PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 7, July 2018, Pages 2853–2865
http://dx.doi.org/10.1090/proc/13047
Article electronically published on March 19, 2018

THE DOUADY-EARLE EXTENSIONS ARE NOT ALWAYS HARMONIC

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(Communicated by Michael Wolf)

ABSTRACT. In this paper we give examples to show that the Douady-Earle extensions are not always harmonic. Furthermore, we discuss the criterion for the Douady-Earle extensions to be harmonic.

1. INTRODUCTIONS

It's an interesting and important problem of finding nice extensions to $\mathbb{H}^n$ of quasiconformal (or quasisymmetric, if $n = 2$) self-homeomorphisms of $\mathbb{S}^{n-1}$. Quasiconformal extensions were first constructed by Beurling and Ahlfors [3] in dimension $n = 2$; higher dimensional extensions were given by Tukia and Välsälä [22], and Tukia [21] produced a version that is compatible with the action of a group of Möbius transformations. Douady and Earle [4] constructed conformally natural extensions in all dimensions.

Since the development of the basic results by Eells-Sampson [7], Hartman [9] (see also [2]), Schoen-Yau [17] and Sampson [15], harmonic maps are used extensively in the study of Teichmüller space. See for example [5], [20] and [24]. Using the fact that the Hopf differential of a harmonic map between two Riemann surfaces is holomorphic, Wolf [24] proved that the Teichmüller space T_g of a compact surface of genus $g > 1$ is homeomorphic to the space of holomorphic quadratic differentials of a fixed compact Riemann surface of the same genus. Many important properties of T_g can be studied by this parametrization; see for example [24], [10].

On the other hand, Wan [23] extended the above results to a more general setting. Actually Wan [23] constructed a map from the space of bounded (with respect to the Poincaré metric) holomorphic quadratic differentials on the unit disk to the universal Teichmüller space. He showed that the map is continuous and open (see [23] and [19] for details). More generally, Hardt and Wolf [8] showed that the set of quasiconformal (quasisymmetric, if $n = 2$) maps $f : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ which admit a quasiconformal harmonic extension is open in the set of quasiconformal (quasisymmetric, resp.) self-maps of $\mathbb{S}^{n-1}$. In [8] the authors stated that H. L. Royden raised the problem of finding a harmonic extension, which might then also enjoy compatibility with Möbius transformations.

Schoen [16] gave a conjecture as follows: *For every quasisymmetric homeomorphism between the unit circle, there exists a quasiconformal harmonic extension of*

Received by the editors June 27, 2015.
2010 *Mathematics Subject Classification.* Primary 32G15, 30F30, 30F60.
Key words and phrases. Douady-Earle extension, harmonic map, quasiconformal map, quasisymmetric map.
The work was partially supported by NSFC No. 11271378, No. 11771456.

it between the unit disks. Li and Tam ([11, 12]) first constructed harmonic quasiconformal extensions under some additional assumptions of smoothness of the boundary maps and a lower bound on its energy density (see also Akutagawa [1]). Non-uniqueness properties of such extensions were also identified by Li-Tam ([12], [13]). They showed that the group compatibility properties would require some care. Similar non-uniqueness phenomena for some infinite energy harmonic maps of hyperbolic surfaces of finite volume were also found by Wolf [25].

McMullen [14, Appendix B] showed that the first variations of the Douady-Earle extensions are harmonic. It is natural to ask if the Douady-Earle extensions of quasisymmetric maps are always harmonic. An affirmative answer would imply Schoen's conjecture. The authors learnt from a number of people who publicly commented that the Douady-Earle extension is "known" to be harmonic, giving this relationship folklore status.

In this paper, we construct examples to show that

Theorem 1.1. *There exists a quasisymmetric map $f : \mathbb{S}^1 \to \mathbb{S}^1$ whose Douady-Earle extension is not harmonic.*

We note that recently Markovic claimed that he proved Schoen's conjecture.

2. Preliminaries

Let $\mathbb{D} := \{z \in \mathbb{C}; |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$, endowed with the Poincaré metric

$$\rho(z)|dz|^2 = \frac{4|dz|^2}{(1-|z|^2)^2}.$$

Denote $\mathbb{S}^1 = \partial\mathbb{D}$ and $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{S}^1$.

A C^2 map $\omega : \mathbb{D} \to \mathbb{D}$ is called *harmonic* if it satisfies the Euler-Lagrange equation

$$\tau(\omega) = \omega_{z\overline{z}} + (\log\rho)_\omega\ \omega_z\omega_{\overline{z}} = 0. \tag{1}$$

Remark 2.1. The Euler-Lagrange equation (1) is conformally invariant on the domain and isometrically invariant on the range. So if we compose a harmonic map with Möbius transformations, the resulting map will remain harmonic.

Let $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation-preserving circle homeomorphism. We can lift ϕ to a self-homeomorphism of $\mathbb{R}$, denoted by $\tilde{\phi}$, such that $0 \le \tilde{\phi}(0) < 1$ and $\tilde{\phi}(x+1) = \tilde{\phi}(x) + 1$. We say that ϕ is *quasisymmetric* if there is a constant $K \ge 1$ such that

$$K^{-1} \le \frac{\tilde{\phi}(x+t) - \tilde{\phi}(x)}{\tilde{\phi}(x) - \tilde{\phi}(x-t)} \le K,$$

for all $x \in \mathbb{R}$ and $t > 0$.

Any quasisymmetric map admits a quasiconformal extension to the unit disk [3]. By [8], [19], a quasisymmetric map $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ with distortion constant K sufficiently close to 1 admits quasiconformal harmonic extension to the unit disk.

