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EXAMPLES OF MEASURES WITH SLOW DECAY OF THE SPHERICAL MEANS OF THE FOURIER TRANSFORM

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ABSTRACT. Many authors have studied the rate of average decay of the Fourier transform of measures because of its relationship with the Falconer's conjecture. Although examples have been given showing that the spherical average of the Fourier transform cannot decay always too fast, they usually do not exhibit a single measure decaying sufficiently slow on the whole space. We recover known results using instead single measures.

1. INTRODUCTION

A classical theorem of Steinhaus [19] states that if $E \subset \mathbb{R}^n$ is a set of positive Lebesgue measure, then for every sufficiently small $\delta > 0$ there exist $x, y \in E$ such that $|x - y| = \delta$. If we define the distance set of E as the collection of all distances between points in E, i.e.,

$$\Delta(E) := \{|x - y| \mid x, y \in E\},$$

then the Steinhaus theorem says that $[0, \epsilon) \subset \Delta(E)$ for some sufficiently small $\epsilon > 0$.

In [6] Falconer investigated in more detail the size of the distance set of more general sets $E \subset \mathbb{R}^n$, not necessarily of positive Lebesgue measure. He showed that there are Cantor-like sets of Hausdorff dimension less than $n/2\,(n \geq 2)$ whose distance set has Lebesgue measure zero. Hence, he was led to conjecture that if a set $E \subset \mathbb{R}^n\,(n \geq 2)$ has dimension strictly greater than $n/2$, then the Lebesgue measure of $\Delta(E)$ is positive. The conjecture remains undecided in every dimension, though Orponen in [16] proved it for self-similar fractals in the plane, up to an additional condition. If $n = 1$, then there exist one-dimensional sets such that $\Delta(E)$ has Lebesgue measure zero.

Mattila devised a method to estimate the size of $\Delta(E)$ that depends on the average rate of decay of the Fourier transform of a measure μ supported in E, or more precisely depends on

$$\sigma(\mu)(r) := \int_{S^{n-1}} |\hat{\mu}(r\omega)|^2 \, d\sigma(\omega),$$

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where σ is the standard measure on the sphere. For more details see [13, 15] and the references therein. Related to the Hausdorff dimension is the s-energy of a set,

$$I_s(\mu) := \iint |x-y|^{-s}\, d\mu(x)d\mu(y),$$

where μ is a measure supported in the set; see [14, chp. 8]. The success of Mattila's approach lies in obtaining the best possible rate of decay of measures with finite s-energy, hence we define

$$\alpha(s) := \sup\{\alpha \mid \sigma(\mu)(r) \leq Cr^{-\alpha} \text{ for } r>0,\ \operatorname{supp}\mu \subset B(0,1) \text{ and } I_s(\mu)<\infty\}.$$

It is not very hard to prove that $\alpha(s) \leq s$ for $0 < s < n$ and that $\alpha(s) = s$ for $0 < s \leq (n-1)/2$; see [15].

The value of $\alpha(s)$ for $\mathbb{R}^2$ was investigated by Bourgain in [1] using the theory of restriction of the Fourier transform to the sphere, and the sharp value was finally settled by Wolff [20], asserting that

$$\alpha(s) = \begin{cases} s & \text{for } 0 < s < \frac{1}{2}, \\ \frac{1}{2} & \text{for } \frac{1}{2} \leq s < 1, \\ \frac{s}{2} & \text{for } 1 \leq s < 2. \end{cases}$$

The proof of Wolff was simplified and generalized by Erdoğan [4, 5], improving the lower bound of $\alpha(s)$ in higher dimensions. Lucà and Rogers in [12] pushed even further the lower bound. Examples of Sjölin [17] show that $\alpha(s) \leq s/2+n/2-1$ for $1 \leq n-2 \leq s < n$, and this bound was not improved for several years until the work of Iosevich and Rudnev [9]. They show how to construct a sequence of measures $d\mu_k = \rho_k\, dx$, where ρ_k is a smooth positive function, such that $I_s(\mu_k) = 1$ and $\sigma(\mu_k)(r_k) \geq Cr_k^{-\frac{n-2}{n}s-1}$ for a sequence $r_k \to \infty$; hence $\alpha(s) \leq \frac{n-2}{n}s+1$. Essential to their argument is the count of lattice points intersecting spheres of large radius. Other proof using the same essential idea, but arguing by a duality principle, is due to Lucà and Rogers [12].

