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# COHOMOLOGY RINGS OF MODULI OF POINT CONFIGURATIONS ON THE PROJECTIVE LINE

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ABSTRACT. We describe the Chow rings of moduli spaces of ordered configurations of points on the projective line for arbitrary (sufficiently generic) stabilities. As an application, we exhibit such a moduli space admitting two small desingularizations with non-isomorphic cohomology rings.

## 1. Introduction

One of the classical examples of geometric invariant theory is the moduli space of ordered point configurations on the projective line [9, Chap. 3]. Recall that this is the space of semi-stable ordered $m$-tuples of points in $\mathbb{P}^1$ modulo projective equivalence, that is, modulo the action of the group $\mathrm{PGL}_2$. Here semi-stability is typically understood with respect to the symmetric stability: a tuple of points is semi-stable if at most $m/2$ points coincide. The resulting moduli space is an irreducible normal projective variety of dimension $m-3$. It is smooth for odd $m$, and singular with isolated singularities in case $m$ is even.

The (intersection) Betti numbers of this moduli space are determined in [9, Exs. 8.11, 8.15]. An explicit coordinatization is described in [6]. In the disguise of polygon spaces, the rational cohomology ring is described by generators and relations in [5]. In the case of the symmetric stability, a description of the rational Chow ring is given in [2] using an interpretation as a moduli space of quiver representations.

In the present paper, we first use the approach via moduli spaces of quiver representations (the necessary prerequisites being recalled in Section 3) to give a unified presentation of the rational Chow ring for arbitrary (sufficiently generic) stabilities in Section 5; see Theorem 13. We first describe the rational Chow ring of the quotient stack of all ordered tuples by projective equivalence and then determine the remaining relations arising from the open embedding of the moduli space.

As our main application of this description, we show that the moduli spaces of an even number of points (with respect to symmetric stability) admit two small desingularizations with non-isomorphic rational cohomology rings; see Corollary 18. The existence of such spaces is a classical topic of intersection homology theory, disproving the existence of a natural ring structure in intersection homology. The classical example of such a space is the Schubert variety

$$\{V \in \mathrm{Gr}_2(\mathbb{C}^5) \mid \dim(V \cap \mathbb{C}^3) \geq 1\};$$

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see [4, Ex. 2]. Using a general analysis of stability conditions in Section 4, we single out two stabilities deforming the symmetric one. It is shown in [10] that the corresponding moduli spaces provide small resolutions of singularities. Using the explicit description of their Chow rings, we prove the claim purely algebraically in Sections 6 and 7.

## 2. The moduli space of points on the projective line

We are concerned with the action of the group $\mathrm{PGL}_2$ on the product $(\mathbb{P}^1)^m$. Let $r_1, \dots, r_m$ be even positive integers and consider the line bundles $L_i = \pi_i^* \mathcal{O}(1)$, where $\pi_i : (\mathbb{P}^1)^m \to \mathbb{P}^1$ is the projection to the $i^{\text{th}}$ factor. Let $L = L_1^{r_1} \otimes \ldots \otimes L_m^{r_m}$. The line bundle $L$ possesses a $\mathrm{PGL}_2$-linearization (see [9, Sect. 3.1]). Let

$$X^{\mathrm{st}}(L) \subseteq X^{\mathrm{sst}}(L) \subseteq (\mathbb{P}^1)^m$$

be the open subsets of properly stable and semi-stable points, respectively, according to [9, Defs. 1.7, 1.8]. Note that we depart from Mumford's notation here. He denotes the set of properly stable points by $X^{\mathrm{st}}_{(0)}(L)$. We do this because in the world of quiver representations, a stable representation will actually be a properly stable point of the representation variety (with respect to a suitable line bundle); see [7, Def. 2.1]. Let $r = (r_1 + \ldots + r_m)/2$ and define $\theta_i = r_i/r$. We obtain a sequence $\theta = (\theta_1, \dots, \theta_m)$ of positive rationals which sum to 2. Mumford's numerical criterion [9, Thm. 2.1] (cf. also [9, Prop. 3.4]) yields that semi-stability and proper stability can be characterized by $\theta$:

**Proposition 1.** *Let $p = (p_1, \dots, p_m)$ be a tuple of points on $\mathbb{P}^1$. The point configuration $p$ lies in $X^{\mathrm{sst}}(L)$ (resp. $X^{\mathrm{st}}(L)$) if and only if not all $p_i$ (with $i \in I$) agree whenever $I \subseteq \{1, \dots, m\}$ is a subset with $\theta_I > 1$ (resp. $\theta_I \geq 1$).*

In the above proposition, $\theta_I$ is defined as $\sum_{i \in I} \theta_i$. This characterization shows that the sets $X^{\mathrm{sst}}(L)$ and $X^{\mathrm{st}}(L)$ depend only on $\theta$. Hence we denote $M^{\theta-\mathrm{sst}} = X^{\mathrm{sst}}(L) /\!\!/ \mathrm{PGL}_2$ and $M^{\theta-\mathrm{st}} = X^{\mathrm{st}}(L) / \mathrm{PGL}_2$. The quotient map $\pi : X^{\mathrm{sst}}(L) \to M^{\theta-\mathrm{sst}}$ is a categorical quotient, and $M^{\theta-\mathrm{st}}$ is an open subset of $M^{\theta-\mathrm{sst}}$. The restriction of $\pi$ to $X^{\mathrm{st}}(L) \to M^{\theta-\mathrm{st}}$ is a geometric quotient, even a principal $\mathrm{PGL}_2$-fiber bundle in the étale topology. It follows that $M^{\theta-\mathrm{st}}$ is smooth. If every semi-stable point $p \in (\mathbb{P}^1)^m$ is already properly stable for $L$, then we write $M^\theta$ for $M^{\theta-\mathrm{sst}} = M^{\theta-\mathrm{st}}$. If there exist semi-stable points that are not properly stable, then one can show that $M^{\theta-\mathrm{sst}}$ is singular.

Of particular interest is the symmetric stability condition, i.e., $L = L_1^2 \otimes \ldots \otimes L_m^2$; its associated sequence of rationals is $\theta^0 = (2/m, \dots, 2/m)$. In this case a point configuration $p = (p_1, \dots, p_m)$ is semi-stable (resp. properly stable) if and only if no more than $\lfloor m/2 \rfloor$ (resp. no more than $\lceil m/2 \rceil - 1$) of the $p_i$'s coincide. If $m$ is odd, then semi-stability and proper stability agree, but if $m$ is even, then $M^{\theta^0-\mathrm{sst}}$ is singular.

We are going to construct—when $m$ is even—two small desingularizations of $M^{\theta^0-\mathrm{sst}}$ whose cohomology rings are not isomorphic. Our strategy is to identify $M^{\theta-\mathrm{sst}}$ (resp. $M^{\theta-\mathrm{st}}$)—for an arbitrary $\theta$—with moduli spaces of quiver representations. This identification allows us to construct small desingularizations using Theorem 2, and we are able to derive presentations of the cohomology rings with Theorem 3 that help us to prove that the ring structures on the cohomology of the two resolutions of singularities that we find differ.