Suppose that $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ is a quasisymmetric homeomorphism that admits a quasiconformal harmonic extension $H(\phi) : \mathbb{D} \to \mathbb{D}$, which is unique by [11], [12]. By Remark 2.1, for any $g, h \in \mathrm{Aut}(\mathbb{D})$, $H(g \circ \phi \circ h) = g \circ H(\phi) \circ h$; that is, $H(\phi)$ is *conformally natural.*

Consider a homeomorphism $\phi : \mathbb{S}^1 \to \mathbb{S}^1$. Douady and Earle [4] defined an extension $E(\phi) : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ of ϕ by mapping each $z \in \mathbb{D}$ to the unique solution $\omega \in \mathbb{D}$ such that

$$F(z,\omega) = \frac{1}{2\pi}\int_{\mathbb{S}^1} \frac{\phi(\xi)-\omega}{1-\overline{\omega}\phi(\xi)} \frac{|d\xi|}{|z-\xi|^2} = 0. \tag{2}$$

By [4], F is real-analytic in $\mathbb{D}\times\mathbb{D}$, and the map $E(\phi) : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is a homeomorphism whose restriction to $\mathbb{D}$ is a real-analytic diffeomorphism. If $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ is quasisymmetric, then $E(\phi)(z) = \omega(z) : \mathbb{D} \to \mathbb{D}$ is quasiconformal. The Douady-Earle extension is conformally natural, i.e.

$$E(g \circ \phi \circ h) = g \circ E(\phi) \circ h$$

for any $g, h \in \mathrm{Aut}(\mathbb{D})$.

Let $V(\mathbb{S}^1)$ and $V(\mathbb{D})$ be the space of continuous vector fields on $\mathbb{S}^1$ and $\mathbb{D}$, respectively. If $f \in V(\mathbb{S}^1)$ and t is close to 0, there is a one-parameter group of homeomorphisms $\phi_t : \mathbb{S}^1 \to \mathbb{S}^1$ such that

$$\phi_t(u) = u + tf(u) + o(t), \quad u \in S^1,$$

uniformly in u. For t close to 0, the Douady-Earle extension $E(\phi_t)$ of ϕ_t satisfies

$$E(\phi_t)(z) = z + tL(f)(z) + o(t),\ z \in \mathbb{D},\ L(f)(z) \in \mathbb{R}^2.$$

This induces a linear map $L : V(\mathbb{S}^1) \to V(\mathbb{D}), f \mapsto L(f)$, which is conformally natural. Earle [6] and McMullen [14] proved that any conformally natural linear map from $V(\mathbb{S}^1)$ to $V(\mathbb{D})$ is unique up to a multiplicity of a constant. Thus the first variations of the Douady-Earle extensions are harmonic (see [14, page 206]). Thus it seems reasonable to guess that the Douady-Earle extensions are harmonic.

3. Proof of Theorem 1.1

In this section we give a family of quasisymmetric maps between $\mathbb{S}^1$ whose Douady-Earle extensions are not harmonic.

Given a homeomorphism $\phi : \mathbb{S}^1 \to \mathbb{S}^1$, we denote its Douady-Earle extension by $\omega(z) : \mathbb{D} \to \mathbb{D}$. Then $\omega = \omega(z)$ and $z \in \mathbb{D}$ satisfy the equation (2).

We rewrite (2) as

$$F(z, \overline{z}, \omega, \overline{\omega}) = 0.$$

Differentiating it by z and $\overline{z}$, respectively, we have

$$\begin{cases} F_z + F_\omega \omega_z + F_{\overline{\omega}}\overline{\omega}_z = 0, \\ F_{\overline{z}} + F_\omega \omega_{\overline{z}} + F_{\overline{\omega}}\overline{\omega}_{\overline{z}} = 0. \end{cases} \tag{3}$$

From (3), we have

$$\begin{cases} \omega_z = \frac{\overline{F}_z F_{\overline{\omega}} - F_z \overline{F}_{\overline{\omega}}}{F_\omega \overline{F}_{\overline{\omega}} - \overline{F}_\omega F_{\overline{\omega}}}, \\ \\ \overline{\omega}_z = \frac{F_z \overline{F}_\omega - \overline{F}_z F_\omega}{F_\omega \overline{F}_{\overline{\omega}} - \overline{F}_\omega F_{\overline{\omega}}}. \end{cases} \tag{4}$$

We denote by

$$A(z,\omega) := F_z(z,\omega) = -\frac{1}{2\pi}\int_{\mathbb{S}^1} \frac{\phi(\xi)-\omega}{1-\overline{\omega}\phi(\xi)} \frac{|d\xi|}{(z-\xi)^2(\overline{z}-\overline{\xi})},$$

$$B(z,\omega) := F_{\overline{z}}(z,\omega) = -\frac{1}{2\pi}\int_{\mathbb{S}^1} \frac{\phi(\xi)-\omega}{1-\overline{\omega}\phi(\xi)} \frac{|d\xi|}{(z-\xi)(\overline{z}-\overline{\xi})^2},$$

$$C(z,\omega) := F_\omega(z,\omega) = -\frac{1}{2\pi}\int_{\mathbb{S}^1}\frac{1}{1-\overline{\omega}\phi(\xi)}\frac{|d\xi|}{|z-\xi|^2},$$

$$D(z,\omega) := F_{\overline{\omega}}(z,\omega) = -\frac{1}{2\pi}\int_{\mathbb{S}^1}\frac{\phi(\xi)(\phi(\xi)-\omega)}{(1-\overline{\omega}\phi(\xi))^2}\frac{|d\xi|}{|z-\xi|^2}.$$

Let

$$O(z,\omega) := C(z,\omega)\overline{C}(z,\omega) - D(z,\omega)\overline{D}(z,\omega),$$

$$P(z,\omega) := \overline{B}(z,\omega)D(z,\omega) - A(z,\omega)\overline{C}(z,\omega),$$

$$Q(z,\omega) := A(z,\omega)\overline{D}(z,\omega) - \overline{B}(z,\omega)C(z,\omega).$$

