PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 7, July 2018, Pages 2867–2876
http://dx.doi.org/10.1090/proc/13144
Article electronically published on March 14, 2018

ASYMPTOTIC TEICHMÜLLER SPACE OF A CLOSED SET OF THE RIEMANN SPHERE

YI QI AND YAN WU

(Communicated by Jeremy Tyson)

ABSTRACT. The asymptotic Teichmüller space $AT(E)$ of a closed subset E of the Riemann sphere $\hat{\mathbb{C}}$ with at least 4 points and the natural asymptotic Teichmüller metric are introduced. It is proved that $AT(E)$ is isometrically isomorphic to the product space of the asymptotic Teichmüller spaces of the connected components of $\hat{\mathbb{C}} \setminus E$ and the Banach space of the Beltrami coefficients defined on E. Furthermore, it is proved that there is a complex Banach manifold structure on $AT(E)$.

1. Introduction

Let X be a hyperbolic Riemann surface, that is, X is a Riemann surface with a universal covering surface conformally equivalent to the open unit disk $\mathbb{D}$. Let $L^\infty(X)$ be the Banach space of Beltrami differentials μ on X with essential supremum norm

$$\|\mu\|_\infty = ess \sup_{z\in X} |\mu(z)| < \infty$$

and let $M(X)$ be its open unit ball centered at $\mu \equiv 0$.

For each $\mu \in M(X)$, there is a quasiconformal mapping $f^\mu : X \to f^\mu(X)$ with Beltrami coefficient μ, which is determined uniquely up to postcomposition by a conformal homeomorphism of $f^\mu(X)$. Two elements μ and ν in $M(X)$ are said to be Teichmüller equivalent, denoted by $\mu \sim \nu$, if there is a conformal mapping $\phi : f^\mu(X) \to f^\nu(X)$ such that $(f^\nu)^{-1} \circ \phi \circ f^\mu$ is homotopic to the identity mapping of X modulo the boundary ∂X of X. The Teichmüller space $T(X)$ of X is defined as the space of Teichmüller equivalent classes; that is,

$$T(X) := M(X)/\sim = \{[\mu],\ \ \mu \in M(X)\},$$

where $[\mu]$ is the Teichmüller equivalent class containing μ.

The asymptotic Teichmüller equivalence " $\approx$ " has the same definition as the Teichmüller equivalence with one exception: the conformal mapping ϕ is replaced by an asymptotically conformal mapping. A mapping ϕ is said to be asymptotically conformal if it is quasiconformal and if, for every $\epsilon > 0$, there is a compact subset C of X such that the dilatation of f outside of C is less than $1+\epsilon$. The asymptotic

Received by the editors July 30, 2014 and, in revised form, January 26, 2016.
2010 *Mathematics Subject Classification.* Primary 32G15; Secondary 30C62, 30F60.
Key words and phrases. Teichmüller space, Teichmüller space of a closed set, asymptotic Teichmüller space of a closed set, quasiconformal mapping.
This research was partially supported by the National Natural Science Foundation of China (Grant Nos.11371045, 11701250) and the Fundamental Research Funds for the Central University.
Yan Wu is the corresponding author.

Teichmüller space $AT(X)$ of X is defined as the space of asymptotic Teichmüller equivalent classes; that is,

$$AT(X) := M(X)/\approx = \{[[\mu]],\ \ \mu \in M(X)\},$$

where $[[\mu]]$ is the asymptotic equivalence class containing μ.

The asymptotic Teichmüller space $AT(X)$ is of interest only when X is of infinite analytic type. Otherwise, it consists of one point. So we always assume in the following that X is of infinite analytic type.

Let E be a closed subset of the Riemann sphere $\hat{\mathbb{C}}$ with at least 4 points. Assume without lose of generality that $0, 1, \infty \in E$.

Let $L^\infty(\mathbb{C})$ be the Banach space of Beltrami differentials defined on $\mathbb{C}$ and let $M(\mathbb{C})$ be its open unit ball. For every $\mu \in M(\mathbb{C})$, there is a quasiconformal self-homeomorphism w^μ of $\mathbb{C}$ with Beltrami coefficient μ, which is determined uniquely up to postcomposition by a Möbius transformation. Two Beltrami differentials μ and ν are said to be E-Teichmüller equivalent and denoted by $\mu \overset{E}{\sim} \nu$ if there exists a Möbius transformation ϕ of $\hat{\mathbb{C}}$ such that $(w^\nu)^{-1} \circ \phi \circ w^\mu$ is isotopic to the identity mapping $id_{\hat{\mathbb{C}}}$ of $\hat{\mathbb{C}}$ modulo E.