Moreover, the work of Iosevich and Rudnev [10] on signed measures suggests the possibility of getting the bound

$$\alpha(s) \leq \begin{cases} s & \text{for } 0 < s < \frac{n-1}{2}, \\ \frac{n-1}{2} & \text{for } \frac{n-1}{2} \leq s < \frac{n}{2}, \\ \frac{n-1}{n}s & \text{for } \frac{n}{2} \leq s < n; \end{cases}$$

however, problems concerning coherence patterns prevented us from upgrading this bound to positive measures. Further results and generalizations can be found in [2, 3, 7, 8, 11, 18].

In this note we show that it is not necessary to use a sequence of measures to prove the theorem of Iosevich and Rudnev [9], namely $\alpha(s) \leq \frac{n-2}{n}s+1$, but a single one does.

Theorem 1.1. *If $n/2 < s < n$, then there exists a measure μ with finite t-energy for $t < s$, such that $\sigma(\mu)(r_k) \geq Cr_k^{-\frac{n-2}{n}s-1}$ for a sequence $r_k \to \infty$.*

Frostman's lemma allows us to construct measures of finite t-energy on a set of dimension $s > t$. Hence, we will prove first that the Fourier transform of measures supported in some Cantor-like sets, just the same sets used by Falconer to state his conjecture, concentrates in balls around lattice points, so that the theorem will be a consequence of counting lattice points on spheres of large radius.

Let $q_1, q_2, \dots$ be a sequence of integers increasing rapidly, say $q_{k+1} > q_k^k$, and define the sets $E_s = \bigcap_{k=1}^{\infty} E_{s,k} \subset \mathbb{R}$ for $0 < s < 1$, where

$$E_{s,k} := \{x \in [0,1] \mid |x - \frac{p}{q_k}| \leq q_k^{-1/s} \text{ for some } p \in \mathbb{Z}_+\}.$$

It is known that $\dim(E_s) = s$; see for example [6]. Taking cartesian product, we get sets $E = E_{s_1} \times \cdots \times E_{s_n} \subset \mathbb{R}^n$ of dimension $\geq s = s_1 + \cdots + s_n < n$. For simplicity, we assume that $s_i = s/n$ and that the sequence $\{q_k\}$ is the same for each axis, in which case $\dim(E) = s$.

Let μ be a probability measure supported in $E = E_{s/n} \times \cdots \times E_{s/n}$; then we have for the Fourier transform $|\hat{\mu}(\xi)| \geq |\int \cos(2\pi\langle \xi, x\rangle)\, d\mu(x)|$. Since $E \subset E_{s/n,k} \times \cdots \times E_{s/n,k}$, only the points with coordinates $x_i = p_i/q_k + a_i$ for $|a_i| \leq q_k^{-n/s}$ contribute to the integral. Hence for the frequencies $\xi = (N_1 q_k, \dots, N_n q_k)$, where N_i are integers satisfying $1 \leq N_i \leq c q_k^{n/s-1}$, we get $\cos(2\pi\langle \xi, x\rangle) = \cos(2\pi q_k \sum N_i a_i)$, but $|2\pi q_k \sum N_i a_i| \leq 2\pi n c$, so choosing c sufficiently small we get

$$|\hat{\mu}(\xi)| \geq \int \cos(2\pi\langle \xi, x\rangle)\, d\mu(x) \geq \frac{1}{2}\mu(\mathbb{R}^n) = \frac{1}{2}.$$

This is basically what we need to know about the Fourier transform of measures supported in our sets.

2. Proof of Theorem 1.1

Assume that E is as before and that μ is a Frostman measure such that $\mu(\mathbb{R}^n) = 1$, hence $|\hat{\mu}(\xi)| \geq \frac{1}{2}$ if $\xi \in q_k\mathbb{Z}^n \cap [0, c q_k^{n/s}]^n$. Since μ is supported in a bounded set, by the uncertainty principle we can assume that $|\hat{\mu}(\xi)| \geq \frac{1}{4}$ in $\xi \in q_k\mathbb{Z}^n \cap [0, c q_k^{n/s}]^n + B(0,\rho)$, where $B(0,\rho)$ is a ball of sufficiently small radius $\rho \sim 1$. In other words, $\hat{\mu}$ concentrates around balls in the lattice $q_k\mathbb{Z}^n \cap [0, c q_k^{n/s}]^n$.