Then by (4), we have

$$\omega_z = \frac{\overline{B}(z,\omega)D(z,\omega) - A(z,\omega)\overline{C}(z,\omega)}{C(z,\omega)\overline{C}(z,\omega) - D(z,\omega)\overline{D}(z,\omega)} = \frac{P(z,\omega)}{O(z,\omega)}, \tag{5}$$

$$\overline{\omega}_z = \frac{A(z,\omega)\overline{D}(z,\omega) - \overline{B}(z,\omega)C(z,\omega)}{C(z,\omega)\overline{C}(z,\omega) - D(z,\omega)\overline{D}(z,\omega)} = \frac{Q(z,\omega)}{O(z,\omega)}. \tag{6}$$

It follows from (5) and (6) that

$$\omega_{z\overline{z}} = \frac{O(z,\omega)P_{\overline{z}}(z,\omega) - O_{\overline{z}}(z,\omega)P(z,\omega)}{O^2(z,\omega)}, \tag{7}$$

where

$$\begin{aligned}
O_{\overline{z}}(z,\omega) &= C(z,\omega)\overline{C}_{\overline{z}}(z,\omega) + \overline{C}(z,\omega)C_{\overline{z}}(z,\omega)\\
&\quad -D(z,\omega)\overline{D}_{\overline{z}}(z,\omega) - \overline{D}(z,\omega)D_{\overline{z}}(z,\omega),\\
P_{\overline{z}}(z,\omega) &= \overline{B}_{\overline{z}}(z,\omega)D(z,\omega) + \overline{B}(z,\omega)D_{\overline{z}}(z,\omega)\\
&\quad -A_{\overline{z}}(z,\omega)\overline{C}(z,\omega) - A(z,\omega)\overline{C}_{\overline{z}}(z,\omega).
\end{aligned}$$

The relevant terms are as follows:

$$\begin{aligned}
A_{\overline{z}}(z,\omega) &= \frac{1}{2\pi}\int_{\mathbb{S}^1}\frac{\phi(\xi)-\omega}{1-\overline{\omega}\phi(\xi)}\frac{|d\xi|}{(z-\xi)^2(\overline{z}-\overline{\xi})^2}\\
&\quad +\frac{\omega_{\overline{z}}}{2\pi}\int_{\mathbb{S}^1}\frac{1}{1-\overline{\omega}\phi(\xi)}\frac{|d\xi|}{(z-\xi)^2(\overline{z}-\overline{\xi})}\\
&\quad -\frac{\overline{\omega}_{\overline{z}}}{2\pi}\int_{\mathbb{S}^1}\frac{\phi(\xi)(\phi(\xi)-\omega)}{(1-\overline{\omega}\phi(\xi))^2}\frac{|d\xi|}{(z-\xi)^2(\overline{z}-\overline{\xi})},\\
B_z(z,\omega) &= F_{\overline{z}z}(z,\omega) = F_{z\overline{z}}(z,\omega) = A_{\overline{z}}(z,\omega),\\
C_z(z,\omega) &= \frac{1}{2\pi}\int_{\mathbb{S}^1}\frac{1}{1-\overline{\omega}\phi(\xi)}\frac{|d\xi|}{(z-\xi)^2(\overline{z}-\overline{\xi})}\\
&\quad -\frac{\overline{\omega}_z}{2\pi}\int_{\mathbb{S}^1}\frac{\phi(\xi)}{(1-\overline{\omega}\phi(\xi))^2}\frac{|d\xi|}{(z-\xi)(\overline{z}-\overline{\xi})},\\
C_{\overline{z}}(z,\omega) &= \frac{1}{2\pi}\int_{\mathbb{S}^1}\frac{1}{1-\overline{\omega}\phi(\xi)}\frac{|d\xi|}{(z-\xi)(\overline{z}-\overline{\xi})^2}\\
&\quad -\frac{\overline{\omega}_{\overline{z}}}{2\pi}\int_{\mathbb{S}^1}\frac{\phi(\xi)}{(1-\overline{\omega}\phi(\xi))^2}\frac{|d\xi|}{(z-\xi)(\overline{z}-\overline{\xi})},
\end{aligned}$$

$$\begin{aligned}
D_z(z,\omega) &= -\frac{1}{2\pi}\int_{\mathbb{S}^1}\frac{\phi(\xi)(\phi(\xi)-\omega)}{(1-\overline{\omega}\phi(\xi))^2}\frac{|d\xi|}{(z-\xi)^2(\overline{z}-\overline{\xi})}\\
&\quad -\frac{\omega_z}{2\pi}\int_{\mathbb{S}^1}\frac{\phi(\xi)}{(1-\overline{\omega}\phi(\xi))^2}\frac{|d\xi|}{(z-\xi)(\overline{z}-\overline{\xi})}\\
&\quad +\frac{\overline{\omega}_z}{\pi}\int_{\mathbb{S}^1}\frac{\phi^2(\xi)(\phi(\xi)-\omega)}{(1-\overline{\omega}\phi(\xi))^3}\frac{|d\xi|}{(z-\xi)(\overline{z}-\overline{\xi})},\\
D_{\overline{z}}(z,\omega) &= -\frac{1}{2\pi}\int_{\mathbb{S}^1}\frac{\phi(\xi)(\phi(\xi)-\omega)}{(1-\overline{\omega}\phi(\xi))^2}\frac{|d\xi|}{(z-\xi)(\overline{z}-\overline{\xi})^2}\\
&\quad -\frac{\omega_{\overline{z}}}{2\pi}\int_{\mathbb{S}^1}\frac{\phi(\xi)}{(1-\overline{\omega}\phi(\xi))^2}\frac{|d\xi|}{(z-\xi)(\overline{z}-\overline{\xi})}\\
&\quad +\frac{\overline{\omega}_{\overline{z}}}{\pi}\int_{\mathbb{S}^1}\frac{\phi^2(\xi)(\phi(\xi)-\omega)}{(1-\overline{\omega}\phi(\xi))^3}\frac{|d\xi|}{(z-\xi)(\overline{z}-\overline{\xi})}.
\end{aligned}$$

To make the above computation feasible, we will consider those homeomorphisms $\phi:\mathbb{S}^1\to\mathbb{S}^1$ satisfying $\omega(0)=0$. This is equivalent to

$$\int_{\mathbb{S}^1}\phi(\xi)|d\xi|=0. \tag{8}$$

Let $\mathcal{H}$ be the set of homeomorphisms between the unit circles. The subset of $\mathcal{H}$ consists of homeomorphisms $\phi:\mathbb{S}^1\to\mathbb{S}^1$ such that $\omega(0)=0$ will be denoted by $\mathcal{H}_0$.

