PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 6, June 2018, Pages 2373–2380
http://dx.doi.org/10.1090/proc/13985
Article electronically published on January 29, 2018

INDECOMPOSABLE GENERALIZED WEIGHT MODULES OVER THE ALGEBRA OF POLYNOMIAL INTEGRO-DIFFERENTIAL OPERATORS

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(Communicated by Kailash Misra)

Abstract. For the algebra $\mathbb{I}_1 = K\langle x, \frac{d}{dx}, \int\rangle$ of polynomial integro-differential operators over a field K of characteristic zero, a classification of indecomposable, generalized weight $\mathbb{I}_1$-modules of finite length is given. Each such module is an infinite dimensional uniserial module. Ext-groups are found between indecomposable generalized weight modules; it is proven that they are finite dimensional vector spaces.

1. Introduction

Throughout, ring means an associative ring with 1; module means a left module; $\mathbb{N} := \{0, 1, \dots\}$ is the set of natural numbers; $\mathbb{N}_+ := \{1, 2, \dots\}$ and $\mathbb{Z}_{\leq 0} := -\mathbb{N}$; K is a field of characteristic zero and K^* is its group of units; $P_1 := K[x]$ is a polynomial algebra in one variable x over K; $\partial := \frac{d}{dx}$; $\mathrm{End}_K(P_1)$ is the algebra of all K-linear maps from P_1 to P_1, and $\mathrm{Aut}_K(P_1)$ is its group of units (i.e. the group of all the invertible linear maps from P_1 to P_1); the subalgebras $A_1 := K\langle x, \partial\rangle$ (the algebra of polynomial differential operators in one variable) and $\mathbb{I}_1 := K\langle x, \partial, \int\rangle$ of $\mathrm{End}_K(P_1)$ are called the (first) *Weyl algebra* and the *algebra of polynomial integro-differential operators* respectively where $\int : P_1 \to P_1$, $p \mapsto \int p\, dx$, is the *integration*, i.e., $\int : x^n \mapsto \frac{x^{n+1}}{n+1}$ for all $n \in \mathbb{N}$. The first Weyl algebra A_1 is the simplest noncommutative deformation of the polynomial algebra with two variables that has numerous applications. The algebra $\mathbb{I}_1$ is its natural extension by the integration operator. It is neither left nor right Noetherian and not a domain. Moreover, it contains infinite direct sums of nonzero left and right ideals, [4]. This shows significant differences between A_1 and $\mathbb{I}_1$.

Simple A_1-modules were classified in [11] (see also [2]). Weight and generalized weight (with respect to the action of $x\partial$) modules form important classes of modules over A_1 and its generalizations. Simple modules over certain generalized Weyl algebras were classified in [2]. Classifications of (various classes of) simple weight modules over algebras that are close to the (generalized) Weyl algebras are given in [8, 10, 12, 13, 15–18, 20]. Simple $\mathbb{I}_1$-modules were classified in [6]. The main goal of the paper is to classify indecomposable generalized weight $\mathbb{I}_1$-modules of finite length. Such classification is contained in Theorem 2.5. A similar classification

Received by the editors January 28, 2017, and, in revised form, August 27, 2017.
2010 *Mathematics Subject Classification.* Primary 16D60, 16D70, 16P50, 16U20.
Key words and phrases. The algebra of polynomial integro-differential operators, generalized weight module, indecomposable module, simple module.

was obtained in [3] for the generalized Weyl algebras where a completely different approach was taken. Ext-groups are found between indecomposable generalized weight $\mathbb{I}_1$-modules; it is proven that they are finite dimensional vector spaces. Finite dimensionality of Ext-groups of simple modules over the (first) Weyl algebra A_1 was proven in [19]. Finite dimensionality of Ext-groups of simple modules over the generalized Weyl algebras was proven in [1].

Properties of the algebras $\mathbb{I}_n := \mathbb{I}_1^{\otimes n}$ of polynomial integro-differential operators in arbitrary many variables are studied in [4] and [7]. In the case $n = 1$, for a more general setting see also [14]. The automorphism groups $\mathrm{Aut}_{K-\mathrm{alg}}(\mathbb{I}_n)$ are found in [5]. The weak homological dimension of the algebra $\mathbb{I}_n$ is n, [4].