## 3. Moduli of quiver representations

A quiver $Q$ is a finite oriented graph. Denote its set of vertices by $Q_0$ and its set of arrows by $Q_1$. A complex representation $M$ of $Q$ consists of complex vector spaces $M_i$ for every $i \in Q_0$ and linear maps $M_a : M_i \to M_j$ attached to every arrow $a : i \to j$. There is an obvious notion of a homomorphism of representations yielding an abelian category of all (complex) representations of $Q$. Our representations will always be assumed to be finite-dimensional (i.e., every $M_i$ is a finite-dimensional vector space). In this case, we can define the dimension vector $\underline{\dim} M = (\dim M_i)_{i \in Q_0}$. Fix a dimension vector $d \in \mathbb{Z}_{\geq 0}^{Q_0}$ and consider the vector space

$$R(Q,d) = \bigoplus_{a:i\to j} \operatorname{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j}).$$

Its elements can be regarded as representations of $Q$ of dimension vector $d$. On $R(Q,d)$ we have an action of the complex linear algebraic group $\mathrm{GL}_d = \prod_{i \in Q_0} \mathrm{GL}_{d_i}$ by change of basis. The diagonally embedded multiplicative group acts trivially whence the $\mathrm{GL}_d$-action descends to an action of $\mathrm{PGL}_d = \mathrm{GL}_d / \mathbb{C}^\times$. The orbits of this group action are in one-to-one correspondence with the isomorphism classes of (complex) representations of $Q$ of dimension vector $d$. We can also interpret this set as the set of $\mathbb{C}$-valued points of the quotient stack $[R(Q,d)/\mathrm{PGL}_d]$.

If we want the quotient to carry a "nicer" geometric structure, we have to impose a stability condition. In King's article [7] Mumford's criterion (cf. [9, Thm. 2.1]) is translated to a purely algebraic condition. Fix a dimension vector $d$. Let $\theta : \mathbb{Q}^{Q_0} \to \mathbb{Q}$ be a linear map for which $\theta(d) = 0$. A representation $M$ of $Q$ of dimension vector $d$ is called $\theta$-semi-stable ($\theta$-stable)[1] if $\theta(\underline{\dim} M') \leq 0$ (resp. $\theta(\underline{\dim} M') < 0$) for every proper, non-zero subrepresentation $M'$ of $M$. The category of $\theta$-semi-stable representations is an abelian finite length category; the simple objects of this category are the $\theta$-stable representations. We consider the open subsets $R(Q,d)^{\theta-\mathrm{st}} \subseteq R(Q,d)^{\theta-\mathrm{sst}} \subseteq R(Q,d)$. The categorical quotient $M(Q,d)^{\theta-\mathrm{sst}} = R(Q,d)^{\theta-\mathrm{sst}} /\!\!/ \mathrm{PGL}_d$ parametrizes isomorphism classes of $\theta$-polystable representations of dimension vector $d$; these are the semi-simple objects of the category of semi-stable representations. The image of the stable locus $R(Q,d)^{\theta-\mathrm{st}}$ under the quotient map $R(Q,d)^{\theta-\mathrm{sst}} \to M(Q,d)^{\theta-\mathrm{sst}}$ is an open subset $M(Q,d)^{\theta-\mathrm{st}}$, and the restriction $R(Q,d)^{\theta-\mathrm{st}} \to M(Q,d)^{\theta-\mathrm{st}}$ is a geometric $\mathrm{PGL}_d$-quotient in the sense of Mumford [9, Def. 0.6], even a principal fiber bundle in the étale topology. In particular, $M(Q,d)^{\theta-\mathrm{st}}$ is smooth. Its points are in one-to-one correspondence with isomorphism classes of $\theta$-stable representations of dimension vector $d$. In the case that the quiver has no oriented cycles the variety $M(Q,d)^{\theta-\mathrm{sst}}$ is projective (see [7, Prop. 4.3]).

If every $\theta$-semi-stable representation of dimension $d$ is stable, then the moduli spaces $M(Q,d)^{\theta-\mathrm{sst}}$ and $M(Q,d)^{\theta-\mathrm{st}}$ agree; we write $M(Q,d)^{\theta}$ in this case. For example, this is the case if $d$ is $\theta$-coprime, which means that $\theta(e) \neq 0$ for every dimension vector $0 \leq e \leq d$, unless $e = 0$ or $e = d$. In this context $e \leq d$ means $e_i \leq d_i$ for every $i \in Q_0$. If $d$ is $\theta$-coprime for some stability condition $\theta$, then $d$ is necessarily indivisible (i.e., $\gcd(d_i \mid i \in Q_0) = 1$). Conversely, if $d$ is indivisible, we find a stability condition for which $d$ is coprime.

[1] Note that in [2] $\theta$-stability of a representation $M$ was defined by $\theta(M') > 0$ for every subrepresentation $M'$. We use the opposite sign convention to match it with Mumford's definition of stability.

However, if there are properly semi-stable points for $\theta$, then the variety $M(Q,d)^{\theta-\mathrm{sst}}$ is typically singular. The paper [10] deals with the question of when small desingularizations can be constructed.

Recall that a small desingularization of a variety[2] $X$ is a proper birational map $f: Y \to X$ from a smooth variety for which there exists a stratification $X = \bigsqcup X_i$ into locally closed subsets $X_i$ over each of which $f$ is étale locally trivial and such that

$$\dim f^{-1}(x) \leq \frac{1}{2}\operatorname{codim}_X(X_i)$$

for every $x \in X_i$, the estimate being strict for all strata but the dense open one. The idea for constructing small desingularizations of $M(Q,d)^{\theta-\mathrm{sst}}$ is to find a stability condition $\theta'$ "close to" $\theta$ which is sufficiently generic.

**Definition** ([10, Def. 3.1])**.** Let $d$ be a dimension vector of $Q$ and let $\theta$ be a stability condition such that $\theta(d) = 0$. A stability condition $\theta'$ of $Q$ with $\theta'(d) = 0$ is called a deformation of $\theta$ with respect to $d$ if the following conditions hold for every proper, non-zero subdimension vector $0 \leq e \leq d$:

(1) $\theta(e) < 0$ implies $\theta'(e) < 0$, and
(2) $\theta'(e) \leq 0$ implies $\theta(e) \leq 0$.

A deformation $\theta'$ of $\theta$ with respect to $d$ is called generic if $d$ is $\theta'$-coprime.

We need to introduce the Euler form of the quiver $Q$. It is the bilinear form $\chi_Q : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$ defined by

$$\chi_Q(d,e) = \sum_{i \in Q_0} d_i e_i - \sum_{\alpha: i \to j} d_i e_j.$$

In general $\chi_Q$ is not symmetric. But if it is, or rather its restriction to the kernel of the stability condition, small desingularizations can be obtained from a generic deformation of the stability condition:

**Theorem 2** ([10, Thm. 4.3])**.** *Let $Q$ be a quiver, let $d$ be an indivisible dimension vector of $Q$, and let $\theta$ be a stability condition with $\theta(d) = 0$ such that a $\theta$-stable representation of dimension $d$ exists. Suppose that $\chi_Q$ is a symmetric bilinear form on* $\ker(\theta)$*. Then the natural morphism $p : M(Q,d)^{\theta'} \to M(Q,d)^{\theta-\mathrm{sst}}$ induced by a generic deformation $\theta'$ of $\theta$ is a small desingularization.*

We want to recall a description of the equivariant Chow ring with rational coefficients $A^*_{\mathrm{PGL}_d}(R(Q,d)^{\theta-\mathrm{sst}})_{\mathbb{Q}}$; see [1, Sect. 2.6] for the definition of equivariant Chow rings. For simplicity we will always use rational coefficients, although it is not always necessary. Let $T_d$ be the maximal torus of $\mathrm{GL}_d$ that consists of invertible diagonal matrices and let $\mathrm{P}T_d$ be the quotient by the diagonally embedded $\mathbb{C}^\times$. The character group of $T_d$ is the free group generated by $x_{i,r}$ with $i \in Q_0$ and $r = 1, \dots, d_i$, and the character group of $\mathrm{P}T_d$ is the subgroup $X(\mathrm{P}T_d) = \{\sum_{i,r} a_{i,r} x_{i,r} \mid \sum_{i,r} a_{i,r} = 0\}$. The equivariant Chow ring $A^*_{\mathrm{P}T_d}(\mathrm{pt})_{\mathbb{Q}}$ is the subring $\mathbb{Q}[x_{j,s} - x_{i,r} \mid i,j \in Q_0,\ r = 1, \dots, d_i, s = 1, \dots, d_j]$ of the polynomial ring $A^*_{T_d}(\mathrm{pt})_{\mathbb{Q}} = \mathbb{Q}[x_{i,r} \mid i \in Q_0,\ r = 1, \dots, d_i]$. The ring $A^*_{\mathrm{P}T_d}(\mathrm{pt})_{\mathbb{Q}}$ is itself a polynomial ring as $X(\mathrm{P}T_d)$ is a free abelian group, but we don't want to choose