To show that a certain Douady-Earle extension ω is not harmonic, it suffices to show that the Euler-Lagrange equation (1) is not true at some point, for example, at the point $z=0$. Since

$$\rho(\omega)|d\omega|^2=\frac{4|d\omega|^2}{(1-|\omega|^2)^2}$$

we have

$$(\log\rho)_\omega=\frac{2\overline{\omega}}{1-|\omega|^2}.$$

Then Euler-Lagrange equation (1) is equivalent to

$$\tau(\omega)=\omega_{z\overline{z}}+\frac{2\overline{\omega}}{1-|\omega|^2}\omega_z\omega_{\overline{z}}=0. \tag{9}$$

For $\phi\in\mathcal{H}_0$, to show that ω doesn't satisfy (9) at $z=0$, it's equivalent to show that $\omega_{z\overline{z}}(0)\neq 0$. From (7), we have

$$\omega_{z\bar{z}}(0)=\frac{O(0,0)P_{\bar{z}}(0,0)-O_{\bar{z}}(0,0)P(0,0)}{O^2(0,0)}. \tag{10}$$

We conclude that

Proposition 3.1 (Criterion of harmonicity). *For $\phi\in\mathcal{H}_0$, the Douady-Earle extension ω of ϕ is not harmonic at the point $z=0$ if and only if*

$$O(0,0)P_{\bar{z}}(0,0)\neq O_{\bar{z}}(0,0)P(0,0).$$

Letting $\phi\in\mathcal{H}_0$, the previous terms can be simplified as follows:

$$\begin{cases} A(0,0)=\frac{1}{2\pi}\int_{\mathbb{S}^1}\overline{\xi}\phi(\xi)|d\xi|, \quad B(0,0)=\frac{1}{2\pi}\int_{\mathbb{S}^1}\xi\phi(\xi)|d\xi|,\\ \\ C(0,0)=-1, \quad D(0,0)=\frac{1}{2\pi}\int_{\mathbb{S}^1}\phi^2(\xi)|d\xi|. \end{cases} \tag{11}$$

$$
\begin{cases}
A_{\overline{z}}(0,0) = \frac{\overline{\omega_{\overline{z}}}}{2\pi} \int_{\mathbb{S}^1} \overline{\xi}\phi^2(\xi)|d\xi|, \\
B_z(0,0) = A_{\overline{z}}(0,0), \quad C_z(0,0) = C_{\overline{z}}(0,0) = 0, \\
D_z(0,0) = \frac{1}{2\pi} \int_{\mathbb{S}^1} \overline{\xi}\phi^2(\xi)|d\xi| + \frac{\overline{\omega_z}}{\pi} \int_{\mathbb{S}^1} \phi^3(\xi)|d\xi|, \\
D_{\overline{z}}(0,0) = \frac{1}{2\pi} \int_{\mathbb{S}^1} \xi\phi^2(\xi)|d\xi| + \frac{\overline{\omega_{\overline{z}}}}{\pi} \int_{\mathbb{S}^1} \phi^3(\xi)|d\xi|.
\end{cases} \tag{12}
$$

Now we construct a family of homeomorphisms $f_t : \mathbb{S}^1 \to \mathbb{S}^1$. Let

$$
\begin{cases}
a_1(t) = 1 + t, \\
a_2(t) = 1 - (1+\sqrt{3})t, \\
a_3(t) = 1 + \sqrt{3}t
\end{cases}
$$

where $-\frac{\sqrt{3}}{3} < t < \frac{\sqrt{3}-1}{2}$. For $z = e^{i\theta} \in \mathbb{S}^1$, we define

$$
f_t(e^{i\theta}) := \begin{cases}
e^{i\frac{1}{a_1(t)}\theta}, \text{ for } 0 \le \theta \le a_1(t)\frac{\pi}{6}, \\
e^{i(1-\frac{a_1(t)}{a_2(t)})\frac{\pi}{6}} e^{i\frac{1}{a_2(t)}\theta}, \text{ for } a_1(t)\frac{\pi}{6} \le \theta \le (a_1(t)+a_2(t))\frac{\pi}{6}, \\
e^{i(1-\frac{1}{a_3(t)})\frac{\pi}{2}} e^{i\frac{1}{a_3(t)}\theta}, \text{ for } \frac{\pi}{2} - a_3(t)\frac{\pi}{6} \le \theta \le \frac{\pi}{2} + a_3(t)\frac{\pi}{6}, \\
e^{i(4-\frac{3\pi}{a_2(t)}-\frac{a_3(t)}{a_2(t)})\frac{\pi}{6}} e^{i\frac{1}{a_2(t)}\theta}, \text{ for } \frac{\pi}{2} + a_3(t)\frac{\pi}{6} \le \theta \le \frac{\pi}{2} + (a_2(t)+a_3(t))\frac{\pi}{6}, \\
e^{i(1-\frac{1}{a_1(t)})\pi} e^{i\frac{1}{a_1(t)}\theta}, \text{ for } \pi - a_1(t)\frac{\pi}{6} \le \theta \le \pi, \\
e^{i\theta}, \text{ for } \pi \le \theta \le 2\pi.
\end{cases}
$$

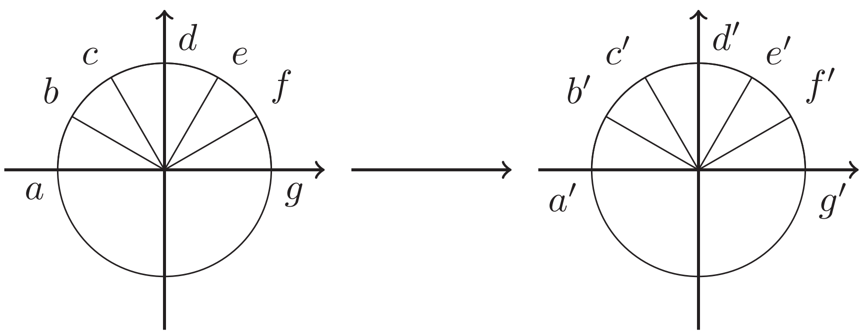

FIGURE 1. The family of homeomorphism $f_t : \mathbb{S}^1 \to \mathbb{S}^1$.