We use now a pigeonholing argument to count lattice points on spheres of certain large radius r_k, although number theoretic reasonings are also possible. The number of lattice points $\xi \in q_k\mathbb{Z}^n$ lying in the annulus $\frac{1}{10} c q_k^{n/s} \leq |\xi| \leq c q_k^{n/s}$ is $\sim q_k^{n(n/s-1)}$. On the other hand, for a lattice point in the annulus we have $|\xi|^2 = q_k^2(N_1^2 + \cdots + N_n^2) \in q_k^2\mathbb{Z} \cap [\frac{c^2}{100} q_k^{2n/s}, c^2 q_k^{2n/s}]$, hence the number of distinct distances from the origin to the lattice points is $\lesssim q_k^{2(n/s-1)}$. Since the points are distribuited among the different distances, then we can find a sphere S_k centered at the origin and of radius $r_k \sim q_k^{n/s}$ such that the number of lattice points on it is $\gtrsim q_k^{(n-2)(n/s-1)} \sim r_k^{(n-2)(1-s/n)}$. In terms of spherical means, we have

$$\sigma(\mu)(r_k) \geq \frac{1}{16 r_k^{n-1}} \int_{S_k \cap (q_k\mathbb{Z}^n + B(0,\rho))} d\sigma \gtrsim \rho^{n-1} r_k^{(n-2)(1-s/n)-(n-1)},$$

where σ is the standard measure on S_k. We conclude thus that $\sigma(\mu)(r_k) \gtrsim r_k^{-\frac{n-2}{n}s-1}$ for a sequence $r_k \to \infty$, which is what we wanted to prove.

Remark 2.1. These arguments can be extended to the case of non-euclidean distances, as done by Iosevich and Rudnev [9].

It is surprising that the sets used by Falconer to state his conjecture do not match the best known upper bound for two and three dimensions.

If $s \le n/2$, then the fact that $|\hat{\mu}(\xi)| \ge \frac{1}{4}$ in balls around lattice points worsen the average decay to $\sigma(\mu)(r_k) \gtrsim r_k^{-s}$ for a sequence $r_k \to \infty$ because a ball can intersect many distances.

We can modify the construction of the set E in dimensions two or three to get a measure whose Fourier transform decays slower. In the case $\mathbb{R}^3$, for example, we construct first a set $E' = E_{s'/2} \times E_{s'/2}$, for $0 < s' < 2$, in the plane spanned by the first two coordinates; as above we have chosen a sequence $\{q_k\}$ increasing rapidly to define $E_{s'/2}$. Now we construct a set $E'' = E_{s''} \subset \mathbb{R}$ for $s'' = 1 - \epsilon$, using instead the sequence $l_k = \lfloor q_k^{4s''/s'} \rfloor$ to define $E_{s''}$. The dimension of $E = E' \times E''$ is $\ge s = s' + s''$ and, by similar calculations as we did before, the Fourier transform of a measure supported in E satisfies $|\hat{\mu}(\xi)| \ge \frac{1}{4}$ for

$$\xi = (\xi_1, \xi_2, \xi_3) \in (q_k\mathbb{Z}^2 \cap [0, c^{1/2} q_k^{2/s'}]^2) \times (l_k\mathbb{Z} \cap [0, c l_k^{1/s''}]) + B(0, \rho),$$

where $c > 0$ is a small constant. Let S_k be a sphere of radius[1] $r_k = c l_k^{1/s''}$ and notice that $r_k = c q_k^{4/s'}$. We see then that $|\hat{\mu}|$ concentrates on a slab $[0, r_k^{1/2}]^2 \times \{r_k\}$ of width $\rho \sim 1$. The shape of $\hat{\mu}$ may remind the reader of the classical Knapp example, hence we can think of S_k as being essentially flat at scale $r_k^{1/2}$ and then the main contribution of $|\hat{\mu}|$ to the spherical mean lies in a cap of radius $r_k^{1/2}$, consisting of all the balls $(q_k\mathbb{Z}^2 \cap [0, c^{1/2} q_k^{2/s'}]^2) \times \{r_k\} + B(0, \rho)$. By direct computation we get that $\sigma(\mu)(r_k) \gtrsim r_k^{-\frac{s+\epsilon}{2} - \frac{1}{2}}$. Since ϵ can be made arbitrarily small, we conclude that $\alpha(s) \le \frac{s}{2} + \frac{1}{2}$ for $1 < s < 3$, which coincides with the known bound. This is basically the Knapp example in disguise.

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