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WILLMORE INEQUALITY ON HYPERSURFACES IN HYPERBOLIC SPACE

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Abstract. In this article, we prove a geometric inequality for star-shaped and mean-convex hypersurfaces in hyperbolic space by inverse mean curvature flow. This inequality can be considered as a generalization of Willmore inequality for a closed surface in hyperbolic 3-space.

1. Introduction

The classical isoperimetric inequality and its generalization, the Alexandrov-Fenchel inequalities, play an important role in different branches of geometry. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain with boundary Σ. Then the classical isoperimetric inequality is

$$|\Sigma| \geq n^{\frac{n-1}{n}} \omega_{n-1}^{\frac{1}{n}} |\Omega|^{\frac{n-1}{n}}, \tag{1.1}$$

and equality in (1.1) holds if and only if Ω is a geodesic ball.

For $k \in \{1, \cdots, n-1\}$, we denote by p_k the normalized k-th order mean curvature of Σ, and set $p_0 = 1$ by convention. The celebrated Alexandrov-Fenchel inequalities [1, 2, 15] for convex hypersurface $\Sigma^{n-1} \subset \mathbb{R}^n$ are

$$\frac{1}{\omega_{n-1}} \int_\Sigma p_k d\mu \geq \left(\frac{1}{\omega_{n-1}} \int_\Sigma p_j d\mu \right)^{\frac{n-1-k}{n-1-j}}, \quad 0 \leq j < k \leq n-1, \tag{1.2}$$

and equality in (1.2) holds if and only if Σ is a geodesic sphere.

Since the isoperimetric inequality holds for non-convex domains, it is natural to extend the original Alexandrov-Fenchel inequality to non-convex domains; see [7–9, 20, 21, 28, 31], etc. We should also mention that the Willmore inequality, which is a weaker form of Alexandrov-Fenchel inequality, has been established for closed surfaces in $\mathbb{R}^3$; see e.g. [10, 26, 30]. More precisely, for any closed surface $\Sigma \subset \mathbb{R}^3$, the Willmore inequality is

$$\int_\Sigma p_1^2 d\mu \geq \omega_2 = 4\pi, \tag{1.3}$$

and equality in (1.3) holds if and only if Σ is a geodesic sphere.

It is interesting to establish the Alexandrov-Fenchel inequalities for hypersurfaces in hyperbolic space; see [4, 16]. Recently, the following hyperbolic Alexandrov-Fenchel inequalities were obtained by Ge-Wang-Wu [17, 18] and Wang-Xia [32].

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Theorem A ([17, 18, 32]). *Let $k \in \{1, \cdots, n-1\}$. Any horospherical convex hypersurface $\Sigma \subset \mathbb{H}^n$ satisfies*

$$\int_\Sigma p_k d\mu \geq \omega_{n-1} \left[\left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{2}{k}} + \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{2(n-1-k)}{k(n-1)}} \right]^{\frac{k}{2}}. \tag{1.4}$$

Equality in (1.4) *holds if and only if* Σ *is a geodesic sphere.*

Inequality (1.4) was proved in [17] for $k = 4$ and in [18] for general even k. For $k = 1$, (1.4) was proved in [18] with a help of a result of Cheng and Zhou [12]. For general integer k, (1.4) was proved in [32].

For $k = 2$, inequality (1.4) was proved by Li-Wei-Xiong [25] under a weaker condition that Σ is star-shaped and 2-convex (i.e., $p_2 > 0$). More precisely,

Theorem B ([25]). *Any star-shaped and* 2*-convex hypersurface* $\Sigma \subset \mathbb{H}^n$ $(n \geq 3)$ *satisfies*

$$\int_\Sigma p_2 d\mu \geq \omega_{n-1}^{\frac{2}{n-1}} |\Sigma|^{\frac{n-3}{n-1}} + |\Sigma|. \tag{1.5}$$

Equality in (1.5) *holds if and only if* Σ *is a geodesic sphere.*

The Willmore inequality (1.3) has also been generalized to closed surface $\Sigma \subset \mathbb{H}^3$; see e.g. [11, 27, 29, 33].

Theorem C ([11, 27, 29, 33]). *Any closed surface* $\Sigma \subset \mathbb{H}^3$ *satisfies*

$$\int_\Sigma (p_1^2 - 1) d\mu \geq \omega_2 = 4\pi. \tag{1.6}$$

Equality in (1.6) *holds if and only if* Σ *is a geodesic sphere.*

To generalize the hyperbolic Willmore inequality to higher dimension, the positivity of the functional $\int_\Sigma (p_1^2 - 1) d\mu$ has already been known. This follows from the optimal Reilly inequality for submanifolds of hyperbolic space, which was achieved by El Soufi and Ilias [14].