As shown in Figure 1, f_t is the identity on the lower-half circle. On the upper-half circle, let $a,\ b,\ c,\ d,\ e,\ f,\ g$ be the points on the source with $\angle aob = \angle fog = a_1(t)\frac{\pi}{6}$, $\angle boc = \angle eof = a_2(t)\frac{\pi}{6}$, $\angle cod = \angle doe = a_3(t)\frac{\pi}{6}$. And let $a',\ b',\ c',\ d',\ e',\ f',\ g'$ be the points on the upper-half circle which divide the half circle into six parts with equal arc length. Then $f_t(a) = a'$, $f_t(b) = b'$, $f_t(c) = c'$, $f_t(d) = d'$, $f_t(e) = e'$, $f_t(f) = f'$, $f_t(g) = g'$. Each f_t is piecewise linear with respect to the angle.

Define $g_t : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ by

$$
g_t(z) = g_t(re^{i\theta}) := rf_t(e^{i\theta}), \text{ for } z = re^{i\theta} \in \overline{\mathbb{D}}
$$

with $0 \le r \le 1$ and $0 \le \theta < 2\pi$. Then g_t is a homeomorphic extension of f_t. Direct computation shows that the maximal quasiconformal dilatation K_t of g_t satisfies

$$K_t \le \max_{1\le j\le 3} |\frac{1+a_j(t)}{1-a_j(t)}|.$$

So $g_t(z)$ is a quasiconformal homeomorphism between $\overline{\mathbb{D}}$ for $-\frac{\sqrt{3}}{3} < t < \frac{\sqrt{3}-1}{2}$. This implies that $f_t : \mathbb{S}^1 \to \mathbb{S}^1$ is a quasisymmetric map and has quasiconformal extension.

Lemma 3.2. *The quasisymmetric homeomorphism $f_t : \mathbb{S}^1 \to \mathbb{S}^1$ satisfies* (8).

Proof. Note that

$$\begin{aligned}
&\int_{\mathbb{S}^1} f_t(\xi)|d\xi| = \int_0^{2\pi} f_t(e^{i\theta})d\theta \\
= \ & \int_0^{a_1(t)\frac{\pi}{6}} e^{i\frac{1}{a_1(t)}\theta} d\theta + \int_{a_1(t)\frac{\pi}{6}}^{(a_1(t)+a_2(t))\frac{\pi}{6}} e^{i(1-\frac{a_1(t)}{a_2(t)})\frac{\pi}{6}} e^{i\frac{1}{a_2(t)}\theta} d\theta \\
&+ \int_{\frac{\pi}{2}-a_3(t)\frac{\pi}{6}}^{\frac{\pi}{2}+a_3(t)\frac{\pi}{6}} e^{i(1-\frac{1}{a_3(t)})\frac{\pi}{2}} e^{i\frac{1}{a_3(t)}\theta} d\theta \\
&+ \int_{\frac{\pi}{2}+a_3(t)\frac{\pi}{6}}^{\frac{\pi}{2}+(a_2(t)+a_3(t))\frac{\pi}{6}} e^{i(4-\frac{3\pi}{a_2(t)}-\frac{a_3(t)}{a_2(t)})\frac{\pi}{6}} e^{i\frac{1}{a_2(t)}\theta} d\theta \\
&+ \int_{\pi-a_1(t)\frac{\pi}{6}}^{\pi} e^{i(1-\frac{1}{a_1(t)})\pi} e^{i\frac{1}{a_1(t)}\theta} d\theta + \int_\pi^{2\pi} e^{i\theta} d\theta \\
= \ & \int_0^{\frac{\pi}{6}} a_1(t)e^{i\theta}d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} a_2(t)e^{i\theta}d\theta + \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} a_3(t)e^{i\theta}d\theta \\
&+ \int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} a_2(t)e^{i\theta}d\theta + \int_{\frac{5\pi}{6}}^{\pi} a_1(t)e^{i\theta}d\theta + \int_\pi^{2\pi} e^{i\theta}d\theta \\
= \ & -ia_1(t)(\frac{\sqrt{3}}{2}-1) - ia_2(t)(\frac{1}{2}-\frac{\sqrt{3}}{2}) - ia_3(t)(-\frac{1}{2}) - i \\
= \ & 0.
\end{aligned}$$

□

We prove Theorem 1.1 by showing that

Theorem 3.3. *For $|t| \neq 0$ sufficiently small, the Douady-Earle extension of f_t is not harmonic at $z = 0$.*

Proof. Since

$$\int_{\mathbb{S}^1} f_t(\xi)|d\xi| = 0,$$

we have $\omega^t(0) = 0$, where $\omega^t : \mathbb{D} \to \mathbb{D}$ denotes the Douady-Earle extension of f_t. In the following, A^t denotes $A(z, \omega^t)$ and so on, as we defined in the discussion before