[2]Variety means irreducible here and in the following.

a basis here. The equivariant Chow ring $A^*_{\mathrm{PGL}_d}(R(Q,d))_{\mathbb{Q}} \cong A^*_{\mathrm{PGL}_d}(\mathrm{pt})_{\mathbb{Q}}$ agrees with the ring

$$A^*_{\mathrm{P}T}(\mathrm{pt})_{\mathbb{Q}}^{W_d} = \mathbb{Q}[x_{j,s} - x_{i,r} \mid i,j \in Q_0,\ r = 1,\dots,d_i, s = 1,\dots,d_j]^{W_d},$$

where $W_d = \prod_i S_{d_i}$ acts by permutation of the variables $x_{i,r}$. These variables are characters of a maximal torus of $\mathrm{GL}_d$, and $W_d$ is the corresponding Weyl group. The ring $A^*_{\mathrm{PGL}_d}(R(Q,d)^{\theta-\mathrm{sst}})_{\mathbb{Q}}$ is a quotient of $A^*_{\mathrm{PGL}_d}(R(Q,d))_{\mathbb{Q}}$. The kernel of the pull-back $A^*_{\mathrm{PGL}_d}(R(Q,d))_{\mathbb{Q}} \to A^*_{\mathrm{PGL}_d}(R(Q,d)^{\theta-\mathrm{sst}})_{\mathbb{Q}}$ can be described by tautological relations in the sense of [2]. A $\theta$-forbidden decomposition of $d$ is a decomposition $d = p + q$ into dimension vectors for which $\theta(p) > 0$. For such a decomposition we consider the element

$$f^{p,q} = \prod_{\alpha: i \to j} \prod_{r=1}^{p_i} \prod_{s=p_j+1}^{d_j} (x_{j,s} - x_{i,r})$$

and the principal ideal $I^{p,q} = (f^{p,q})$ in the ring $A^*_{\mathrm{P}T_d}(\mathrm{pt})_{\mathbb{Q}}^{W_p \times W_q}$. Let

$$\rho^{p,q} : A^*_{\mathrm{P}T_d}(\mathrm{pt})_{\mathbb{Q}}^{W_p \times W_q} \to A^*_{\mathrm{P}T_d}(\mathrm{pt})_{\mathbb{Q}}^{W_d}$$

be the $A^*_{\mathrm{P}T_d}(\mathrm{pt})_{\mathbb{Q}}^{W_d}$-linear map defined by

$$\rho^{p,q}(f) = \sum_{\sigma} \sigma f \cdot \prod_i \prod_{r=1}^{p_i} \prod_{s=p_i+1}^{d_i} (x_{i,\sigma_i(s)} - x_{i,\sigma_i(r)})^{-1},$$

where $\sigma = (\sigma_i)_i \in W_d$ ranges over all $(p,q)$-shuffles; that means each $\sigma_i$ is a $(p_i,q_i)$-shuffle permutation.

**Theorem 3.** *The equivariant Chow ring $A^*_{\mathrm{PGL}_d}(R(Q,d)^{\theta-\mathrm{sst}})_{\mathbb{Q}}$ is isomorphic to the quotient of the ring $A^*_{\mathrm{P}T_d}(\mathrm{pt})_{\mathbb{Q}}^{W_d}$ by the ideal*

$$\sum_{p,q} \rho^{p,q}(I^{p,q}).$$

*Proof.* This can be proved using the same arguments as in [3, Thm. 8.1]. There the $\mathrm{GL}_d$-equivariant Chow ring of the semi-stable locus is considered, but the arguments can be applied for the $\mathrm{PGL}_d$-equivariant situation as well. □

*Remark* 4. It can be shown analogously to [3, Thm. 5.1] that the equivariant cycle map $A^*_{\mathrm{PGL}_d}(R(Q,d)^{\theta-\mathrm{sst}})_{\mathbb{Q}} \to H^*_{\mathrm{PGL}_d}(R(Q,d)^{\theta-\mathrm{sst}};\mathbb{Q})$ is an isomorphism. In the case that $Q$ is acyclic and $d$ is $\theta$-coprime this is shown in [8, Thm. 3].

## 4. Moduli of point configurations as quiver moduli

Let $m \geq 3$. Fix a sequence $\theta = (\theta_1,\dots,\theta_m)$ of positive rational numbers which sum to 2 and consider the quotients $M^{\theta-\mathrm{sst}}$ and $M^{\theta-\mathrm{st}}$ introduced in Section 2.

Let $U_m$ be the $m$-subspace quiver. It consists of $m$ sources and one sink—formally $Q_0 = \{1,\dots,m,\infty\}$ with $\infty$ being the sink—and has one arrow pointing from every source to the sink. Let $d = (1,\dots,1;2)$, i.e., $d_i = 1$ for $i = 1,\dots,m$ and $d_\infty = 2$, i.e., the dimension vector with ones at every source and two at the sink.

Pictorially, the quiver and the dimension vector are given by

$$\begin{tikzcd}
 & & \overset{2}{\bullet} & \\
\underset{1}{\bullet} \arrow[urr] & \underset{1}{\bullet} \arrow[ur] & \cdots & \underset{1.}{\bullet} \arrow[ul]
\end{tikzcd}$$

Let $R = R(U_m, d) \cong (\mathbb{C}^2)^m$. The group $\mathrm{GL}_d$ is $(\mathbb{C}^\times)^m \times \mathrm{GL}_2(\mathbb{C})$, and it acts on the vector space $R$ via $(t_1, \ldots, t_m, g).(v_1, \ldots, v_m) = (t_1^{-1} g v_1, \ldots, t_m^{-1} g v_m)$. Abbreviate $G = \mathrm{PGL}_d = ((\mathbb{C}^\times)^m \times \mathrm{GL}_2)/\mathbb{C}^\times$.

From the rational numbers $\theta_1, \ldots, \theta_m$ we construct a stability condition for $U_m$ which we, by slight abuse of notation, also denote by $\theta$. The value of $\theta$ at the $i^{\text{th}}$ source is $\theta_i$, while the value at the sink is defined as $\theta_\infty = -1$. This stability condition satisfies $\theta(d) = 0$.

**Lemma 5.** *Let $v = (v_1, \ldots, v_m)$ be a point of $R$. The representation $v$ is semi-stable (resp. stable) if and only if every $v_i$ is non-zero and $\{v_i \mid i \in I\}$ spans $\mathbb{C}^2$ whenever $I \subseteq \{1, \ldots, m\}$ is a subset satisfying $\theta_I > 1$ (resp. $\theta_I \geq 1$).*

*Proof.* Let $(v_1, \ldots, v_m)$ be semi-stable. If $v_i$ were 0, then there would be a subrepresentation of dimension vector $e_i = (0, \ldots, 1, \ldots, 0; 0)$ (with a one in the $i^{\text{th}}$ position). But $\theta(e_i) = \theta_i > 0$, which contradicts semi-stability. For a subset $I$ with $\theta_I > 1$ we apply a similar argument: Suppose that the span of $\{v_i \mid i \in I\}$ were a proper subspace of $\mathbb{C}^2$. Then this would yield a subrepresentation of dimension vector $d' = \sum_{i \in I} e_i + e_\infty$. So $\theta(d') = \theta_I - 1 > 0$. Again a contradiction. Conversely, suppose that $v$ satisfies the numerical conditions of the lemma. The only subdimension vectors of $d = (1, \ldots, 1; 2)$ which have a positive $\theta$-value are of the form $\sum_{i \in I} e_i + n e_\infty$ with either $n = 0$ and $I \neq \emptyset$ or $n = 1$ and $\theta_I > 1$. All of these subdimension vectors cannot belong to subrepresentations of $v$. The characterization of stability can be shown in just the same way. $\square$