If w^μ and w^ν are normalized to fix the points 0, 1 and ∞, then $\mu \overset{E}{\sim} \nu$ if and only if $(w^\nu)^{-1} \circ w^\mu$ is isotopy to the identity mapping $id_{\hat{\mathbb{C}}}$ modulo E.

The Teichmüller space $T(E)$ of the closed subset E can be defined as the set of all E-Teichmüller equivalent classes of Beltrami coefficients defined on $\hat{\mathbb{C}}$; that is,

$$T(E) := M(\mathbb{C})/ \overset{E}{\sim} = \{[\mu]_E,\ \ \mu \in M(\mathbb{C})\},$$

where $[\mu]_E$ is the E-Teichmüller equivalent class containing μ.

There are natural Teichmüller metrics on $T(X)$, $AT(X)$ and $T(E)$, respectively. Moreover, there are complex Banach manifold structures on $T(X)$, $AT(X)$ and $T(E)$ respectively, such that the canonical projections

$$\Phi : M(X) \to T(X); \mu \mapsto [\mu]$$

and

$$\Phi_E : M(\mathbb{C}) \to T(E); \mu \mapsto [\mu]_E$$

are holomorphic split submersions and

$$\Psi : M(X) \to AT(X); \mu \mapsto [[\mu]]$$

is holomorphic.

Teichmüller space $T(X)$ was introduced by Teichmüller [16] in the 1940s and was fully studied by Ahlfors and Bers and their "family" since the 1960s [1–3]. The asymptotic Teichmüller space $AT(X)$ was introduced by Gardiner and Sullivan [13] for the unit disk $\mathbb{D}$, and by Earle, Gardiner and Lakic [4, 6, 7] for the arbitrary Riemann surfaces of infinite analytic type (see [12] also). Teichmüller space $T(E)$ of a closed subset E was introduced by Lieb [15]. We refer to [5, 8, 10, 11, 14] for more information and details on these Teichmüller spaces.

The purpose of this paper is to study a new relative of Teichmüller space, the asymptotic Teichmüller space $AT(E)$ of a closed subset E of the Riemann sphere $\hat{\mathbb{C}}$ with at least 4 points. It is a composition of the asymptotic Teichmüller space and the Teichmüller space of a closed subset of $\hat{\mathbb{C}}$.

The paper is organized as follows. In section 2, the notion of the asymptotic Teichmüller space $AT(E)$ and the asymptotic Teichmüller metric on $AT(E)$ are given. Furthermore, it is proved that $AT(E)$ is isometrically isomorphic to the product space of the asymptotic Teichmüller spaces of the connected components of $\hat{\mathbb{C}} \setminus E$ and the Banach space of the Beltrami coefficients defined on E. In section 3, it is proved that there is a natural complex Banach manifold structure on $AT(E)$.

2. Asymptotic Teichmüller space of a closed set

A quasiconformal self-homeomorphism ϕ of $\hat{\mathbb{C}}$ is called E-asymptotically conformal, if for every $\epsilon > 0$ there is a compact subset C of $\hat{\mathbb{C}}$ with $E \subset \hat{\mathbb{C}} - C$ such that $K_z(\phi) < 1 + \epsilon$ for all $z \in \hat{\mathbb{C}} - C$.

Let $QC(\hat{\mathbb{C}})$ be the set of all quasiconformal self-homeomorphisms of $\hat{\mathbb{C}}$. Two elements f and $g \in QC(\hat{\mathbb{C}})$ are called E-asymptotically Teichmüller equivalent and denoted by $f \overset{E}{\approx} g$, if there is an $f(E)$-asymptotically conformal self-homeomorphism ϕ of $\hat{\mathbb{C}}$ such that $\phi \circ f$ is isotopic to g modulo E. Then we define the asymptotic Teichmüller space $AT(E)$ of the closed subset E to be the set of all E-asymptotically Teichmüller equivalent classes of quasiconformal self-homeomorphisms of $\hat{\mathbb{C}}$; that is,

$$AT(E) = QC(\hat{\mathbb{C}})/\overset{E}{\approx} = \{[[f]]_E : f \in QC(\hat{\mathbb{C}})\},$$

where $[[f]]_E$ is the E-asymptotically Teichmüller equivalent class containing f.