2. Classification of indecomposable, generalized weight $\mathbb{I}_1$-modules of finite length

In this section, a classification of indecomposable, generalized weight $\mathbb{I}_1$-modules of finite length is given (Theorem 2.5).

As an abstract algebra, the algebra $\mathbb{I}_1$ is generated by the elements ∂, $H := \partial x$, and $\int$ (since $x = \int H$) that satisfy the defining relations, [4, Proposition 2.2] (where $[a, b] := ab - ba$):

$$\partial \int = 1, \;\; [H, \int] = \int, \;\; [H, \partial] = -\partial, \;\; H(1 - \int \partial) = (1 - \int \partial)H = 1 - \int \partial.$$

The elements of the algebra $\mathbb{I}_1$,

$$e_{ij} := \int^i \partial^j - \int^{i+1} \partial^{j+1}, \;\; i, j \in \mathbb{N}, \tag{1}$$

satisfy the relations $e_{ij}e_{kl} = \delta_{jk}e_{il}$ where δ_{jk} is the Kronecker delta function. Notice that $e_{ij} = \int^i e_{00}\partial^j$. The matrices of the linear maps $e_{ij} \in \mathrm{End}_K(K[x])$ with respect to the basis $\{x^{[s]} := \frac{x^s}{s!}\}_{s\in\mathbb{N}}$ of the polynomial algebra $K[x]$ are the elementary matrices, i.e.,

$$e_{ij} * x^{[s]} = \begin{cases} x^{[i]} & \text{if } j = s, \\ 0 & \text{if } j \neq s. \end{cases}$$

Let $E_{ij} \in \mathrm{End}_K(K[x])$ be the usual matrix units; i.e., $E_{ij} * x^s = \delta_{js}x^i$ for all $i, j, s \in \mathbb{N}$. Then

$$e_{ij} = \frac{j!}{i!}E_{ij}, \tag{2}$$

$Ke_{ij} = KE_{ij}$, and $F := \bigoplus_{i,j\geq 0} Ke_{ij} = \bigoplus_{i,j\geq 0} KE_{ij} \simeq M_\infty(K)$, the algebra (without 1) of infinite dimensional matrices.

$\mathbb{Z}$-grading on the algebra $\mathbb{I}_1$ and the canonical form of an integro-differential operator, [4]. The algebra $\mathbb{I}_1 = \bigoplus_{i\in\mathbb{Z}} \mathbb{I}_{1,i}$ is a $\mathbb{Z}$-graded algebra ($\mathbb{I}_{1,i}\mathbb{I}_{1,j} \subseteq \mathbb{I}_{1,i+j}$ for all $i, j \in \mathbb{Z}$) where

$$\mathbb{I}_{1,i} = \begin{cases} D_1 \int^i = \int^i D_1 & \text{if } i > 0, \\ D_1 & \text{if } i = 0, \\ \partial^{|i|} D_1 = D_1 \partial^{|i|} & \text{if } i < 0, \end{cases} \tag{3}$$

the algebra $D_1 := K[H] \bigoplus \bigoplus_{i\in\mathbb{N}} Ke_{ii}$ is a commutative non-Noetherian subalgebra of $\mathbb{I}_1$, and $He_{ii} = e_{ii}H = (i + 1)e_{ii}$ for $i \in \mathbb{N}$ (notice that $\bigoplus_{i\in\mathbb{N}} Ke_{ii}$ is the direct sum of nonzero ideals of D_1); $(\int^i D_1)_{D_1} \simeq D_1$, $\int^i d \mapsto d$; ${}_{D_1}(D_1\partial^i) \simeq D_1$, $d\partial^i \mapsto d$, for all $i \geq 0$ since $\partial^i \int^i = 1$. Notice that the maps $\cdot \int^i : D_1 \to D_1 \int^i$, $d \mapsto d\int^i$, and $\partial^i \cdot : D_1 \to \partial^i D_1$, $d \mapsto \partial^i d$, have the same kernel $\bigoplus_{j=0}^{i-1} Ke_{jj}$.