**Proposition 6.** *There is a natural isomorphism $M^{\theta-\mathrm{sst}} \cong M^{\theta-\mathrm{sst}}(U_m, d)$ which restricts to an isomorphism $M^{\theta-\mathrm{st}} \cong M^{\theta-\mathrm{st}}(U_m, d)$.*

*Proof.* Let $L$ be the corresponding line bundle on $(\mathbb{P}^1)^m$ (uniquely determined by $\theta$ up to a positive power). The set $R^{\theta-\mathrm{sst}}(U_m, d)$ is contained in $(\mathbb{C}^2 - \{0\})^m$ by Lemma 5. It is easy to see, using Proposition 1 and Lemma 5, that $R^{\theta-\mathrm{sst}}(U_m, d)$ is precisely the inverse image of $X^{\mathrm{sst}}(L)$ under the map $\varpi : (\mathbb{C}^2 - \{0\})^m \to (\mathbb{P}^1)^m$. The same holds for the stable loci. As $\varpi$ is a geometric quotient by the group $(\mathbb{C}^\times)^m$, it induces an isomorphism $R^{\theta-\mathrm{sst}}(U_m, d) /\!\!/ G \xrightarrow{\cong} X^{\mathrm{sst}}(L) /\!\!/ \mathrm{PGL}_2$ which restricts to an isomorphism $R^{\theta-\mathrm{st}}(U_m, d)/G \xrightarrow{\cong} X^{\mathrm{st}}(L)/\mathrm{PGL}_2$. $\square$

Lemma 5 shows that the subsets $I \subseteq \{1, \ldots, m\}$ with $\theta_I > 1$ play an important role. We call those subsets $\theta$-forbidden. Denote by $\mathcal{I}^\theta$ the set of all $\theta$-forbidden subsets.

**Lemma 7.** *Let $\theta = (\theta_1, \ldots, \theta_m)$ be a sequence of positive rational numbers with $\theta_1 + \ldots + \theta_m = 2$. Interpret it as a stability condition for $U_m$.*

(1) *The $\theta$-stable locus is non-empty if and only if $\theta_i < 1$ for every $1 = 1, \ldots, m$.*
(2) *The dimension vector $(1, \ldots, 1; 2)$ is $\theta$-coprime if and only if $\theta_I := \sum_{i \in I} \theta_i \neq 1$ for every non-empty proper subset $I$ of $\{1, \ldots, m\}$.*

(3) *Let $\theta'$ be another such stability condition. In this case $\theta'$ is a deformation of $\theta$ with respect to* $(1,\dots,1;2)$ *if and only if*
 (a) $\theta_I < 1$ *implies* $\theta'_I < 1$ *and*
 (b) $\theta'_I \leq 1$ *implies* $\theta_I \leq 1$

*for every proper non-empty subset* $I \subsetneq \{1,\dots,m\}$.

*Proof.* All three claims follow from Lemma 5. To show the first assertion assume $\theta_i \geq 1$ for some $i$. For any $v = (v_1,\dots,v_m) \in R$, the span of $v_i$ is at most one-dimensional, so $v$ is not stable. Conversely if all $\theta_i < 1$, then a point $v$ consisting of pairwise linearly independent vectors is stable. Assertions (2) and (3) are also direct applications of Lemma 5. □

From now on we will deal with the case where $m$ is even. Say $m = 2n$. The symmetric stability condition is $\theta^0 = (1/n,\dots,1/n)$. We will consider two generic deformations $\theta^+$ and $\theta^-$ of $\theta^0$ given by

$$\begin{aligned}\theta^+ &= (\frac{1}{n}+\varepsilon, \frac{1}{n}-\frac{\varepsilon}{2n-1},\dots,\frac{1}{n}-\frac{\varepsilon}{2n-1};-1),\\ \theta^- &= (\frac{1}{n}-\varepsilon, \frac{1}{n}+\frac{\varepsilon}{2n-1},\dots,\frac{1}{n}+\frac{\varepsilon}{2n-1};-1)\end{aligned}$$

for a sufficiently small rational number $\varepsilon$. We analyze the forbidden subsets for these three stability conditions.

**Lemma 8.** *Let $m = 2n$. For $\varepsilon$ sufficiently small $d = (1,\dots,1;2)$ is coprime for both $\theta^+$ and $\theta^-$, and the sets of forbidden subsets for $\theta^0$, $\theta^+$, and $\theta^-$ are*

$$\begin{aligned}\mathcal{I}^{\theta^0} &= \{I \mid |I| > n\},\\ \mathcal{I}^{\theta^+} &= \mathcal{I}^{\theta^0} \sqcup \{I \mid |I| = n \text{ and } 1 \in I\},\\ \mathcal{I}^{\theta^-} &= \mathcal{I}^{\theta^0} \sqcup \{I \mid |I| = n \text{ and } 1 \notin I\}.\end{aligned}$$

*As a consequence $\theta^+$ and $\theta^-$ are generic deformations of $\theta^0$.*

*Proof.* If $I \subseteq \{1,\dots,m\}$ is a subset with $k$ elements, then the $\theta^+$-value of $I$ is

$$\theta^+_I = \begin{cases} \frac{k}{n} + \frac{2n-k}{2n-1}\varepsilon & \text{if } 1 \in I,\\ \frac{k}{n} - \frac{k}{2n-1}\varepsilon & \text{if } 1 \notin I.\end{cases}$$

If $\varepsilon$ is smaller than $\frac{2n-1}{n(n+1)}$, which ensures that $\frac{n-1}{n}+\frac{n+1}{2n-1}\varepsilon < 1$ and $\frac{n+1}{n}-\frac{n+1}{2n-1}\varepsilon > 1$, then $\theta^+_I \neq 1$ and, moreover, the set of $\theta^+$-forbidden subsets is as asserted. The proof for $\theta^-$ works in the same fashion. □

An immediate consequence of Lemma 8 and Theorem 2 is the following.

**Proposition 9.** *With the notation as in Lemma 8, both $M^{\theta^+}$ and $M^{\theta^-}$ are small desingularizations of $M^{\theta^0-\mathrm{sst}}$.*

*Proof.* To make sure Theorem 2 is applicable we need to check that the Euler form restricted to $\ker(\theta^0)$ is symmetric. Let $d = (d_1,\dots,d_{2n};d_\infty)$ and let $e = (e_1,\dots,e_{2n};e_\infty)$ be contained in the kernel of $\theta^0$. That means $nd_\infty = \sum_{i=1}^{2n} d_i$ and

$ne_\infty = \sum_{i=1}^{2n} e_i$. Thus

$$\chi_{U_{2n}}(d,e) = \sum_{i=1}^{2n} d_i e_i + d_\infty e_\infty - \sum_{i=1}^{2n} d_i e_\infty$$
$$= \sum_{i=1}^{2n} d_i e_i + d_\infty e_\infty - \frac{1}{n} \sum_{i,j=1}^{2n} d_i e_j = \chi_{U_{2n}}(e,d).$$

□

## 5. Chow rings of moduli of point configurations

In order to compute and compare the Chow rings of the two small desingularizations obtained in Proposition 9 it will be useful to also discuss Chow rings of semi-stable moduli stacks. In fact, $A^*(M^{\theta^+})_{\mathbb{Q}}$ and $A^*(M^{\theta^-})_{\mathbb{Q}}$ will turn out to be quotients of the Chow ring of the $\theta^0$-semi-stable moduli stack (see Corollary 15). To compute an explicit presentation of all these rings, we will apply Theorem 3 to moduli of points on $\mathbb{P}^1$, i.e., moduli of representation of $U_m$ with dimension vector $d = (1, \ldots, 1; 2)$; cf. also [2, Cor. 29]. The equivariant Chow ring $A^*_{\mathrm{PT}_d}(\mathrm{pt})_{\mathbb{Q}}$ is the subring $\mathbb{Q}[y_j - x_i \mid i = 1, \ldots, m,\ j = 1, 2]$ of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_m, y_1, y_2]$. Denote $y = y_1 + y_2$ and $z = y_1 y_2$.