Without loss of generality, assume 0, 1 and $\infty \in E$. The asymptotic Teichmüller space of E can also be defined as the set of all E-asymptotically Teichmüller equivalent classes of Beltrami coefficients in $M(\mathbb{C})$, where two Beltrami differentials $\mu, \nu \in M(\mathbb{C})$ are said to be E-asymptotically Teichmüller equivalent, denoted by $\mu \overset{E}{\approx} \nu$, if there exists a $w^\mu(E)$-asymptotically conformal mapping ϕ such that $(w^\nu)^{-1} \circ \phi \circ w^\mu$ is isotopic to the identity mapping $id_{\hat{\mathbb{C}}}$ of $\hat{\mathbb{C}}$ modulo E. Then the asymptotic Teichmüller space $AT(E)$ is defined as

$$AT(E) := M(\mathbb{C})/\overset{E}{\approx} = \{[[\mu]]_E, \ \ \mu \in M(\mathbb{C})\},$$

where $[[\mu]]_E$ is the E-asymptotically equivalent class containing μ.

Let $f : \mathbb{C} \to \mathbb{C}$ be a quasiconformal homeomorphism. The boundary dilatation of f over E is defined as

$$H_E(f) := \inf\{K(f|_{\mathbb{C} \setminus C}) : C \text{ is a compact subset of } \mathbb{C} \text{ and } E \subset \mathbb{C} \setminus C\}.$$

Set

$$d_{AT(E)}([[\mu]]_E, [[\nu]]_E) = \frac{1}{2} \inf\{\log H_{w^\mu(E)}(h)\}$$

for any two points $[[\mu]]_E$ and $[[\nu]]_E \in AT(E)$, where the infimum is taken over all quasiconformal self-homeomorphisms h of $\mathbb{C}$ that are isotopic to $w^\nu \circ (w^\mu)^{-1}$ modulo $w^\mu(E)$. Then $d_{AT(E)}$ is a metric on $AT(E)$, which is called the Teichmüller metric in this paper.

Let Ψ_E denote the natural projection of $M(\mathbb{C})$ onto $AT(E)$ which maps μ to the E-asymptotically equivalent class $[[\mu]]_E$. Then Ψ_E is continuous.

It is clear that an E-Teichmüller equivalence implies an E-asymptotically Teichmüller equivalence. Thus there is a well-defined natural mapping

$$\begin{array}{rccc} \pi: & T(E) & \longrightarrow & AT(E) \\ & [\mu]_E & \longmapsto & [[\mu]]_E, \end{array}$$

which is continuous under Teichmüller metrics.

Let $\hat{\mathbb{C}} \setminus E = \bigcup_{i\in I} X_i$, where each X_i is a connected component of $\hat{\mathbb{C}} \setminus E$ and I is an at most countable index set. Let

$$T(\hat{\mathbb{C}} \setminus E) := \left\{ ([\mu_i])_{i\in I} \in \prod_{i\in I} T(X_i) : \sup_{i\in I} d_{T(X_i)}([0],[\mu_i]) < \infty \right\}.$$

Then $T(\hat{\mathbb{C}} \setminus E)$ has a natural metric

$$d_{T(\hat{\mathbb{C}}\setminus E)}(([\mu_i])_{i\in I}, ([\nu_i])_{i\in I}) := \sup_{i\in I} d_{T(X_i)}([\mu_i],[\nu_i]).$$

Let

$$AT(\hat{\mathbb{C}} \setminus E) := \left\{ ([[\mu_i]])_{i\in I} \in \prod_{i\in I} AT(X_i) : ([\mu_i])_{i\in I} \in T(\hat{\mathbb{C}} \setminus E) \right\}.$$

Then there is a natural metric

$$d_{AT(\hat{\mathbb{C}}\setminus E)}(([[\mu_i]])_{i\in I}, ([[\nu_i]])_{i\in I}) := \sup_{i\in I} d_{AT(X_i)}([[\mu_i]],[[\nu_i]])$$

on $AT(\hat{\mathbb{C}} \setminus E)$ as well.