Theorem D ([14]). *Let* (M^m, g) $(m \geq 2)$ *be a compact and connected Riemannian manifold isometrically immersed in* $\mathbb{H}^n$ *by* ϕ*. Then*

$$\lambda_1(M) \leq \frac{m}{|M|} \int_M (|H|^2 - 1) d\mu, \tag{1.7}$$

where $\lambda_1(M)$ *is the first non-zero eigenvalue of Laplacian of* (M, g) *and* H *is the mean curvature vector of* M*. Furthermore, equality in* (1.7) *holds if and only if* $\phi(M)$ *is minimally immersed in a geodesic sphere of radius* $\sinh^{-1}(\sqrt{\frac{m}{\lambda_1(M)}})$.

Inspired by these previous results, we prove the Willmore inequality for star-shaped and mean-convex (i.e., $p_1 > 0$) hypersurfaces in hyperbolic space.

Theorem 1. *Any star-shaped and mean-convex hypersurface* $\Sigma \subset \mathbb{H}^n$ $(n \geq 3)$ *satisfies*

$$\int_\Sigma (p_1^2 - 1) d\mu \geq \omega_{n-1}^{\frac{2}{n-1}} |\Sigma|^{\frac{n-3}{n-1}}. \tag{1.8}$$

Equality in (1.8) *holds if and only if* Σ *is a geodesic sphere.*

We expect that the inequality (1.8) will be useful in defining the Hawking mass for hypersurfaces in $\mathbb{H}^n$. In [24], the *Hawking mass* for a closed embedded surface Σ in $\mathbb{H}^3$ is defined as

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}\left[1 - \frac{1}{4\pi}\int_\Sigma (p_1^2 - 1)d\mu\right].$$

We now give the outline of the proof of Theorem 1. In fact, we prove it in a more general setting. Motivated by [6, 13, 25], we adopt the inverse curvature flow (ICF)

$$\partial_t X = \frac{1}{(n-1)p_k^{1/k}}\nu$$

in our proof. When $k = 1$, this flow is just inverse mean curvature flow, which has been used by Huisken and Ilmanen [22, 23] to prove the Riemannian Penrose inequality in general relativity. We start from a given star-shaped and k-convex hypersurface Σ in hyperbolic space and evolve it by ICF. By the convergence results of Gerhardt [19], this flow exists for all time, and the evolving hypersurface Σ_t with $\Sigma_0 = \Sigma$ remains star-shaped and k-convex for all $t \geq 0$.

We next consider the quantity

$$Q_k(t) := |\Sigma_t|^{-\frac{n-3}{n-1}}\int_{\Sigma_t}(p_k^{2/k} - 1)d\mu.$$

We study the limit of $Q_k(t)$ as $t \to \infty$. Notice that the roundness estimate for Σ_t is not strong enough to calculate the limit of $Q_k(t)$. However, similar to [6, 25], we are able to give a positive lower bound for the limit of $Q_k(t)$, which will be used to establish the monotonicity of $Q_k(t)$. Finally, we prove that for $k = 1, 2$, $Q_k(t)$ is monotone decreasing under ICF. From this, Theorem 1 follows immediately.

2. Preliminaries

In this article, we consider the hyperbolic space $\mathbb{H}^n = \mathbb{R}^+ \times \mathbb{S}^{n-1}$ equipped with the metric

$$\overline{g} = dr^2 + \sinh^2 r g_{\mathbb{S}^{n-1}},$$

where $g_{\mathbb{S}^{n-1}}$ is the standard round metric on the unit sphere $\mathbb{S}^{n-1}$. Let $\Sigma \subset \mathbb{H}^n$ be a closed hypersurface with its unit outward normal vector ν. The second fundamental form h of Σ is defined by

$$h(X, Y) = \langle \overline{\nabla}_X \nu, Y\rangle$$

for any $X, Y \in T\Sigma$. The principal curvature $\kappa = (\kappa_1, \cdots, \kappa_{n-1})$ comprises the eigenvalues of h with respect to the induced metric g on Σ. For $k \in \{1, \cdots, n-1\}$, the normalized k-th elementary symmetric polynomial of κ is defined as

$$p_k(\kappa) := \frac{1}{\binom{n-1}{k}}\sum_{i_1<i_2<\cdots<i_k}\kappa_{i_1}\cdots\kappa_{i_k},$$

which can also be viewed as a function of the second fundamental form $h_i^j = g^{jk}h_{ki}$. For simplicity, we write p_k for $p_k(\kappa)$. The following lemma can be regarded as a normalized version of Lemma 2 in [25].