**Lemma 10.** *The ring* $A^*_{\mathrm{PGL}_d}(\mathrm{pt})_{\mathbb{Q}} = \mathbb{Q}[y_j - x_i \mid i = 1, \ldots, m,\ j = 1, 2]^{S_2}$ *is generated by the algebraically independent elements* $X_i = \frac{1}{2}y - x_i$ *(all* $i = 1, \ldots, m$*) and* $Y = \frac{1}{4}y^2 - z$.

*Proof.* It is obvious that $X_i$ and $Y$ are elements of the ring in question, and it is easy to show that they are algebraically independent. The fact that the generating series of that ring is $(1-q)^{-m}(1-q^2)^{-1}$ concludes the proof. □

First of all, we consider the moduli stack $\mathcal{M}$ of $m$ points in $\mathbb{P}^1$ up to $\mathrm{PGL}_2$-action. We define $U = \{(v_1, \ldots, v_m) \in (\mathbb{C}^2)^m \mid v_i \neq 0 \text{ (all } i)\}$. It is an open subset of $R(U_m, d) = (\mathbb{C}^2)^m$, and $\mathcal{M}$ is precisely the quotient stack $[U/\mathrm{PGL}_d]$. The Chow ring of a quotient stack $[X/G]$, for a linear algebraic group $G$ acting on an algebraic scheme $X$, is defined as the equivariant Chow ring $A^*_G(X)$; see the discussion in [1, Sect. 5.3] for why this is reasonable. The identification of $\mathcal{M}$ with $[U/\mathrm{PGL}_d]$ shows that the Chow ring $A^*(\mathcal{M})_{\mathbb{Q}} = A^*_{\mathrm{PGL}_2}((\mathbb{P}^1)^m)_{\mathbb{Q}}$ is isomorphic to $A^*([U/\mathrm{PGL}_d])_{\mathbb{Q}} = A^*_{\mathrm{PGL}_d}(U)_{\mathbb{Q}}$, which is a quotient of $\mathbb{Q}[X_1, \ldots, X_m, Y]$. We show the following.

**Lemma 11.** *The Chow ring* $A^*(\mathcal{M})_{\mathbb{Q}} = A^*_{\mathrm{PGL}_2}((\mathbb{P}^1)^m)_{\mathbb{Q}}$ *with rational coefficients is isomorphic to* $\mathbb{Q}[X_1, \ldots, X_m, Y]/(X_i^2 - Y \mid i = 1, \ldots, m)$.

*Proof.* The proof works in the same fashion as the proof of [3, Thm. 8.1]. We give it for completeness. The complement of $U$ inside $R = (\mathbb{C}^2)^m$ is the union $\bigcup_{i=1}^m Z_i$ of subspaces $Z_i = \{(v_1, \ldots, v_m) \mid v_i = 0\}$. The class $[Z_i]$ in the equivariant Chow ring $A^*_{\mathrm{PGL}_d}(R(U_m, d)) = \mathbb{Q}[y_j - x_i \mid i, j]$ is $(y_1 - x_i)(y_2 - x_i)$. We see that

$$(y_1 - x_i)(y_2 - x_i) = z - x_i y + x_i^2 = X_i^2 - Y.$$

Using the fact that the map $\bigoplus_i A^*_{\mathrm{PGL}_d}(Z_i)_{\mathbb{Q}} \to A^*_{\mathrm{PGL}_d}(R(U_m, d))_{\mathbb{Q}}$ surjects onto the kernel of the map $A^*_{\mathrm{PGL}_d}(R(U_m, d))_{\mathbb{Q}} \to A^*(\mathcal{M})_{\mathbb{Q}}$ completes the proof. □

We consider a non-trivial stability condition $\theta$ given by rational numbers $\theta_1, \ldots, \theta_m$ as described in the previous section. The quotient stack

$$\mathcal{M}^{\theta-\mathrm{sst}} = [R(U_m, d)^{\theta-\mathrm{sst}} / \mathrm{PGL}_d]$$

is an open substack of $\mathcal{M}$. Therefore $A^*(\mathcal{M}^{\theta-\mathrm{sst}})$ is a quotient of $\mathcal{A} = A^*(\mathcal{M})_{\mathbb{Q}} = \mathbb{Q}[X_1, \ldots, X_m]/(X_i^2 = X_j^2)$.

For every subset $I \subseteq \{1, \ldots, m\}$ we define

$$f^I = \prod_{i \in I} (y_2 - x_i);$$

it is an element of $A^*_{\mathrm{P}T_d}(\mathrm{pt})_{\mathbb{Q}}$. Let $I \in \mathcal{I}^\theta$; i.e., $I$ is a $\theta$-forbidden subset. For such $I$, the dimension vector $d_I = (d_{I,1}, \ldots, d_{I,m}; 1)$—with $d_{I,i} = 1$ if $i \in I$ and $d_{I,i} = 0$ otherwise—has a positive $\theta$-value, hence gives a $\theta$-forbidden decomposition $d = d_I + (d - d_I)$. The polynomial attached to this forbidden decomposition is precisely $f^I$. Applying Theorem 3 yields that the kernel of $A^*(\mathcal{M})_{\mathbb{Q}} \to A^*(\mathcal{M}^{\theta-\mathrm{sst}})_{\mathbb{Q}}$ is generated by the elements

$$\rho(f^I) = \sum_{J \subsetneq I} (-1)^{|J|} x_J \sum_{\nu=0}^{|I-J|-1} y_1^\nu y_2^{|I-J|-1-\nu},$$
$$\rho(f^I(y_2 - y_1)) = \sum_{J \subseteq I} (-1)^{|J|} x_J (y_1^{|I-J|} + y_2^{|I-J|}),$$

where $\rho : A^*_{\mathrm{P}T_d}(\mathrm{pt})_{\mathbb{Q}} \to A^*_{\mathrm{P}T_d}(\mathrm{pt})^{S_2}$ is the symmetrization map.

**Lemma 12.** *Let $I \subseteq \{1, \ldots, m\}$ be a subset of cardinality $k$. Then*

$$\rho(f^I) = \sum_{\nu=0}^{\lfloor \frac{k-1}{2} \rfloor} e_{k-1-2\nu}(X_i \mid i \in I) Y^\nu,$$
$$\frac{1}{2}\rho(f^I(y_2 - y_1)) = \sum_{\nu=0}^{\lfloor \frac{k}{2} \rfloor} e_{k-2\nu}(X_i \mid i \in I) Y^\nu.$$

*Proof.* We prove the two equalities asserted in the lemma by induction on $k$. It suffices to check them for the sets $I = \{1, \ldots, k\}$. Let $R_k = \rho(f^{\{1,\ldots,k\}})$ and $S_k = \frac{1}{2}\rho(f^{\{1,\ldots,k\}}(y_2 - y_1))$ and denote by $\tilde{R}_k$, resp. $\tilde{S}_k$, the right-hand sides of the equations, i.e.,

$$\tilde{R}_k = \sum_{\nu=0}^{\lfloor \frac{k-1}{2} \rfloor} e_{k-1-2\nu}(X_1, \ldots, X_k) Y^\nu, \qquad \tilde{S}_k = \sum_{\nu=0}^{\lfloor \frac{k}{2} \rfloor} e_{k-2\nu}(X_1, \ldots, X_k) Y^\nu.$$

We see that $R_0 = 0 = \tilde{R}_0$ and $S_0 = 1 = \tilde{S}_1$. Obviously the expressions $\tilde{R}_k$ and $\tilde{S}_k$ satisfy the relations

$$\tilde{R}_k = X_k \tilde{R}_{k-1} + \tilde{S}_{k-1}, \qquad \tilde{S}_k = X_k \tilde{S}_{k-1} + Y \tilde{R}_{k-1}.$$

To complete the proof, it suffices to show that these relations hold for $R_k$ and $S_k$ as well. This is an easy but lengthy computation. We give it here for completeness.

