb{S}$ has size l,*

$$\frac{1}{|\mathbb{S}|^2}\sum_{\mathcal{J},\mathcal{J}'\in\mathbb{S}} |\mathcal{J}\cap\mathcal{J}'|^t \ge \operatorname{tr}(D_{t,l,m}^2),$$

and equality holds if and only if $\mathbb{S}$ is a t-(m,l,λ) block design with λ as in Theorem 3.13.

Levenshteĭn's classical bounds in polynomial metric spaces [25] show that when $\mathbb{S}$ is chosen to be a maximal constant-weight Johnson code with weight $l = m/2$ and size $2m-2$, it is a 2-$(m, m/2, m/2-1)$ block design. In more detail, $\mathbb{S}$ has minimal Hamming distance $2l - 2l^2/m = m/2$, Levenshteĭn's standard substitution function is linear, and his dimensionality constants are $r_0 = 1$ and $r_1 = \binom{2m-2}{1} - 1 = 2m-3$. Hence, equality is achieved in bound (1.24) of [25], from which it follows that bound (1.26) is saturated as well, implying that $\mathbb{S}$ is a 2-design.

We can now deduce that subsets of coordinate projections in a maximal orthoplectic fusion frame constructed from a maximal set of mutually unbiased bases realize block 2-designs. For the sake of keeping the exposition self-contained, we have chosen to incorporate the combinatorial part of the proof in the form of the tensor-based characterization in Corollary 3.16.

Theorem 3.17. *Let $\{\mathcal{B}_k\}_{k\in K}$ be a maximal set of mutually unbiased bases for $\mathbb{F}^m$, so $|K| = k_{\mathbb{F}}(m)$, and let $\mathbb{S} \subset [\![m]\!]$ be a collection of subsets, each with size l. If $\mathcal{F} = \left\{P_{\mathcal{J}}^{(k)} : k \in K, \mathcal{J} \in \mathbb{S}\right\}$ is a $(2d_{\mathbb{F}}(m), m/2, m)$-fusion frame, where each $P_{\mathcal{J}}^{(k)}$ is the $\mathcal{J}$-coordinate projection with respect to $\mathcal{B}_k$, then $\mathcal{F}$ is a Grassmannian 2-design if and only if $\mathbb{S}$ is a 2-$(m, m/2, m/2-1)$ block design.*

Proof. First, let $\mathcal{F}$ be a Grassmannian 2-design. By Corollary 2.1 and the choice of $l = m/2$,

$$\sum_{k,k'\in K}\sum_{\mathcal{J},\mathcal{J}'\in\mathbb{S}} \left(\operatorname{tr}(P_{\mathcal{J}}^{(k)}P_{\mathcal{J}'}^{(k')})\right)^2 = \frac{m^2 d_{\mathbb{F}}(m)(d_{\mathbb{F}}(m)+1)}{4}.$$

From the assumption on the size $k_{\mathbb{F}}(m) = |K|$ and $d_{\mathbb{F}}(m) = (m-1)k_{\mathbb{F}}(m)$, the set $\mathbb{S}$ has size $|\mathbb{S}| = 2(m-1)$. Since the orthormal bases are unitarily equivalent and each pair of them is mutually unbiased, we can obtain the sum for the squared Hilbert-Schmidt inner products belonging to one basis,

$$\begin{aligned}\sum_{\mathcal{J},\mathcal{J}'\in\mathbb{S}} \left(\operatorname{tr}(P_{\mathcal{J}}^{(k)}P_{\mathcal{J}'}^{(k)})\right)^2 &= \frac{m^2 d_{\mathbb{F}}(m)(d_{\mathbb{F}}(m)+1)}{4k_{\mathbb{F}}(m)} - (k_{\mathbb{F}}(m)-1)4(m-1)^2\frac{m^2}{16}\\ &= \frac{(m-1)m^2}{4}\big(k_{\mathbb{F}}(m)(m-1)+1-(k_{\mathbb{F}}(m)-1)(m-1)\big) = \frac{(m-1)m^3}{4}.\end{aligned}$$