Each element a of the algebra $\mathbb{I}_1$ is the unique finite sum

$$a = \sum_{i>0} a_{-i}\partial^i + a_0 + \sum_{i>0} \int^i a_i + \sum_{i,j\in\mathbb{N}} \lambda_{ij} e_{ij} \tag{4}$$

where $a_k \in K[H]$ and $\lambda_{ij} \in K$. This is the *canonical form* of the polynomial integro-differential operator [4].

Let

$$v_i := \begin{cases} \int^i & \text{if } i > 0, \\ 1 & \text{if } i = 0, \\ \partial^{|i|} & \text{if } i < 0. \end{cases}$$

Then $\mathbb{I}_{1,i} = D_1 v_i = v_i D_1$, and an element $a \in \mathbb{I}_1$ is the unique finite sum

$$a = \sum_{i\in\mathbb{Z}} b_i v_i + \sum_{i,j\in\mathbb{N}} \lambda_{ij} e_{ij}, \tag{5}$$

where $b_i \in K[H]$ and $\lambda_{ij} \in K$. So, the set $\{H^j\partial^i, H^j, \int^i H^j, e_{st} \,|\, i \geq 1; j, s, t \geq 0\}$ is a K-basis for the algebra $\mathbb{I}_1$. The multiplication in the algebra $\mathbb{I}_1$ is given by the rule:

$$\int H = (H-1)\int, \quad H\partial = \partial(H-1), \quad \int e_{ij} = e_{i+1,j},$$

$$e_{ij}\int = e_{i,j-1}, \quad \partial e_{ij} = e_{i-1,j}, \quad e_{ij}\partial = e_{i,j+1},$$

$$He_{ii} = e_{ii}H = (i+1)e_{ii}, \quad i \in \mathbb{N},$$

where $e_{-1,j} := 0$ and $e_{i,-1} := 0$.

The algebra $\mathbb{I}_1$ has the only proper ideal $F = \bigoplus_{i,j\in\mathbb{N}} Ke_{ij} \simeq M_\infty(K)$ and $F^2 = F$. The factor algebra $\mathbb{I}_1/F$ is canonically isomorphic to the skew Laurent polynomial algebra $B_1 := K[H][\partial, \partial^{-1}; \tau]$, $\tau(H) = H + 1$, via $\partial \mapsto \partial$, $\int \mapsto \partial^{-1}$, $H \mapsto H$ (where $\partial^{\pm 1}\alpha = \tau^{\pm 1}(\alpha)\partial^{\pm 1}$ for all elements $\alpha \in K[H]$). The algebra B_1 is canonically isomorphic to the (left and right) localization $A_{1,\partial}$ of the Weyl algebra A_1 at the powers of the element ∂ (notice that $x = \partial^{-1}H$).

An $\mathbb{I}_1$-module M is called a *weight* module if $M = \bigoplus_{\lambda\in K} M_\lambda$ where $M_\lambda := \{m \in M \,|\, Hm = \lambda m\}$. An $\mathbb{I}_1$-module M is called a *generalized weight* module if $M = \bigoplus_{\lambda\in K} M^\lambda$ where $M^\lambda := \{m \in M \,|\, (H-\lambda)^n m = 0 \text{ for some } n = n(m)\}$. The set $\mathrm{Supp}(M) := \{\lambda \in K \,|\, M^\lambda \neq 0\}$ is called the *support* of the generalized weight module M. For all $\lambda \in K$ and $n \geq 1$,

$$\partial^n M^\lambda \subseteq M^{\lambda-n} \text{ and } \int^n M^\lambda \subseteq M^{\lambda+n}.$$

Let $0 \to N \to M \to L \to 0$ be a short exact sequence of $\mathbb{I}_1$-modules. Then M is a generalized weight module iff so are the modules N and L, and in this case

$$\mathrm{Supp}(M) = \mathrm{Supp}(N) \cup \mathrm{Supp}(L).$$

For each $\mathbb{I}_1$-module M, there is a short exact sequence of $\mathbb{I}_1$-modules

$$0 \to FM \to M \to \overline{M} := M/FM \to 0 \tag{6}$$

where

(i) $F \cdot FM = FM$, and

(ii) $F \cdot \overline{M} = 0$,

and the properties (i) and (ii) determine the short exact sequence (6) uniquely; i.e., if $0 \to M_1 \to M \to M_2 \to 0$ is a short exact sequence of $\mathbb{I}_1$-modules such that $FM_1 = M_1$ and $FM_2 = 0$, then $M_1 \simeq FM$ and $M_2 \simeq \overline{M}$.