Lemma 2. *Let* $(T_{k-1})^i_j := \frac{\partial p_k}{\partial h^j_i}$ *and* $(h^2)^j_i := h^\ell_i h^j_\ell$. *Then we have*

$$\sum_{i,j}(T_{k-1})^j_i h^i_j = kp_k, \quad \sum_{i,j}(T_{k-1})^j_i \delta^i_j = kp_{k-1},$$
$$\sum_{i,j}(T_{k-1})^j_i (h^2)^i_j = (n-1)p_1 p_{k-1} - (n-1-k)p_{k+1}.$$

Moreover, if $\kappa \in \Gamma^+_k$, *we have the following Newton-MacLaurin inequalities:*

$$\frac{p_{k-1}p_{k+1}}{p_k^2} \le 1, \quad \frac{p_1 p_{k-1}}{p_k} \ge 1, \tag{2.1}$$

and equality holds in (2.1) *at a given point if and only if* Σ *is umbilical at this point.*

We now consider the inverse curvature flow (ICF)

$$\partial_t X = \frac{1}{(n-1)p_k^{1/k}}\nu, \tag{2.2}$$

where $\Sigma_t = X(t,\cdot)$ is a family of hypersurfaces in $\mathbb{H}^n$, ν is the unit outward normal to $\Sigma_t = X(t,\cdot)$. Let $d\mu_t$ be its area element on Σ_t. By the divergence free property of T_{k-1}, we list the following evolution equations.

Lemma 3. *Under ICF* (2.2), *we have*

$$\begin{aligned}\partial_t p_k =& -\frac{1}{n-1}\nabla^i\left[(T_{k-1})^j_i \nabla_j\left(\frac{1}{p_k^{1/k}}\right)\right]\\ &-\frac{1}{(n-1)p_k^{1/k}}[(n-1)p_1p_k - (n-1-k)p_{k+1} - kp_{k-1}],\end{aligned} \tag{2.3}$$

$$\partial_t d\mu = \frac{p_1}{p_k^{1/k}} d\mu. \tag{2.4}$$

In [19], Gerhardt investigated the inverse curvature flow of star-shaped hypersurfaces in hyperbolic space and proved the following long-time existence and convergence result.

Theorem 4 ([19]). *If the initial hypersurface is star-shaped and* k*-convex, then the solution for inverse curvature flow* (2.2) *exists for all time* t *and preserves the condition of star-shapedness and* k*-convexity. Moreover, the hypersurfaces become strictly convex exponentially fast and more and more totally umbilical in the sense of*

$$|h^j_i - \delta^j_i| \le Ce^{-\frac{t}{n-1}}, \quad t > 0;$$

i.e., the principal curvatures are uniformly bounded and converge exponentially fast to one.

3. The asymptotic behavior of monotone quantity

We define the quantity

$$Q_k(t) := |\Sigma_t|^{-\frac{n-3}{n-1}} \int_{\Sigma_t} (p_k^{2/k} - 1)d\mu,$$

where $|\Sigma_t|$ is the area of Σ_t. In this section, we estimate the lower bound of the limit of $Q_k(t)$. First of all, we recall Lemma 7 of [25], which is an application of the sharp Sobolev inequality on $\mathbb{S}^{n-1}$ due to Beckner [3].

Lemma 5. *For every positive function f on $\mathbb{S}^{n-1}$, we have*

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} f^{n-3} dvol_{\mathbb{S}^{n-1}} + \frac{n-3}{n-1}\int_{\mathbb{S}^{n-1}} f^{n-5}|\nabla f|^2 dvol_{\mathbb{S}^{n-1}} \\ \geq \omega_{n-1}^{\frac{2}{n-1}}\left(\int_{\mathbb{S}^{n-1}} f^{n-1} dvol_{\mathbb{S}^{n-1}}\right)^{\frac{n-3}{n-1}}.\end{aligned} \tag{3.1}$$