The average of the Hilbert-Schmidt inner products of the $|\mathbb{S}| = 2(m-1)$ coordinate projections belonging to each basis is then

$$\frac{1}{|\mathbb{S}|^2}\sum_{\mathcal{J},\mathcal{J}'\in\mathbb{S}} \left(\operatorname{tr}(P_{\mathcal{J}}^{(k)}P_{\mathcal{J}'}^{(k)})\right)^2 = \frac{1}{|\mathbb{S}|}\left(l^2 + (n-2)\frac{l^4}{m^2}\right).$$

Specializing the expression $\operatorname{tr}\left(D_{2,l,m}^2\right) = l^2(l^2-2l+m)/m(m-1)$ to $l = m/2$ shows that equality holds in Corollary 3.16 for $t = 2$, so $\mathbb{S}$ is a block 2-design. The parameter of the design then follows from $\lambda = |\mathbb{S}|\frac{l(l-1)}{m(m-1)} = m/2-1$.

Conversely, if $\mathbb{S}$ is a 2-$(m, m/2, m/2-1)$ block design, then equality holds in the inequality in Corollary 3.16. Since the squared inner product between any two coordinate projections belonging to different bases equals l^4/m^2, the lower bound from Corollary 3.16 is equivalent to a lower bound for the squared inner products among the coordinate projections belonging to all mutually unbiased bases, and both bounds are saturated. Using the preceding two identities shows that this implies that equality holds in the inequality in Theorem 2.9 and hence $\mathcal{F}$ is a Grassmannian 2-design. ☐

4. A family of maximal orthoplectic fusion frames

In this section, we construct a family of $\{0,1\}$-matrices, $\{S_r\}_{r\in\mathbb{N}}$, and show that they generate maximally orthoplectic block 1-designs and therefore generate maximal orthoplectic fusion frames by Theorem 3.11. Consequently, by Theorem 3.17 they are 2-designs. We give an independent proof of this fact to illustrate the rigidity in the construction of these matrices.

We recall from Corollary 2.7 that a necessary condition for the existence of a maximal orthoplectic fusion frame is that the subspace dimension is $l = \frac{m}{2}$. In order to exploit the existence of maximal sets of mutually unbiased bases in prime power dimensions, it is natural to focus on the case where m is a power of two. We construct the block 1-designs in terms of the associated *incidence matrices.*

Definition 4.1. The **incidence matrix** S associated with a sequence $\mathbb{S} = \{\mathcal{J}_1, \mathcal{J}_2, \dots, \mathcal{J}_n\}$ of subsets of $[\![m]\!]$ is an $m \times n$ matrix whose (a, b)-th entry is

$$S_{a,b} = \begin{cases} 1, & a \in \mathcal{J}_b, \\ 0, & \text{otherwise.} \end{cases}$$

Let $S_1 = I_2$. For $r \in \mathbb{N}$, let $F_r = I_{(2^r-1)} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and let 1_r denote the $2^r \times 1$ matrix of all ones and let 0_r be the $2^r \times 1$ matrix of all zeros. For $r \geq 2$, define S_r recursively and blockwise by

$$S_r = \left(B_r^{(i)} \ B_r^{(ii)} \ B_r^{(iii)} \right),$$

where

$$B_r^{(i)} = \begin{pmatrix} 1_{r-1} & 0_{r-1} \\ 0_{r-1} & 1_{r-1} \end{pmatrix}, B_r^{(ii)} = \begin{pmatrix} S_{r-1} \\ S_{r-1} \end{pmatrix}, \text{ and } B_r^{(iii)} = \begin{pmatrix} S_{r-1} \\ S_{r-1}F_{r-1} \end{pmatrix}.$$

If c_r and ρ_r denote the number of columns and rows of S_t, respectively, then we have the recurrence relation

$$c_1 = 2, c_{r+1} = 2c_r + 2,$$

which has the solution $c_r = 2^{r+1} - 2$. By the construction of S_r, $\rho_{r+1} = 2\rho_r = 2^r$, so S_r is a $2^r \times (2^{r+1} - 2)$ matrix.