For each $j \in \mathbb{N}$, the $\mathbb{I}_1$-submodule $V_j = \bigoplus_{i\in\mathbb{N}} Ke_{ij}$ of F is isomorphic to $K[x]$. In particular, the $\mathbb{I}_1$-module $F = \bigoplus_{j\in\mathbb{N}} V_j \simeq K[x]^{(\mathbb{N})}$ is a semisimple $\mathbb{I}_1$-module. Hence, we have

$$FM \simeq K[x]^{(I)}; \tag{7}$$

i.e. the $\mathbb{I}_1$-module FM is isomorphic to the direct sum of I copies of the simple weight $\mathbb{I}_1$-module $K[x]$ for some index set I. Clearly, $\overline{M}$ is a B_1-module.

The indecomposable $\mathbb{I}_1$-modules $M(n, \lambda)$. For $\lambda \in K$ and a natural number $n \geq 1$, consider the B_1-module

$$M(n, \lambda) := B_1 \otimes_{K[H]} K[H]/(H - \lambda)^n. \tag{8}$$

Clearly,

$$M(n, \lambda) \simeq B_1/B_1(H - \lambda)^n \simeq \mathbb{I}_1/(F + \mathbb{I}_1(H - \lambda)^n). \tag{9}$$

The $\mathbb{I}_1$-module/B_1-module $M(n, \lambda)$ is a generalized weight module with $\operatorname{Supp} M(n, \lambda) = \lambda + \mathbb{Z}$,

$$M(n, \lambda) = \bigoplus_{i\in\mathbb{Z}} M(n, \lambda)^{\lambda+i} \text{ and } \dim M(n, \lambda)^{\lambda+i} = n \text{ for all } i \in \mathbb{Z}. \tag{10}$$

For an algebra A, we denote by $A - \text{Mod}$ its module category. The next proposition describes the set of indecomposable, generalized weight $\mathbb{I}_1$-modules of finite length M with $FM = 0$.

Proposition 2.1.

(1) *$M(n, \lambda)$ is an indecomposable, generalized weight $\mathbb{I}_1$-module of finite length n.*

(2) *$M(n, \lambda) \simeq M(m, \mu)$ if and only if $n = m$ and $\lambda - \mu \in \mathbb{Z}$.*

(3) *Let M be a generalized weight B_1-module of length n (i.e., let M be a generalized weight $\mathbb{I}_1$-module of length n such that $FM = 0$, by (6)). Then M is indecomposable if and only if $M \simeq M(n, \lambda)$ for some $\lambda \in K$.*

Proof. (1) Since $(B_1)_{K[H]} = \bigoplus_{i\in\mathbb{Z}} \partial^i K[H]$ is a free right $K[H]$-module, the functor

$$B_1 \otimes_{K[H]} - \ : K[H] - \text{Mod} \to B_1 - \text{Mod}, \ \ N \mapsto B_1 \otimes_{K[H]} N,$$

is an exact functor. The $K[H]$-module $K[H]/(H - \lambda)^n$ is indecomposable; hence the B_1-module $M(n, \lambda)$ is indecomposable and a generalized weight of length n.

(2) ($\Rightarrow$) Suppose that $\mathbb{I}_1$-modules $M(n, \lambda)$ and $M(m, \mu)$ are isomorphic. Then $\operatorname{Supp}(M(n, \lambda)) = \operatorname{Supp}(M(m, \mu))$, i.e., $\lambda + \mathbb{Z} = \mu + \mathbb{Z}$, i.e., $\lambda - \mu \in \mathbb{Z}$. Then $n = m$ by (10).

($\Leftarrow$) Suppose that $k := \lambda - \mu \in \mathbb{Z}$ and $n = m$. We may assume that $k \geq 1$. Using the equality $(H - \lambda)^n \partial^k = \partial^k (H - \lambda - k)^n = \partial^k (H - \mu)^n$, we see that the B_1-homomorphism

$$M(n, \lambda) = B_1/B_1(H - \lambda)^n \to M(n, \mu) = B_1/B_1(H - \mu)^n,$$
$$1 + B_1(H - \lambda)^n \mapsto \partial^k + B_1(H - \mu)^n,$$

is an isomorphism with the inverse given by the rule $1 + B_1(H-\mu)^n \mapsto \partial^{-k} + B_1(H-\lambda)^n$.