Moreover, equality in (3.1) *holds if and only if f is a constant.*

Proposition 6. *Under ICF* (2.2), *we have*

$$\liminf_{t\to\infty} Q_k(t) \geq \omega_{n-1}^{\frac{2}{n-1}}. \tag{3.2}$$

Proof. Recall that star-shaped hypersurfaces can be written as graphs of function $r = r(t,\theta)$, $\theta \in \mathbb{S}^{n-1}$. Denote $\lambda(r) = \sinh(r)$; then $\lambda'(r) = \cosh(r)$. We next define a function $\varphi(\theta) = \Phi(r(\theta))$, where $\Phi(r)$ is a positive function satisfying $\Phi' = \frac{1}{\lambda}$. Let $\theta = \{\theta^j\}$, $j = 1, \cdots, n-1$, be a coordinate system on $\mathbb{S}^{n-1}$ and φ_i, φ_{ij} be the covariant derivatives of φ with respect to the metric $g_{\mathbb{S}^{n-1}}$. Define

$$v = \sqrt{1 + |\nabla\varphi|^2_{\mathbb{S}^{n-1}}}.$$

From [19], we know that

$$\lambda = O(e^{\frac{t}{n-1}}), \quad |\nabla\varphi|_{\mathbb{S}^{n-1}} + |\nabla^2\varphi|_{\mathbb{S}^{n-1}} = O(e^{-\frac{t}{n-1}}). \tag{3.3}$$

Since $\lambda' = \sqrt{1+\lambda^2}$, we have

$$\lambda' = \lambda\left(1 + \frac{1}{2}\lambda^{-2} + O(e^{-\frac{4t}{n-1}})\right). \tag{3.4}$$

From (3.3), we also have

$$\frac{1}{v} = 1 - \frac{1}{2}|\nabla\varphi|^2_{\mathbb{S}^{n-1}} + O(e^{-\frac{4t}{n-1}}). \tag{3.5}$$

In terms of φ, we can express the metric and the second fundamental form of Σ as

$$\begin{aligned} g_{ij} &= \lambda^2(\sigma_{ij} + \varphi_i\varphi_j), \\ h_{ij} &= \frac{\lambda'}{v\lambda} g_{ij} - \frac{\lambda}{v}\varphi_{ij}, \end{aligned}$$

where $\sigma_{ij} = g_{\mathbb{S}^{n-1}}(\partial_{\theta^i}, \partial_{\theta^j})$. Denote $a_i = \sum_k \sigma^{ik}\varphi_{ki}$ and note that $\sum_i a_i = \Delta_{\mathbb{S}^{n-1}}\varphi$. By (3.3), the principal curvatures of Σ_t take the following form

$$\kappa_i = \frac{\lambda'}{v\lambda} - \frac{a_i}{v\lambda} + O(e^{-\frac{4t}{n-1}}), \quad i = 1, \cdots, n-1.$$

Then we have

$$p_k = \left(\frac{\lambda'}{v\lambda}\right)^k - \frac{k}{n-1}\left(\frac{\lambda'}{v\lambda}\right)^{k-1}\frac{\Delta_{\mathbb{S}^{n-1}}\varphi}{v\lambda} + O(e^{-\frac{4t}{n-1}}).$$

By using (3.4) and (3.5), we get

$$p_k = \left(1 + \frac{k}{2\lambda^2} - \frac{k|\nabla\varphi|^2_{\mathbb{S}^{n-1}}}{2}\right) - \frac{k}{n-1}\frac{\Delta_{\mathbb{S}^{n-1}}\varphi}{\lambda} + O(e^{-\frac{4t}{n-1}}).$$

Hence, we have

$$p_k^{2/k}-1=\frac{1}{\lambda^2}-|\nabla\varphi|^2_{\mathbb{S}^{n-1}}-\frac{2}{n-1}\frac{\Delta_{\mathbb{S}^{n-1}}\varphi}{\lambda}+O(e^{-\frac{4t}{n-1}}).$$

On the other hand,

$$\sqrt{\det g}=\left[\lambda^{n-1}+O(e^{\frac{(n-3)t}{n-1}})\right]\sqrt{\det g_{\mathbb{S}^{n-1}}}.$$