Furthermore, if $\tilde{c}_r^{(j)}$ denotes the number of ones in the jth column of S_r and $\tilde{\rho}_r^{(j)}$ denotes the number of ones in the j-th row of S_r, then it is straightforward to verify, by construction, that both of these values are independent of j. In particular, $\tilde{c}_r^{(j)} = 2^{r-1}$ for each j and $\tilde{\rho}_r^{(j)} = 2^r - 1$ for each j in the index set of columns or rows, respectively. We record this as a lemma.

Lemma 4.1. *Each column of S_r has exactly 2^{r-1} ones among its entries, and each row of S_r has exactly $2^r - 1$ ones among its entries.*

Next, we examine the inner products among the columns $\{s_j\}_{j \in [\![c_r]\!]}$, of S_r, noting that these are encoded in the matrix

$$S_r^* S_r = \left(\langle s_b, s_a \rangle \right)_{a,b=1}^{c_r}.$$

We write $J_{x,y}$ for the $x \times y$ matrix whose entries all equal 1.

Lemma 4.2. *For each $r \in \mathbb{N}$, the matrix $G = S_r^* S_r$ is of the form*

$$G = 2^{r-2} \cdot \left[J_{c_r,c_r} + I_{c_r/2} \otimes \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right].$$

Proof. We prove the claimed form of G by induction. The claim is true for $r = 1$, so let $r > 1$, and assume the claim holds for $r - 1$.

Using the block structure in the definition of S_r, we have

$$G = \begin{pmatrix} \left(B_r^{(i)}\right)^* B_r^{(i)} & \left(B_r^{(i)}\right)^* B_r^{(ii)} & \left(B_r^{(i)}\right)^* B_r^{(iii)} \\ \left(B_r^{(ii)}\right)^* B_r^{(i)} & \left(B_r^{(ii)}\right)^* B_r^{(ii)} & \left(B_r^{(ii)}\right)^* B_r^{(iii)} \\ \left(B_r^{(iii)}\right)^* B_r^{(i)} & \left(B_r^{(iii)}\right)^* B_r^{(ii)} & \left(B_r^{(iii)}\right)^* B_r^{(iii)} \end{pmatrix}.$$

A direct application of the definition of $B_r^{(i)}$ and Lemma 4.1 gives the values of the first row and first column of blocks in G:

$$G = \begin{pmatrix} 2^{r-1} \cdot I_2 & 2^{r-2} \cdot J_{2,c_{r-1}} & 2^{r-2} \cdot J_{2,c_{r-1}} \\ 2^{r-2} \cdot J_{c_{r-1},2} & \left(B_r^{(ii)}\right)^* B_r^{(ii)} & \left(B_r^{(ii)}\right)^* B_r^{(iii)} \\ 2^{r-2} \cdot J_{c_{r-1},2} & \left(B_r^{(iii)}\right)^* B_r^{(ii)} & \left(B_r^{(iii)}\right)^* B_r^{(iii)} \end{pmatrix}.$$

A direct application of the induction hypothesis gives us the center block,

$$\left(B_r^{(ii)}\right)^* B_r^{(ii)} = S_{r-1}^* S_{r-1} + S_{r-1}^* S_{r-1} = 2^{r-2}\left[J_{c_{r-1},c_{r-1}} + I_{(c_{r-1}/2)} \otimes \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right].$$

Next, observe that by the induction assumption and definition of F_{r-1}, we have

$$\begin{aligned} S_{r-1}^* S_{r-1} F_{r-1} &= 2^{r-3}\left[J_{c_{r-1},c_{r-1}} + I_{c_r/2} \otimes \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right]\left[I_{c_{r-1}/2} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \\ &= 2^{r-3}\left[J_{c_{r-1},c_{r-1}} + I_{c_r/2} \otimes \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right], \end{aligned}$$

so it follows that

$$\left(B_r^{(ii)}\right)^* B_r^{(iii)} = S_{r-1}^* S_{r-1} + S_{r-1}^* S_{r-1} F_{r-1} = 2^{r-2} J_{c_{r-1},c_{r-1}},$$