(3) ($\Leftarrow$) This implication follows from statement (2).

($\Rightarrow$) Each indecomposable, generalized weight B_1-module M is of the type $B_1 \otimes_{K[H]} N$ for an indecomposable $K[H]$-module N isomorphic to $K[H]/(H-\lambda)^n$ for some $\lambda \in K$. Therefore, $M \simeq M(n, \lambda)$. $\square$

Lemma 2.2. *Let M be an indecomposable, generalized weight $\mathbb{I}_1$-module. Then* $\mathrm{Supp}(M) \subseteq \lambda + \mathbb{Z}$ *for some $\lambda \in K$.*

Proof. Let $M = \bigoplus_{\mu \in \mathrm{Supp}(M)} M^\mu$ be a generalized weight $\mathbb{I}_1$-module. Then

$$M = \bigoplus_{\mu+\mathbb{Z} \in \mathrm{Supp}(M)/\mathbb{Z}} M_{\mu+\mathbb{Z}}$$

is a direct sum of $\mathbb{I}_1$-submodules $M_{\mu+\mathbb{Z}} := \bigoplus_{i\in\mathbb{Z}} M^{\mu+i}$ where $\mathrm{Supp}(M)/\mathbb{Z}$ is the image of the support $\mathrm{Supp}(M)$ under the abelian group epimorphism $K \to K/\mathbb{Z}$, $\gamma \mapsto \gamma + \mathbb{Z}$. The $\mathbb{I}_1$-module M is indecomposable; hence $M = M_{\lambda+\mathbb{Z}}$ for some $\lambda \in K$, i.e., $\mathrm{Supp}(M) \subseteq \lambda + \mathbb{Z}$. $\square$

The next lemma describes the set of indecomposable, generalized weight $\mathbb{I}_1$-modules M with $FM = M$.

Lemma 2.3. *Let M be an indecomposable, generalized weight $\mathbb{I}_1$-module M. Then the following statements are equivalent:*

(1) $FM = M$.
(2) $M \simeq K[x]$.
(3) $\mathrm{Supp}(M) \subseteq \mathbb{N}$.

Proof. (1) $\Rightarrow$ (2) : If $FM = M$, then $M \simeq K[x]^{(I)}$ for some set I necessarily with $|I| = 1$ since M is indecomposable, i.e., $M \simeq K[x]$.

(2) $\Rightarrow$ (3) : $\mathrm{Supp}(K[x]) = \{1, 2, \ldots\} \subseteq \mathbb{N}$.

(3) $\Rightarrow$ (1) : Suppose that $\mathrm{Supp}(M) \subseteq \mathbb{N}$. Using the short exact sequence of $\mathbb{I}_1$-modules $0 \to FM \to M \to \overline{M} := M/FM \to 0$ we see that $\mathrm{Supp}(M) = \mathrm{Supp}(FM) \cup \mathrm{Supp}(\overline{M})$. Since $\mathrm{Supp}(FM) = \mathrm{Supp}(K[x]^{(I)}) = \{1, 2, \ldots\}$ and $\mathrm{Supp}(\overline{M})$ is an abelian group, we must have $\overline{M} = 0$ (since $\mathrm{Supp}(M) \subseteq \mathbb{N}$), i.e., $M = FM$. $\square$

The following result is a key step in obtaining a classification of indecomposable, generalized weight $\mathbb{I}_1$-modules of finite length.

Theorem 2.4. *Let M be a generalized weight $\mathbb{I}_1$-module of finite length. Then the short exact sequence* (6) *splits.*

Proof. We can assume that $FM \neq 0$ and $\overline{M} \neq 0$. It is obvious that $FM \simeq K[x]^s$ for some $s \geq 1$ and the B_1-module $\overline{M} \simeq \bigoplus_{i=1}^t M(n_i, \lambda_i)$ for some $n_i \geq 1$, $\lambda_i \in K$, and $t \geq 1$. It suffices to show that

$$\mathrm{Ext}^1_{\mathbb{I}_1}(M(n,\lambda), K[x]) = 0 \tag{11}$$

for all $n \geq 1$ and $\lambda \in K$. If $\lambda \in \mathbb{Z}$ we can assume that $\lambda = 0$ by Proposition 2.1.(2).