So we have

$$\begin{aligned}\int_{\Sigma_t}(p_k^{2/k}-1)d\mu&=\int_{\mathbb{S}^{n-1}}\lambda^{n-1}(p_k^{2/k}-1)dvol_{\mathbb{S}^{n-1}}+O(e^{\frac{(n-5)t}{n-1}})\\&=\int_{\mathbb{S}^{n-1}}\left(\lambda^{n-3}-\lambda^{n-1}|\nabla\varphi|^2_{\mathbb{S}^{n-1}}\right)dvol_{\mathbb{S}^{n-1}}\\&\quad-\frac{2}{n-1}\int_{\mathbb{S}^{n-1}}\lambda^{n-2}\Delta_{\mathbb{S}^{n-1}}\varphi dvol_{\mathbb{S}^{n-1}}+O(e^{\frac{(n-5)t}{n-1}})\\&=\int_{\mathbb{S}^{n-1}}\left(\lambda^{n-3}-\lambda^{n-1}|\nabla\varphi|^2_{\mathbb{S}^{n-1}}\right)dvol_{\mathbb{S}^{n-1}}\\&\quad+\frac{2(n-2)}{n-1}\int_{\mathbb{S}^{n-1}}\lambda^{n-3}\langle\nabla\lambda,\nabla\varphi\rangle_{\mathbb{S}^{n-1}}dvol_{\mathbb{S}^{n-1}}+O(e^{\frac{(n-5)t}{n-1}}).\end{aligned}$$

Since $\nabla\lambda=\lambda\lambda'\nabla\varphi$, it follows that $|\nabla\lambda-\lambda^2\nabla\varphi|_{g_{\mathbb{S}^{n-1}}}\le O(e^{-\frac{t}{n-1}})$. We deduce that

$$\int_{\Sigma_t}(p_k^{2/k}-1)d\mu=\int_{\mathbb{S}^{n-1}}\left(\lambda^{n-3}+\frac{n-3}{n-1}\lambda^{n-5}|\nabla\lambda|^2\right)dvol_{\mathbb{S}^{n-1}}+O(e^{\frac{(n-5)t}{n-1}}). \tag{3.6}$$

Moreover,

$$|\Sigma_t|^{\frac{n-3}{n-1}}=\left(\int_{\mathbb{S}^{n-1}}\lambda^{n-1}dvol_{\mathbb{S}^{n-1}}\right)^{\frac{n-3}{n-1}}+O(e^{\frac{(n-5)t}{n-1}}).$$

Using Lemma 5, we achieve

$$\liminf_{t\to\infty}|\Sigma_t|^{-\frac{n-3}{n-1}}\int_{\Sigma_t}(p_k^{2/k}-1)d\mu\ge\omega_{n-1}^{\frac{2}{n-1}}.$$

□

4. Monotonicity

In this section, we show that for $k=1,2$, the quantity $Q_k(t)$ is monotone decreasing under ICF (2.2).

Proposition 7. *Under ICF (2.2), the quantity $Q_k(t)$ is monotone decreasing for $k=1,2$. Moreover, $\frac{d}{dt}Q_k(t)=0$ at some time t if and only if Σ_t is totally umbilical.*

Proof. Under ICF (2.2), by (2.3) we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\Sigma_t}(p_k^{\frac{2}{k}}-1)d\mu\\
=&\int_{\Sigma_t}\frac{2}{k}p_k^{\frac{2}{k}-1}\partial_t p_k+(p_k^{\frac{2}{k}}-1)p_1p_k^{-\frac{1}{k}}d\mu\\
=&-\frac{2}{k(n-1)}\int_{\Sigma_t}p_k^{\frac{2}{k}-1}\nabla^i\left[(T_{k-1})_i^j\nabla_j(p_k^{-\frac{1}{k}})\right]d\mu\\
&-\int_{\Sigma_t}\frac{2p_k^{\frac{1}{k}-1}}{k(n-1)}\left[(n-1)p_1p_k-(n-1-k)p_{k+1}-kp_{k-1}\right]d\mu\\
&+\int_{\Sigma_t}(p_k^{\frac{2}{k}}-1)p_1p_k^{-\frac{1}{k}}d\mu\\
=&-\frac{2}{k^2(n-1)}\left(\frac{2}{k}-1\right)\int_{\Sigma_t}p_k^{\frac{1}{k}-3}(T_{k-1})_i^j\nabla^ip_k\nabla_jp_kd\mu\\
&+\int_{\Sigma_t}\left[p_1p_k^{-\frac{1}{k}}(p_k^{\frac{2}{k}}-1)+\frac{2}{k}p_k^{\frac{1}{k}}\left(\frac{p_{k+1}}{p_k}-p_1\right)-\frac{2}{n-1}p_k^{\frac{1}{k}-1}(p_{k+1}-p_{k-1})\right]d\mu\\
=:&I+II.
\end{aligned}$$

Since $(T_{k-1})_i^j$ is positive definite if $p_k>0$, we get $I\le 0$ for $k=1,2$. To handle the second term, we analyze it for $k=1,2$ separately. By the Newton-MacLaurin inequality, if $p_k>0$, then $p_1\ge p_2^{\frac{1}{2}}\ge\cdots\ge p_k^{\frac{1}{k}}>0$.