Proposition 3.1. By 11 and 12, for t close to 0, we have

$$\begin{aligned}
A^t(0,0) &= \frac{1}{2\pi}\int_{\mathbb{S}^1}\overline{\xi}f_t(\xi)|d\xi| \\
&= \frac{1}{2\pi}[\pi+(\frac{2}{\frac{1}{a_1(t)}-1}-\frac{2}{\frac{1}{a_2(t)}-1})\sin(1-a_1(t))\frac{\pi}{6} \\
&\quad +(\frac{2}{\frac{1}{a_3(t)}-1}-\frac{2}{\frac{1}{a_2(t)}-1})\sin(1-a_3(t))\frac{\pi}{6}] \\
&= 1-(8-\sqrt{3})\frac{\pi^2}{1296}t^2+o(t^2). \\
B^t(0,0) &= \frac{1}{2\pi}\int_{\mathbb{S}^1}\xi f_t(\xi)|d\xi| \\
&= [\frac{1}{2\pi}(\frac{2}{\frac{1}{a_1(t)}+1}-\frac{2}{\frac{1}{a_2(t)}+1})\sin(1+a_1(t))\frac{\pi}{6} \\
&\quad +(\frac{2}{\frac{1}{a_2(t)}+1}-\frac{2}{\frac{1}{a_3(t)}+1})\sin(1+a_3(t))\frac{\pi}{6}] \\
&= \frac{-3+\sqrt{3}}{8\pi}t+(\frac{\sqrt{3}}{8\pi}-\frac{1}{12})t^2+o(t^2). \\
D^t(0,0) &= \frac{1}{2\pi}\int_{S^1}f_t^2(\xi)|d\xi| = \frac{\sqrt{3}}{4\pi}(a_1(t)-a_3(t)) \\
&= \frac{-3+\sqrt{3}}{4\pi}t.
\end{aligned}$$

We denote by

$$\begin{aligned}
I_1(t) &= \int_{\mathbb{S}^1}\overline{\xi}f_t^2(\xi)|d\xi| \\
&= i2\{\frac{1}{\frac{2}{a_1(t)}-1}[1-\sin(1+a_1(t))\frac{\pi}{6}] \\
&\quad +\frac{1}{\frac{2}{a_3(t)}-1}\sin(2-a_3(t))\frac{\pi}{6} \\
&\quad +\frac{1}{\frac{2}{a_2(t)}-1}[\sin(1+a_1(t))\frac{\pi}{6}-\sin(2-a_3(t))\frac{\pi}{6}]-1\} \\
&= (14+2\sqrt{3}-\frac{5}{3}\pi-\frac{7}{3}\sqrt{3}\pi)it^2+o(t^2). \\
I_2(t) &= \int_{\mathbb{S}^1}\xi f_t^2(\xi)|d\xi| \\
&= i2\{\frac{1}{\frac{2}{a_1(t)}+1}[1-\sin(1-a_1(t))\frac{\pi}{6}]-\frac{1}{\frac{2}{a_3(t)}+1}\sin(2+a_3(t))\frac{\pi}{6} \\
&\quad +\frac{1}{\frac{2}{a_2(t)}+1}[\sin(1-a_1(t))\frac{\pi}{6}+\sin(2+a_3(t))\frac{\pi}{6}]-\frac{1}{3}\} \\
&= -\frac{8\sqrt{3}}{9}it+[\frac{-8(1+\sqrt{3})+2(2+\sqrt{3})\pi}{27}]it^2+o(t^2).
\end{aligned}$$

$$\begin{aligned} I_3(t) &= \int_{\mathbb{S}^1} f_t^3(\xi)|d\xi| \\ &= \frac{2}{3}(a_1(t)+a_2(t)-a_3(t)-1)i \\ &= -\frac{4\sqrt{3}}{3}it. \end{aligned}$$

We are going to show that $\omega^t_{z\overline{z}}(0) \neq 0$. By definition,

$$O^t(0,0) = |C^t(0,0)|^2 - |D^t(0,0)|^2 = 1 - \frac{(3-\sqrt{3})^2}{16\pi^2}t^2.$$

$$\begin{aligned} P^t(0,0) &= \overline{B}^t(0,0)D^t(0,0) - A^t(0,0)\overline{C}^t(0,0) \\ &= 1 + [\frac{(3-\sqrt{3})^2}{32\pi^2} - (8-\sqrt{3})\frac{\pi^2}{1296}]t^2 + o(t^2). \end{aligned}$$

$$\begin{aligned} Q^t(0,0) &= A^t(0,0)\overline{D}^t(0,0) - \bar{B}^t(0,0)C^t(0,0) \\ &= \frac{3(-3+\sqrt{3})}{8\pi}t + (\frac{\sqrt{3}}{8\pi} - \frac{1}{12})t^2 + o(t^2). \end{aligned}$$

From (5) and (6), we have

$$\begin{aligned} \overline{\omega}^t_{\overline{z}}(0,0) &= \frac{\overline{P}^t(0,0)}{O^t(0,0)} \\ &= 1 + [\frac{3(3-\sqrt{3})^2}{32\pi^2} - (8-\sqrt{3})\frac{\pi^2}{1296}]t^2 + o(t^2). \\ \overline{\omega}^t_z(0,0) &= \frac{Q^t(0,0)}{O^t(0,0)} \\ &= \frac{3(-3+\sqrt{3})}{8\pi}t + (\frac{\sqrt{3}}{8\pi} - \frac{1}{12})t^2 + o(t^2). \end{aligned}$$

Applying (12), we have

$$\begin{aligned} D^t_{\overline{z}}(0,0) &= \frac{1}{2\pi}I_2(t) + \frac{1}{\pi}\overline{\omega}^t_{\overline{z}}(0,0)I_3(t) \\ &= -\frac{16\sqrt{3}}{9\pi}it + \frac{1}{27\pi}[-4(1+\sqrt{3}) + (2+\sqrt{3})\pi]it^2 + o(t^2). \end{aligned}$$

$$\begin{aligned} \overline{D}^t_{\overline{z}}(0,0) &= \frac{1}{2\pi}\overline{I_1(t)} + \frac{1}{\pi}\omega^t_{\overline{z}}(0,0)\overline{I_3(t)} \\ &= [\frac{3(1-\sqrt{3})}{2\pi^2} - \frac{1}{2\pi}(14+2\sqrt{3} - \frac{5}{3}\pi - \frac{7\sqrt{3}}{3}\pi)]it^2 + o(t^2). \end{aligned}$$

$$\begin{aligned} \overline{B}^t_{\overline{z}}(0,0) &= \bar{A}^t_z(0,0) = \frac{1}{2\pi}\omega^t_z(0,0)\overline{I_1(t)} \\ &= -\frac{1}{2\pi}(14+2\sqrt{3} - \frac{5}{3}\pi - \frac{7}{3}\sqrt{3}\pi)it^2 + o(t^2). \end{aligned}$$