(i) $F(H-\lambda)^n = F$: The equality follows from the equalities $e_{ij}(H-\lambda)^n = e_{ij}(j+1-\lambda)^n$ and the choice of λ.

(ii) $M(n, \lambda) \simeq \mathbb{I}_1/\mathbb{I}_1(H-\lambda)^n$: By (i), $\mathbb{I}_1(H-\lambda)^n \supseteq F(H-\lambda)^n = F$. Hence, by (9),

$$M(n, \lambda) \simeq \mathbb{I}_1/(F + \mathbb{I}_1(H-\lambda)^n) = \mathbb{I}_1/\mathbb{I}_1(H-\lambda)^n.$$

(iii) *The equality* (11) *holds*: Let $M = M(n, \lambda)$. By (ii), the short exact sequence of $\mathbb{I}_1$-modules

$$0 \to \mathbb{I}_1(H-\lambda)^n \to \mathbb{I}_1 \to M \to 0 \tag{12}$$

is a projective resolution of the $\mathbb{I}_1$-module M since the map

$$\cdot(H-\lambda)^n : \mathbb{I}_1 \to \mathbb{I}_1(H-\lambda)^n, \ \ a \mapsto a(H-\lambda)^n,$$

is an isomorphism of $\mathbb{I}_1$-modules, by the choice of λ. Then

$$\mathrm{Ext}^1_{\mathbb{I}_1}(M, K[x]) \simeq Z^1/B^1,$$

where $Z^1 = \mathrm{Hom}_{\mathbb{I}_1}(\mathbb{I}_1(H-\lambda)^n, K[x]) \simeq K[x]$, and $B^1 \simeq (H-\lambda)^n K[x] = K[x]$, by the choice of λ. Hence, the equality (11) holds. The proof of the theorem is complete. □

The next theorem is a classification of indecomposable, generalized weight $\mathbb{I}_1$-modules of finite length.

Theorem 2.5. *Each indecomposable, generalized weight $\mathbb{I}_1$-module of finite length is isomorphic to one of the modules below:*

1. *$K[x]$,*
2. *$M(n, \lambda)$ where $n \geq 1$ and $\lambda \in \Lambda$, where Λ is any fixed subset of K such that the map $\Lambda \to (K/\mathbb{Z})$, $\lambda \mapsto \lambda + \mathbb{Z}$, is a bijection.*

The $\mathbb{I}_1$-modules above are pairwise nonisomorphic, indecomposable, generalized weights, and of finite length.

Proof. The theorem follows from Theorem 2.4, Proposition 2.1, and Lemma 2.3. □

Corollary 2.6. *Every indecomposable, generalized weight $\mathbb{I}_1$-module is a uniserial module.*

Proof. The statement follows from Theorem 2.5. □

3. Homomorphisms and Ext-groups between indecomposable modules

Proposition 3.1.

1. *Let M and N be generalized weight $\mathbb{I}_1$-modules such that* $\mathrm{Supp}(M) \cap \mathrm{Supp}(N) = \emptyset$. *Then* $\mathrm{Hom}_{\mathbb{I}_1}(M, N) = 0$.
2. $\mathrm{Hom}_{\mathbb{I}_1}(M(n, \lambda), K[x]) = 0$.
3. $\mathrm{Hom}_{\mathbb{I}_1}(K[x], M(n, \lambda)) = 0$.
4. $\mathrm{Hom}_{\mathbb{I}_1}(M(n, \lambda), M(m, \lambda)) \simeq \mathrm{Hom}_{K[H]}(K[H]/((H-\lambda)^n), K[H]/((H-\lambda)^m))$ $\simeq K[H]/((H-\lambda)^{\min(n,m)}) \simeq K^{\min(n,m)}$.

Proof. (1) This is obvious.