The expression $X_kR_{k-1} + S_{k-1}$ equals

$$\left(\frac{1}{2}(y_1+y_2) - x_k\right)\sum_{j=0}^{k-2}(-1)^j e_j(x_1,\dots,x_{k-1})\sum_{\nu=0}^{k-j-2} y_1^\nu y_2^{k-j-2-\nu}$$
$$+\frac{1}{2}\sum_{j=0}^{k-1}(-1)^j e_j(x_1,\dots,x_{k-1})(y_1^{k-j-1}+y_2^{k-j-1}),$$

which agrees with

$$\frac{1}{2}\sum_{j=0}^{k-2}(-1)^j e_j(x_1,\dots,x_{k-1})\underbrace{\Big(y_1^{k-j-1}+\sum_{\nu=0}^{k-j-2}(y_1^{\nu+1}y_2^{k-j-2-\nu}+y_1^\nu y_2^{k-j-1-\nu})+y_2^{k-j-1}\Big)}_{=\sum_{\nu=0}^{k-j-1} y_1^\nu y_2^{k-j-1-\nu}}$$
$$-\sum_{j=0}^{k-2}(-1)^j x_k e_j(x_1,\dots,x_{k-1})\sum_{\nu=0}^{k-j-2} y_1^\nu y_2^{k-j-2-\nu},$$

and that equals

$$\sum_{j=0}^{k-1}(-1)^j\Big(e_j(x_1,\dots,x_{k-1})+x_k e_{j-1}(x_1,\dots,x_{k-1})\Big)\sum_{\nu=0}^{k-j-1} y_1^\nu y_2^{k-j-1-\nu},$$

which is just $R_k$ when interpreting $e_{-1}(x_1,\dots,x_{k-1})$ as zero. For $S_k$ the computation reads as follows: The term $X_kS_{k-1}+YR_{k-1}$ is equal to

$$\left(\frac{1}{2}(y_1+y_2)-x_k\right)\cdot\frac{1}{2}\sum_{j=0}^{k-1}(-1)^j e_j(x_1,\dots,x_{k-1})(y_1^{k-j-1}+y_2^{k-j-1})$$
$$+\left(\frac{1}{4}(y_1+y_2)^2-y_1y_2\right)\sum_{j=0}^{k-2}(-1)^j e_j(x_1,\dots,x_{k-1})\sum_{\nu=0}^{k-j-2} y_1^\nu y_2^{k-j-2-\nu},$$

which can be re-written as

$$\frac{1}{4}\sum_{j=0}^{k-1}(-1)^j e_j(x_1,\dots,x_{k-1})(y_1^{k-j}+y_1y_2(y_1^{k-j-2}+y_2^{k-j-2})+y_2^{k-j})$$
$$-\frac{1}{2}\sum_{j=0}^{k-1}(-1)^j x_k e_j(x_1,\dots,x_{k-1})(y_1^{k-j-1}+y_2^{k-j-1})$$
$$+\frac{1}{4}\sum_{j=0}^{k-2}(-1)^j e_j(x_1,\dots,x_{k-1})\underbrace{\sum_{\nu=0}^{k-j-2}(y_1^{\nu+2}y_2^{k-j-2-\nu}-2y_1^{\nu+1}y_2^{k-j-1-\nu}+y_1^\nu y_2^{k-j-\nu})}_{=y_1^{k-j}-y_1y_2(y_1^{k-j-2}+y_2^{k-j-2})+y_2^{k-j}},$$

and that is the same as

$$\frac{1}{2}\sum_{j=0}^{k}(-1)^j\Big(e_j(x_1,\dots,x_{k-1}+x_k e_{j-1}(x_1,\dots,x_{k-1})\Big)(y_1^{k-j}+y_2^{k-j}).$$

This equals $S_k$. Here we formally need to set $e_k(x_1,\dots,x_{k-1})=0$. The lemma is proved. $\square$

We put $R_I = \rho(f^I)$ and $S_I = \frac{1}{2}\rho((y_2 - y_1)f^I)$. Note also that the relations

$$(*) \qquad R_I = X_i R_{I-\{i\}} + S_{I-\{i\}}, \qquad S_I = X_i S_{I-\{i\}} + Y R_{I-\{i\}}$$

imply that we may restrict to minimal forbidden subsets $I$, i.e., minimal elements of $\mathcal{I}^\theta$ with respect to inclusion. Denote the set of those minimal forbidden subsets with $\mathcal{I}^\theta_{\min}$.

**Theorem 13.** *The Chow ring $A^*(\mathcal{M}^{\theta-\mathrm{sst}})_\mathbb{Q}$ is the quotient of the ring*

$$\mathbb{Q}[X_1, \dots, X_m, Y]/(X_i^2 - Y)$$

*by the ideal generated by the elements*

$$R_I = \sum_{\nu=0}^{\lfloor \frac{k-1}{2} \rfloor} e_{k-1-2\nu}(X_i \mid i \in I) Y^\nu \quad \text{and} \quad S_I = \sum_{\nu=0}^{\lfloor \frac{k}{2} \rfloor} e_{k-2\nu}(X_i \mid i \in I) Y^\nu,$$

*with $I \in \mathcal{I}^\theta_{\min}$.*

*Proof.* Let $\mathcal{A} = \mathbb{Q}[X_1, \dots, X_m, Y]/(X_i^2 - Y)$. Applying Lemmas 10, 11, and 12 to Theorem 3 yields $A^*(\mathcal{M}^{\theta-\mathrm{sst}})_\mathbb{Q} = \mathcal{A}/(R_I, S_I \mid I \in \mathcal{I}^\theta)$. The relations $(*)$ show that the ideal $(R_I, S_I \mid I \in \mathcal{I}^\theta_{\min})$ agrees with $(R_I, S_I \mid I \in \mathcal{I}^\theta)$. $\square$

*Remark* 14. Note that if there is a principal bundle quotient $X/G$ for the action of a linear algebraic group on an algebraic scheme $X$, then, by [1, Prop. 8], the equivariant Chow ring $A^*_G(X)$ coincides with the Chow ring $A^*(X/G)$ of the quotient. Therefore, if $\theta$ is a generic stability condition, that means every semi-stable point for $\theta$ is already stable, and the Chow ring of the moduli stack $A^*(\mathcal{M}^{\theta-\mathrm{sst}})_\mathbb{Q} = A^*_{\mathrm{PGL}_d}(R(U_m, d)^{\theta-\mathrm{sst}})_\mathbb{Q}$ equals the Chow ring of the moduli space $M^\theta = R(U_m, d)^{\theta-\mathrm{st}}/\mathrm{PGL}_d$. So in the generic case Theorem 13 actually gives a description of $A^*(M^\theta)_\mathbb{Q}$.