By conjugation, we also have

$$\begin{aligned} A^t_{\bar{z}}(0,0) &= B^t_z(0,0) \\ &= \frac{1}{2\pi}(14+2\sqrt{3}-\frac{5}{3}\pi-\frac{7}{3}\sqrt{3}\pi)it^2+o(t^2). \end{aligned}$$

By direct computation, we have

$$\begin{aligned} P^t_{\bar{z}}(0,0) &= \overline{B}^t_{\bar{z}}(0,0)D^t(0,0)+\overline{B}^t(0,0)D^t_{\bar{z}}(0,0)-A^t_{\bar{z}}(0,0)\overline{C}^t(0,0) \\ &= [\frac{1}{2\pi}(14+2\sqrt{3}-\frac{7}{3}\sqrt{3}\pi-\frac{5}{3}\pi)+\frac{2(\sqrt{3}-1)}{3\pi^2}]it^2+o(t^2). \end{aligned}$$

Since $C^t(0,0)=-1$, $C^t_z(0,0)=C^t_{\bar{z}}(0,0)=0$, it is easy to check that

$$\begin{aligned} O^t_{\bar{z}}(0,0) &= -D^t(0,0)\overline{D}^t_{\bar{z}}(0,0)-\overline{D}^t(0,0)D^t_{\bar{z}}(0,0) \\ &= \frac{4(1-\sqrt{3})}{3\pi^2}it^2+o(t^2). \end{aligned}$$

As a result, we have

$$\begin{aligned} \omega^t_{z\bar{z}}(0) &= \frac{O^t(0,0)P^t_{\bar{z}}(0,0)-O^t_{\bar{z}}(0,0)P^t(0,0)}{(O^t)^2(0,0)} \\ &= [\frac{1}{2\pi}(14+2\sqrt{3}-\frac{7}{3}\sqrt{3}\pi-\frac{5}{3}\pi)+\frac{2(\sqrt{3}-1)}{\pi^2}]it^2+o(t^2) \\ &\approx 0.07378it^2+o(t^2). \end{aligned}$$

Thus we know that when $|t|$ is small enough and $t\neq 0$, $\omega^t_{z\bar{z}}(0)\neq 0$. Thus the Douady-Earle extensions f_t are not harmonic. ☐

Remark 3.4. It is interesting to know whether for any t with $t\neq 0$ and $-\frac{\sqrt{3}}{3}<t<\frac{\sqrt{3}-1}{2}$, the Douady-Earle extension of f_t is harmonic or not. Let $t=-0.5$ and let $\omega(z)$ be the Douady-Earle extension of $f_{-0.5}$. A direct computation deduces that $\omega_{z\bar{z}}(0)\approx 0.01737i$, with error controlled by 10^{-5}. This shows that the Douady-Earle extension of $f_{-0.5}$ is not harmonic.

Our proof of Theorem 3.3 implies the following.

Corollary 3.5. *The Douady-Earle extensions are not always harmonic with respect to the Euclidean metric on* $\mathbb{D}$.

Note that the family of quasisymmetric maps $\{f_t\}$ defined above is piece-wise smooth. We know from [18] that each f_t admits a quasisymmetric harmonic extension to the Poincaré disk.

Endow the set $\mathcal{H}$ of homeomorphisms $\phi:\mathbb{S}^1\to\mathbb{S}^1$ with the compact-open topology. We know that the Douady-Earle extension $E(\phi)$ and the first partial derivatives of $E(\phi)$ depend continuously on $\phi\in\mathcal{H}$ (see [4, pages 30-31]). Denote the subset of $\mathcal{H}_0$ ($\mathcal{H}$) consisting of elements whose Douady-Earle are not harmonic by $\mathcal{H}_0'$ ($\mathcal{H}'$). We have showed that $\mathcal{H}_0'$ is non-empty. Using Proposition 3.1, we have

Corollary 3.6. *The subsets* $\mathcal{H}_0'$ *and* $\mathcal{H}'$ *are non-empty open subsets of* $\mathcal{H}_0$ *and* $\mathcal{H}$, *respectively.*

Remark 3.7. By conformally natural property, it's possible to give a criteria for the harmonicity of any homeomorphism in $\mathcal{H}$ at any point in the unit disk.

It seems reasonable to make the following conjecture:

Conjecture 3.8. *The subset $\mathcal{H}'$ is an open and dense subset of $\mathcal{H}$.*

We conclude this section with the following:

Problem 3.9. Is there a non-trivial homeomorphism between circles whose Douady-Earle extension is harmonic?

4. Criterion of harmonicity by Fourier series

We consider the harmonicity of the Douady-Earle extension of $\phi \in \mathcal{H}_0$ at $z = 0$. For $\phi \in \mathcal{H}_0$, denote its Fourier series by

$$\phi(z) = \sum_{-\infty}^{+\infty} a_n z^n,$$

where

$$a_n = \frac{1}{2\pi} \int_{|z|=1} \phi(z) \bar{z}^n |dz|.$$

By (8), $a_0 = 0$. With the previous notation, we have

$$A(0,0) = a_1,\ B(0,0) = a_{-1},\ C(0,0) = -1,\ D(0,0) = \Sigma_{-\infty}^{+\infty} a_n a_{-n},$$

and

$$O(0,0) = 1 - D(0,0)\bar{D}(0,0),\ P(0,0) = \bar{a}_{-1}D(0,0) + a_1,$$
$$Q(0,0) = a_1\bar{D}(0,0) + \bar{a}_{-1}.$$