(2) This follows from the fact that $FM(n, \lambda) = 0$ and $Fp = K[x]$ for all nonzero elements $p \in K[x]$ (since $K[x]$ is a simple $\mathbb{I}_1$-module, F is an ideal of the algebra $\mathbb{I}_1$ such that $FK[x] = K[x]$).

(3) This follows from the fact that $FK[x] = K[x]$ and $FM(n, \lambda) = 0$: $f(K[x]) = f(FK[x]) = Ff(K[x]) = 0$ for any $f \in \mathrm{Hom}_{\mathbb{I}_1}(K[x], M(n, \lambda))$.

(4) The first isomorphism is obvious. Then the rest follows. □

Proposition 3.2.

(1) $\mathrm{Ext}^1_{\mathbb{I}_1}(K[x], K[x]) = 0$.
(2) $\mathrm{Ext}^1_{\mathbb{I}_1}(M(n,\lambda), K[x]) = 0$.
(3) $\mathrm{Ext}^1_{\mathbb{I}_1}(K[x], M(n,\lambda)) = 0$.
(4) $\mathrm{Ext}^1_{\mathbb{I}_1}(M(n,\lambda), M(m,\mu)) = \begin{cases} K^{\min(n,m)} & \text{if } \lambda - \mu \in \mathbb{Z}, \\ 0 & \text{if } \lambda - \mu \notin \mathbb{Z}. \end{cases}$

Proof. (1) Let $0 \to K[x] \to N \to K[x] \to 0$ be an s.e.s. of $\mathbb{I}_1$-modules. Then $FN = N$ (since $FK[x] = K[x]$) and N is semisimple by (7). Therefore the s.e.s. splits.

(2) See (11).

(3) Let $0 \to M = M(n,\lambda) \to L \to K[x] \to 0$ be an s.e.s. of $\mathbb{I}_1$-modules. Since $FM = 0$, we have $FL = FK[x] \simeq K[x]$ is a submodule of L such that $FL \cap M = 0$ (since otherwise $FL \subseteq M$ by simplicity of the $\mathbb{I}_1$-module $FL \simeq K[x]$, and so $0 \neq K[x] \simeq FL = F^2L \subseteq FM = 0$, a contradiction). Then $FL \oplus M \subseteq L$. Furthermore, $FL \oplus M = L$ since $l_{\mathbb{I}_1}(FL \oplus M) = l_{\mathbb{I}_1}(L)$. This means that the s.e.s. splits.

(4) Let $0 \to M_1 \to M \to M_1 \to 0$ be an s.e.s. of generalized weight $\mathbb{I}_1$-modules. If $\mathrm{Supp}(M_1) \cap \mathrm{Supp}(M_2) = \emptyset$, it splits. In particular, $\mathrm{Ext}^1_{\mathbb{I}_1}(M(n,\lambda), M(m,\mu)) = 0$ if $\lambda - \mu \notin \mathbb{Z}$. If $\lambda - \mu \in \mathbb{Z}$ we can assume that $\lambda = \mu$ (since $M(m,\lambda) \simeq M(m,\mu)$). Using (12), where we assume that $\lambda = 0$ if $\lambda \in \mathbb{Z}$, we see that

$$\begin{aligned}
&\mathrm{Ext}^1_{\mathbb{I}_1}(M(n,\lambda), M(m,\lambda)) \\
&\quad \simeq M(m,\lambda)/(H-\lambda)^n M(m,\lambda) \\
&\quad \simeq \mathrm{coker}((H-\lambda)^n \cdot : K[H]/(H-\lambda)^m \to K[H]/(H-\lambda)^m) \\
&\quad \simeq K[H]/(H-\lambda)^{\min(n,m)} \\
&\quad \simeq K^{\min(n,m)}.
\end{aligned}$$

□

Since the left global dimension of the algebra $\mathbb{I}_1$ is 1, [9], Propositions 3.1 and 3.2 describe all the Ext-groups between indecomposable, generalized weight $\mathbb{I}_1$-modules. This is also obvious from the proofs of the propositions.

Acknowledgments

The first author is grateful to the University of São Paulo for hospitality during his visit and to Fapesp for financial support (processo 2013/24392-5). The third author is supported in part by the CNPq (301320/2013-6) and by the Fapesp (2014/09310-5). The authors would like to thank the referee for the valuable comments and careful reading of the paper.

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