We show how this applies to the stability conditions $\theta^0$, $\theta^+$, and $\theta^-$ in the case that $m = 2n$. A combination of the previous theorem with Lemma 8 yields

**Corollary 15.** *The rings $\mathcal{A}^0 = A^*(\mathcal{M}^{\theta^0-\mathrm{sst}})_\mathbb{Q}$ and $\mathcal{A}^\pm = A^*(M^{\theta^\pm})_\mathbb{Q}$ are quotients of the ring $\mathcal{A} = \mathbb{Q}[X_1, \dots, X_{2n}, Y]/(X_i^2 - Y)$ by ideals $\mathfrak{a}^0$ and $\mathfrak{a}^\pm$, which are given by*

$$\begin{aligned}
\mathfrak{a}^0 &= (R_I, S_I \mid I \subseteq \{1, \dots, 2n\},\ |I| = n+1), \\
\mathfrak{a}^+ &= (R_I, S_I \mid I \subseteq \{1, \dots, 2n\},\ |I| = n,\ 1 \in I) + \mathfrak{a}^0, \\
\mathfrak{a}^- &= (R_I, S_I \mid I \subseteq \{1, \dots, 2n\},\ |I| = n,\ 1 \notin I) + \mathfrak{a}^0.
\end{aligned}$$

## 6. Automorphisms

Consider $\mathcal{A} = \mathbb{Q}[X_1, \dots, X_m, Y]/(X_i^2 - Y) = \mathbb{Q}[X_1, \dots, X_m]/(X_i^2 = X_j^2)$. We want to determine the automorphism group of this graded ring. For this, consider the following automorphisms of the polynomial ring $\mathbb{Q}[X_1, \dots, X_m]$.

- For a non-zero rational $d$, we denote by $m_d$ the dilation with $d$.
- Let $\sigma \in S_m$ be a permutation. The automorphism that sends $X_i$ to $X_{\sigma(i)}$ will be called $\pi_\sigma$.
- Given $i \in \{1, \dots, m\}$, let $\tau_i$ be $\tau_i(f) = f(X_1, \dots, -X_i, \dots, X_m)$.

We verify immediately that the above-mentioned automorphisms of the polynomial ring leave the ideal $(X_i^2 - X_j^2 \mid i,j = 1,\ldots,m)$ invariant. Hence they descend to automorphisms of the ring $\mathcal{A}$. We denote them with the same symbol. The rest of the section will be devoted to the proof of

**Proposition 16.** *If $m > 2$, then the group $\mathrm{Aut}(\mathcal{A})$ is generated by the elements $m_d$ ($d \in \mathbb{Q}^\times$), $\pi_\sigma$ (where $\sigma \in S_m$), and $\tau_i$ ($i \in \{1,\ldots,m\}$).*

*Proof.* Let $\varphi$ be an automorphism of $\mathcal{A}$. It is given by an invertible matrix $A = (a_{ij}) \in \mathrm{GL}_m(\mathbb{Q})$—that means $\varphi(X_j) = \sum_i a_{ij}X_i$—such that $\varphi(X_{j_1})^2 - \varphi(X_{j_2})^2$ is contained in the ideal generated by the expressions $X_{i_1}^2 - X_{i_2}^2$. We compute

$$\varphi(X_{j_1})^2 - \varphi(X_{j_2})^2 = \sum_i (a_{ij_1}^2 - a_{ij_2}^2)x_i^2 + 2\sum_{i_1<i_2}(a_{i_1j_1}a_{i_2j_1} - a_{i_1j_2}a_{i_2j_2})x_{i_1}x_{i_2}.$$

From this we deduce the relations

(a) $\sum_i a_{ij_1}^2 = \sum_i a_{ij_2}^2$,
(b) $a_{i_1j_1}a_{i_2j_1} = a_{i_1j_2}a_{i_2j_2}$

for all $i_1 < i_2$ and all $j_1 < j_2$.

We assume there is an index $j$ for which the $j^{\text{th}}$ column contains two non-zero entries, say $a_{i_1j}a_{i_2j} \neq 0$. From relation (b) we deduce that $a_{i_1j'}a_{i_2j'} \neq 0$ for every other column index $j'$ and

$$a_{i_2j'} = \frac{a_{i_1j}a_{i_2j}}{a_{i_1j'}}.$$

Suppose there were a third non-zero entry $a_{i_3j}$ in the $j^{\text{th}}$ column. We apply relation (b) for $i_1, i_3$, and $i_2, i_3$ and obtain

$$a_{i_3j'} = \frac{a_{i_1j}a_{i_3j}}{a_{i_1j'}}, \qquad a_{i_3j'} = \frac{a_{i_2j}a_{i_3j}}{a_{i_2j'}} = \frac{a_{i_1j'}a_{i_3j}}{a_{i_1j}},$$

so consequently $a_{i_1j} = a_{i_1j'}$, and in a similar vein $a_{i_2j} = a_{i_2j'}$ and $a_{i_3j} = a_{i_3j'}$. As $i_3$ was chosen arbitrarily, we deduce that the $j'^{\text{th}}$ column would have to be equal to the $j^{\text{th}}$, which contradicts the fact that $A$ is invertible. This shows that under the assumption that there is a column which contains more than one non-zero entry, it would have to have precisely two, and every other column would have precisely two non-vanishing entries in the exact same positions. This is absurd because the matrix $A$ is assumed to have at least 3 columns.

Summarizing, $A$ is a matrix with at most one non-zero entry in every column. As the column sums are all the same by relation (a), we can apply a dilation to make it a matrix with entries 0 or $\pm 1$. By regularity of $A$, every row of $A$ has also precisely one non-zero entry. Therefore, up to the application of some $\tau_i$'s, it is a permutation matrix. The proposition is proved. ∎

## 7. The ring structure

**Theorem 17.** *The rings $A^*(M^{\theta^+})_{\mathbb{Q}}$ and $A^*(M^{\theta^-})_{\mathbb{Q}}$ are not isomorphic if $n \geq 3$.*

*Proof.* We first treat the smallest case $n = 3$ since it forms a blueprint of the proof in the general case. We abbreviate $\mathcal{A}^\pm = A^*(M^{\theta^\pm})_{\mathbb{Q}}$. We claim that $\mathcal{A}^-$ contains a non-zero 2-nilpotent element which is homogeneous of degree 1, whereas $\mathcal{A}^+$ does not.

Indeed, for all $2 \le i < j < k \le 6$, we have the degree 2 relation $Y + (X_iX_j + X_iX_k + X_jX_k) = 0$ in $\mathcal{A}^-$, as well as $X_i^2 = Y$. Summing four of these relations, we find $4Y + 2e_2(X_2, X_3, X_4, X_5) = 0$, and thus $(X_2 + X_3 + X_4 + X_5)^2 = 0$ as claimed.

On the other hand, the relations of degree 2 in $\mathcal{A}^+$ are generated by

$$Y + X_1(X_i + X_j) + X_iX_j = 0 \quad \text{for all } 2 \le i < j \le 6,$$

and by $X_i^2 = Y$. Assume that $x = \sum_{i=1}^6 a_iX_i$ is a 2-nilpotent homogeneous element of degree 1. Thus

$$\begin{aligned} 0 &= \sum_{i=1}^6 a_i^2 \underbrace{X_i^2}_{=Y} + 2\sum_{i=2}^6 a_1a_iX_1X_i + \sum_{2\le i<j\le 6} a_ia_j \underbrace{X_iX_j}_{=-Y-X_1(X_i+X_j)} \\ &= (\sum_{i=1}^6 a_i^2 - 2\sum_{2\le i<j\le 6} a_ia_j)Y + 2\sum_{i=2}^6 (a_1a_i - \sum_{\substack{j=2\\ j\ne i}}^6 a_ia_j)X_1X_i, \end{aligned}$$

and hence we find the two conditions

$$\sum_{i=1}^6 a_i^2 = 2\sum_{2\le i<j\le 6} a_ia_j, \qquad\qquad a_i(a_1 + a_i - \sum_{j=2}^6 a_j) = 0$$

for all $i = 2, \dots, 6$. Let $I \subseteq \{2, \dots, 6\}$ be the set of indices $i$ for which $a_i \ne 0$. If $I$ is empty, the first condition implies $a_1 = 0$; thus $x = 0$ as claimed. Otherwise, for $i \in I$, we have $a_i = \sum_{j=2}^6 a_j - a_1 =: c \ne 0$. Denoting $k = |I|$, we thus find $a_1 = (k-1)c$, and the first condition yields $(k-1)^2c^2 + kc^2 = k(k-1)c^2$, a contradiction.