Set

$$E_1 = \omega_z(0,0) = \frac{P(0,0)}{O(0,0)} = \frac{\bar{a}_{-1}D(0,0) + a_1}{1 - D(0,0)\bar{D}(0,0)},$$
$$E_2 = \bar{\omega}_z(0,0) = \frac{Q(0,0)}{O(0,0)} = \frac{a_1\bar{D}(0,0) + \bar{a}_{-1}}{1 - D(0,0)\bar{D}(0,0)},$$
$$D_1 = \frac{1}{2\pi} \int_{\mathbb{S}^1} \bar{\xi}\phi^2(\xi)|d\xi| = \Sigma_{-\infty}^{+\infty} a_n a_{1-n},$$
$$D_2 = \frac{1}{2\pi} \int_{\mathbb{S}^1} \xi\phi^2(\xi)|d\xi| = \Sigma_{-\infty}^{+\infty} a_n a_{-n-1}.$$

Then

$$A_{\bar{z}}(0,0) = B_z(0,0) = D_1\bar{E}_1, \quad C_z(0,0) = C_{\bar{z}}(0,0) = 0.$$

Denote by $M_l = \Sigma_{s+t=l} a_s a_t$. Then

$$D_3 = \frac{1}{2\pi} \int_{\mathbb{S}^1} \phi^3(\xi)|d\xi| = \Sigma_{-\infty}^{+\infty} a_{-l} M_l,$$
$$D_z(0,0) = D_1 + 2D_3E_2, \quad D_{\bar{z}}(0,0) = D_2 + 2D_3\bar{E}_1,$$

and

$$\begin{aligned} O_{\bar{z}}(0,0) &= -D(0,0)\bar{D}_{\bar{z}}(0,0) - \bar{D}(0,0)D_{\bar{z}}(0,0) \\ &= -D(0,0)\bar{D}_1 - \bar{D}(0,0)D_2 - 2D(0,0)\bar{D}_3\bar{E}_2 - 2\bar{D}(0,0)D_3\bar{E}_1, \\ P_{\bar{z}(0,0)} &= \bar{B}_{\bar{z}}(0,0)D(0,0) + \bar{B}(0,0)D_{\bar{z}}(0,0) + A_{\bar{z}}(0,0) \\ &= \bar{a}_{-1} + 2\bar{a}_{-1}D_3E_1 + D_1\bar{E}_1 + D(0,0)\bar{D}_1E_1. \end{aligned}$$

As a result,

$$O(0,0) = 1 - D(0,0)\bar{D}(0,0), P(0,0) = a_1 + \bar{a}_{-1}D(0,0).$$

Summarizing the above discussion, we have the following:

$$\begin{aligned} \phi(z) &= \Sigma_{-\infty}^{\infty} a_n z_n, \quad a_0 = 0. \\ O(0,0) &= 1 - |\Sigma_{-\infty}^{+\infty} a_n a_{-n}|^2. \\ P(0,0) &= a_1 + \bar{a}_{-1} \Sigma_{-\infty}^{+\infty} a_n a_{-n}. \end{aligned}$$

$$\begin{aligned} O_{\bar{z}}(0,0) &= -\Sigma_{-\infty}^{+\infty} a_n a_{-n} \Sigma_{-\infty}^{+\infty} \bar{a}_n \bar{a}_{-n} - \Sigma_{-\infty}^{+\infty} \bar{a}_n \bar{a}_{-n} \Sigma_{-\infty}^{+\infty} a_n a_{-n-1} \\ &\quad -2\Sigma_{-\infty}^{+\infty} a_n a_{-n} \Sigma_{-\infty}^{+\infty} \bar{a}_{-n} \bar{M}_n \frac{\bar{a}_{-1} + a_1 \Sigma_{-\infty}^{+\infty} \bar{a}_n \bar{a}_{-n}}{1 - |\Sigma_{-\infty}^{+\infty} a_n a_{-n}|^2} \\ &\quad -2\Sigma_{-\infty}^{+\infty} \bar{a}_n \bar{a}_{-n} \Sigma_{-\infty}^{+\infty} a_{-n} M_n \frac{\bar{a}_1 + a_{-1} \Sigma_{-\infty}^{+\infty} \bar{a}_n \bar{a}_{-n}}{1 - |\Sigma_{-\infty}^{+\infty} a_n a_{-n}|^2}. \end{aligned}$$

$$\begin{aligned} P_{\bar{z}}(0,0) &= \bar{a}_{-1} + 2\bar{a}_{-1} \Sigma_{-\infty}^{+\infty} a_{-n} M_n \frac{a_1 + \bar{a}_{-1} \Sigma_{-\infty}^{+\infty} a_n a_{-n}}{1 - |\Sigma_{-\infty}^{+\infty} a_n a_{-n}|^2} \\ &\quad + \Sigma_{-\infty}^{+\infty} a_n a_{1-n} \frac{\bar{a}_1 + a_{-1} \Sigma_{-\infty}^{+\infty} \bar{a}_n \bar{a}_{-n}}{1 - |\Sigma_{-\infty}^{+\infty} a_n a_{-n}|^2} \\ &\quad + \Sigma_{-\infty}^{+\infty} a_n a_{-n} \Sigma_{-\infty}^{+\infty} \bar{a}_n \bar{a}_{1-n} \frac{a_1 + \bar{a}_{-1} \Sigma_{-\infty}^{+\infty} a_n a_{-n}}{1 - |\Sigma_{-\infty}^{+\infty} a_n a_{-n}|^2}. \end{aligned}$$

Then from (10), we obtain a sufficient and necessary condition of the Douady-Earle extension of ϕ to be harmonic at $z = 0$, that is,

$$O(0,0) P_{\bar{z}}(0,0) - O_{\bar{z}}(0,0) P(0,0) = 0,$$

where $O(0,0), O_{\bar{z}}(0,0), P(0,0)$ and $P_{\bar{z}}(0,0)$ are as above.

Acknowledgements

The second author would like to thank Michael Wolf for bringing this problem to his attention and for his many suggestions and help. He also would like to thank Clifford J. Earle and Luen-Fai Tam for helpful discussions and suggestions.

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