and, by symmetry, we also have

$$\left(B_r^{(iii)}\right)^* B_r^{(ii)} = 2^{r-2} J_{c_{r-1},c_{r-1}}.$$

Finally, observe that $F_{r-1}^* S_{r-1}^* S_{r-1} F_{r-1} = S_{r-1}^* S_{r-1}$, so

$$\begin{aligned} \left(B_r^{(iii)}\right)^* B_r^{(iii)} &= S_{r-1}^* S_{r-1} + F_{r-1}^* S_{r-1}^* S_{r-1} F_{r-1} \\ &= 2^{r-2}\left[J_{c_{r-1},c_{r-1}} + I_{c_{r-1}/2} \otimes \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right]. \end{aligned}$$

This establishes that the nine blocks match the claimed form of G. □

For each $r \in \mathbb{N}$, we let $\mathbb{S}_r$ be the set of blocks in $[\![m]\!]$ defined in accordance with the columns of S_r by $\mathcal{J}_b = \{a : (S_r)_{a,b} = 1\}$, where $(S_r)_{a,b}$ denotes the (a,b)-th entry of S_t.

Finally, we state the main theorem of this section, which summarizes the construction of maximal orthoplectic tight fusion frames.

Theorem 4.2. *Let $r \in \mathbb{N}$, where r is even if $\mathbb{F} = \mathbb{R}$. If $m = 2^r$ and $\{\mathcal{B}_k\}_{k\in K}$ is a maximal collection of mutually unbiased bases for $\mathbb{F}^m$, so $|K| = k_{\mathbb{F}}(m)$, then $\mathcal{F} = \left\{P_{\mathcal{J}}^{(k)} : k \in K, \mathcal{J} \in \mathbb{S}_r\right\}$ forms a tight, maximal orthoplectic fusion frame, where each $P_{\mathcal{J}}^{(k)}$ is the $\mathcal{J}$-coordinate projection with respect to $\mathcal{B}_k$.*

Proof. It follows directly from Lemma 4.1 that $\mathbb{S}_r$ is a 1-$(m, l, m-1)$ block design, where $m = 2^r$, $l = 2^{r-1}$, and $|\mathbb{S}_r| = c_r = 2^{r+1} - 2$. If $\{s_j\}_{j\in[\![c_r]\!]}$ denotes the columns of S_r, then the Gram matrix $S_r^* S_r$ encodes the intersections of the blocks by $\langle s_a, s_b\rangle = |\mathcal{J}_a \cap \mathcal{J}_b|$, so Lemma 4.2 implies that $\max_{a\neq b} |\mathcal{J}_a \cap \mathcal{J}_b| = 2^{r-2}$. This means that $\mathbb{S}_r$ is c-cohesive, where $c = \frac{l^2}{m} = \frac{m}{4}$, and since $|\mathbb{S}_r| = 2^{r+1} - 2 = 2(m-1)$, we conclude that $\mathbb{S}_r$ is a maximally orthoplectic block 1-design.

Finally, using a maximal set of mutually unbiased bases $\{\mathcal{B}_k\}_{k\in K}$ and the maximally orthoplectic block 1-$(m, m/2, m-1)$ design $\mathbb{S}_r$, Theorem 3.11 shows that the set $\mathcal{F} = \{P_{\mathcal{J}}^{(k)} : k \in K,\, \mathcal{J} \in \mathbb{S}_r\}$ forms a maximal orthoplectic fusion frame, where each $P_{\mathcal{J}}^{(k)}$ is the $\mathcal{J}$-coordinate projection with respect to $\mathcal{B}_k$. □

Although Lemma 4.2 only shows that $\mathbb{S}_r$ is an l^2/m-cohesive block 1-design, by the maximality of $\mathcal{F}$, Theorem 3.17 implies that $\mathbb{S}_r$ is a block 2-design.

Corollary 4.3. *The block set* $\mathbb{S}_r$ *is a* 2-$(2^r, 2^{r-1}, 2^{r-1}-1)$ *block design.*

The 2-design property of $\mathbb{S}_r$ does not appear explicitly in the proof of Theorem 4.2. Nevertheless, the additional, implicit structural constraints were useful for finding the incidence matrix S_r, which is central to our construction.

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