We now turn to the general case $n \ge 4$. Using the descriptions in Corollary 15 we see that the generators of $\mathfrak{a}^\pm$ have degree at least $n-1$. We assume there is an isomorphism $\varphi : \mathcal{A}^+ \to \mathcal{A}^-$ of graded algebras. It is induced by an automorphism of the polynomial algebra $\mathbb{Q}[X_1, \dots, X_{2n}, Y]$. As $n-1 \ge 3$, this isomorphism must descend to an automorphism $\varphi$ of the algebra $\mathcal{A}$. We read off the classification in Proposition 16 that $\varphi$ must leave the ideal $(Y)$ invariant. The isomorphism $\varphi : \mathcal{A}^+ \to \mathcal{A}^-$ would hence yield an isomorphism

$$\varphi : \mathcal{A}^+/(Y) \to \mathcal{A}^-/(Y).$$

Abbreviate $\mathcal{B}^\pm = \mathcal{A}^\pm/(Y)$. We show that the rings $\mathcal{B}^+$ and $\mathcal{B}^-$ can't be isomorphic. Both $\mathcal{B}^+$ and $\mathcal{B}^-$ are quotients of $\mathbb{Q}[X_1, \dots, X_{2n}]/(X_i^2) = \mathcal{B}$. The only relations of degree $n-1$ that define $\mathcal{B}^+$ inside $\mathcal{B}$ are

$$e_{n-1}(X_1, X_{i_2}, \dots, X_{i_n}) = 0$$

for all $2 \le i_2 < \ldots < i_n \le 2n$. This shows that a basis of the $(n-1)^{\text{st}}$ homogeneous component of $\mathcal{B}^+$ is given by the monomials $X_J = \prod_{j\in J} X_j$ with $J$ ranging over all subsets of $\{1, \dots, 2n\}$ with $1 \in J$ and $|J| = n-1$. A monomial $X_J$ with $J \subseteq \{2, \dots, 2n\}$ and $|J| = n-1$ can be written in terms of these monomials as

$$X_J = -X_1 \sum_{j\in J} X_{J-\{j\}}.$$

On the other hand the degree $n-1$ part of $\mathcal{B}^-$ is described inside $\mathcal{B}$ by the relations

$$e_{n-1}(X_{i_1}, \dots, X_{i_n}) = 0$$

with $2 \leq j_1 < \ldots < j_n \leq 2n$. This is a system of $\binom{2n-1}{n}$ linearly independent equations in $\binom{2n-1}{n-1}$ variables. Therefore the monomials $X_J$ vanish in $\mathcal{B}^-$ when $J \subseteq \{2, \ldots, 2n\}$ is a subset of $n-1$ elements.

We consider the Zariski-closed subsets $Z^\pm \subset \mathcal{B}_1^\pm \cong \mathbb{A}^{2n}$ of $(n-1)$-nilpotent elements, i.e., $Z^\pm = \{a \in \mathcal{B}_1^\pm \mid a^{n-1} = 0\}$. Write $a = \sum_{1=1}^{2n} a_i X_i$ as a linear combination of the basis elements. We see that $a^{n-1}$ is

$$\sum_{p_1+\ldots+p_{2n}=n-1} \binom{n-1}{p_1 \ldots p_{2n}} a_1^{p_1} \ldots a_{2n}^{p_{2n}} X_1^{p_1} \ldots X_{2n}^{p_{2n}} = \sum_{\substack{J \subseteq \{1,\ldots,2n\} \\ |J|=n-1}} (n-1)!\, a_J X_J$$

because all squares vanish in $\mathcal{B}$. In the ring $\mathcal{B}^-$ the above expression simplifies to

$$(n-1)! \sum_{\substack{K \subseteq \{2,\ldots,2n\} \\ |K|=n-2}} a_1 a_K X_1 X_K,$$

from which we see that $Z^-$ is cut out by equations $a_1 a_{k_2} \ldots a_{k_{n-2}}$. The closed subset $\{a_1 = 0\}$ is an irreducible component of $Z^-$ of dimension $2n-1$. When working in $\mathcal{B}^+$ we obtain for $a \in Z^+$ the equation

$$0 = (n-1)! \sum_{\substack{K \subseteq \{2,\ldots,2n\} \\ |K|=n-2}} a_K \Bigg( a_1 - \sum_{j \in \{2,\ldots,2n\}-K} a_j \Bigg) X_1 X_K.$$

Let $U \subseteq \mathcal{B}_1^+$ be the open subset defined by $a_L \neq 0$ for all $L \subset \{2, \ldots, 2n\}$ with $|L| = 2n-2$. The complement of $U$ is a union of hyperplanes of codimension 2. The intersection $Z^+ \cap U$ is defined by the equations

$$a_1 = \sum_{\nu=1}^{n+1} a_{j_\nu}$$

where $2 \leq j_1 < \ldots < j_{n+1} \leq 2n$. These are $\binom{2n}{n+1}$ linear equations. The codimension of $Z^+ \cap U$ inside $U$ is hence at least 2. The choice of $U$ then assures that there can be no irreducible component of $Z^+$ which is of codimension 1 inside $\mathcal{B}_1^+$.

We have shown that $Z^+$ and $Z^-$ can't be isomorphic as varieties, which shows that the rings $\mathcal{B}^+$ and $\mathcal{B}^-$ are non-isomorphic. This contradicts our assumption that there is an isomorphism $\mathcal{A}^+ \to \mathcal{A}^-$. The theorem is proved. □

Combining this result with the algebraicity of cohomology (see Remark 4), we conclude:

**Corollary 18.** *The rational cohomology rings of the small desingularizations $M^{\theta^+}$ and $M^{\theta^-}$ of $M^{\theta^0-\mathrm{sst}}$ are not isomorphic if $n \geq 3$.*

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## References


[1] Dan Edidin and William Graham, *Equivariant intersection theory*, Invent. Math. **131** (1998), no. 3, 595–634. MR1614555

[2] H. Franzen, *Chow rings of fine quiver moduli are tautologically presented*, Math. Z. **279** (2015), no. 3-4, 1197–1223. MR3318266

[3] H. Franzen and M. Reineke, *Semi-stable Chow-Hall algebras of quivers and quantized Donaldson-Thomas invariants*, Preprint. arXiv:1512.03748, 2015.

[4] M. Goresky and R. MacPherson, *Problems and bibliography on intersection homology*, Intersection cohomology (Bern, 1983), Progr. Math., vol. 50, Birkhäuser Boston, Boston, MA, 1984, pp. 221–233. MR788180

[5] J.-C. Hausmann and A. Knutson, *The cohomology ring of polygon spaces* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **48** (1998), no. 1, 281–321. MR1614965

[6] Benjamin Howard, John Millson, Andrew Snowden, and Ravi Vakil, *The equations for the moduli space of $n$ points on the line*, Duke Math. J. **146** (2009), no. 2, 175–226. MR2477759

[7] A. D. King, *Moduli of representations of finite-dimensional algebras*, Quart. J. Math. Oxford Ser. (2) **45** (1994), no. 180, 515–530. MR1315461

[8] Alastair D. King and Charles H. Walter, *On Chow rings of fine moduli spaces of modules*, J. Reine Angew. Math. **461** (1995), 179–187. MR1324213

[9] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, 3rd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994. MR1304906

[10] Markus Reineke, *Quiver moduli and small desingularizations of some GIT quotients*, Representation theory—current trends and perspectives, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2017, pp. 613–635. MR3644807



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