(i) If $k=1$, then

$$\begin{aligned}
II=&\int_{\Sigma_t}\left[(p_1^2-1)+2(p_2-p_1^2)-\frac{2}{n-1}(p_2-1)\right]d\mu\\
=&\int_{\Sigma_t}\left[\frac{2n-4}{n-1}p_2-p_1^2-\frac{n-3}{n-1}\right]d\mu\\
\le&\frac{n-3}{n-1}\int_{\Sigma_t}(p_1^2-1)d\mu.
\end{aligned}$$

(ii) If $k=2$, then

$$\begin{aligned}
II=&\int_{\Sigma_t}\left[p_1p_2^{-\frac{1}{2}}(p_2-1)+p_2^{\frac{1}{2}}\left(\frac{p_3}{p_2}-p_1\right)-\frac{2}{n-1}p_2^{-\frac{1}{2}}(p_3-p_1)\right]d\mu\\
=&\frac{n-3}{n-1}\int_{\Sigma_t}p_2^{-\frac{1}{2}}(p_3-p_1)d\mu\\
\le&\frac{n-3}{n-1}\int_{\Sigma_t}(p_2-1)d\mu.
\end{aligned}$$

Combining with Proposition 6, we know that the quantity

$$\int_{\Sigma_t}(p_k^{2/k}-1)d\mu$$

is positive under ICF (2.2). By (2.4) we get

$$\frac{d}{dt}|\Sigma_t|=\int_{\Sigma_t}\frac{p_1}{p_k^{1/k}}d\mu\ge|\Sigma_t|.$$

Therefore, for $k = 1, 2$, we have

$$\frac{d}{dt} Q_k(t) \le 0.$$

If the equality holds, then the Newton-MacLaurin inequalities assure equalities everywhere on Σ_t. Therefore Σ_t is totally umbilical. □

Now we complete the proof of Theorem 1.

Proof of Theorem 1. For $k = 1, 2$, since $Q_k(t)$ is monotone decreasing, we have

$$Q_k(0) \ge \liminf_{t\to\infty} Q_k(t) \ge \omega_{n-1}^{\frac{2}{n-1}}.$$

This implies that $\Sigma_0 = \Sigma$ satisfies

$$\int_\Sigma (p_k^{2/k} - 1) d\mu \ge \omega_{n-1}^{\frac{2}{n-1}} |\Sigma|^{\frac{n-3}{n-1}}.$$

Now if we assume that equality in (1.8) is attained, then $Q_k(t)$ is a constant. Then Proposition 7 indicates that Σ_t is totally umbilical and therefore a geodesic sphere. If Σ is a geodesic sphere of radius r, then $|\Sigma| = \omega_{n-1} \sinh^{n-1} r$ and $p_1 = \coth r$. Hence, we have

$$\int_\Sigma (p_k^{2/k} - 1) d\mu = \omega_{n-1} \sinh^{n-1} r (\coth^2 r - 1) = \omega_{n-1}^{\frac{2}{n-1}} |\Sigma|^{\frac{n-3}{n-1}}.$$

Therefore, equality in (1.8) holds on a geodesic sphere. This completes the proof of Theorem 1. □

Remark 8. In fact, for $k = 1$ we prove Theorem 1, and for $k = 2$ we recover Theorem B proved by Li-Wei-Xiong [25].

It is natural to put forward the following question.

Question. For $k \in \{3, \cdots, n-1\}$, let $\Sigma \subset \mathbb{H}^n$ $(n \ge k+1)$ be a star-shaped and k-convex hypersurface. Then

$$\int_\Sigma (p_k^{2/k} - 1) d\mu \ge \omega_{n-1}^{\frac{2}{n-1}} |\Sigma|^{\frac{n-3}{n-1}}. \tag{4.1}$$

Equality in (4.1) holds if and only if Σ is a geodesic sphere.

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