Let $M(E)$ be the open unit ball of $L^\infty(E)$. It is clear that $M(E)$ is a metric space under the following metric:

$$d_{M(E)}(\mu,\nu) = \tanh^{-1}\left\| \frac{\mu-\nu}{1-\mu\overline{\nu}} \right\|_\infty, \quad \forall \mu,\ \nu \in M(E).$$

Therefore, $AT(\hat{\mathbb{C}} \setminus E) \times M(E)$ is a metric space with the natural maximum metric $d = \max\{d_{AT(\hat{\mathbb{C}}\setminus E)}, d_{M(E)}\}$.

Then we have the following theorem.

Theorem 2.1. *There is a well-defined isometry from $AT(E)$ onto $AT(\hat{\mathbb{C}} \setminus E) \times M(E)$; that is,*

$$\begin{array}{rccc} \tilde{\theta}: & AT(E) & \longrightarrow & AT(\hat{\mathbb{C}} \setminus E) \times M(E) \\ & [[\mu]]_E & \longmapsto & (([[\mu_{X_i}]])_{i\in I}, \mu_E). \end{array}$$

Proof. First, we prove that the mapping $\tilde{\theta}$ is well defined.

Indeed, if $\mu \overset{E}{\approx} \nu$, there is an E-asymptotic conformal self-homeomorphism ϕ of $\hat{\mathbb{C}}$ such that $(w^\nu)^{-1} \circ \phi \circ w^\mu$ is isotopic to the identity $id_{\hat{\mathbb{C}}}$ of $\hat{\mathbb{C}}$ modulo E. By restricting to X_i, $(w^\nu)^{-1} \circ \phi \circ w^\mu$ is isotopic to the identity id_{X_i} of X_i modulo the boundary ∂X_i of X_i. This implies that $w^\mu|_{X_i}$ and $w^\nu|_{X_i}$ are asymptotically Teichmüller equivalent for each $i \in I$.

Since $(w^\nu)^{-1} \circ \phi \circ w^\mu$ is isotopic to the identity $id_{\hat{\mathbb{C}}}$ of $\hat{\mathbb{C}}$ modulo E, its Beltrami coefficient is zero almost everywhere in E. By the E-asymptotically conformality of ϕ, we have $\mu = \nu$ almost everywhere in E.

Therefore, we conclude that $\tilde{\theta}$ is well defined.

Secondly, we prove that $\tilde{\theta}$ is a bijection.

For every $(([[\mu_{X_i}]])_{i\in I}, \mu_E) \in AT(\hat{\mathbb{C}} \setminus E) \times M(E)$, we have by definition that $([\mu_{X_i}])_{i\in I} \in T(\hat{\mathbb{C}} \setminus E)$. Then, by a suitable choice of representations μ_{X_i} of $[\mu_{X_i}]$'s, the following Beltrami differential:

$$\mu(z) = \begin{cases} \mu_{X_i}(z), & z \in X_i,\ i \in I; \\ \mu_E(z), & z \in E \end{cases}$$

is an element in $M(\hat{\mathbb{C}})$. So $(([[\mu_{X_i}]])_{i\in I}, \mu_E)$ is the image of $[[\mu]]_E$ under $\tilde{\theta}$, which means that $\tilde{\theta}$ is surjective.

If $[[\mu]]_E,\ [[\nu]]_E \in AT(E)$ and $\tilde{\theta}([[\mu]]_E) = \tilde{\theta}([[\nu]]_E)$, then $\mu, \nu \in M(\hat{\mathbb{C}})$ and $\mu_i \approx \nu_i$, $i \in I$ and $\mu_E = \nu_E$, where $\mu_i = \mu|_{X_i}$, $\nu_i = \nu|_{X_i}$, $\mu_E = \mu|_E$ and $\nu_E = \nu|_E$ are the restrictions of μ and ν to X_i and E respectively.

Let w^μ and w^ν be the normalized quasiconformal self-homeomorphisms of $\hat{\mathbb{C}}$ with Beltrami coefficients μ and ν respectively. Since E is a closed subset of $\hat{\mathbb{C}}$ with at least 4 points, X_i's are hyperbolic. Since $\mu_i \approx \nu_i$, it is clear that there exists an asymptotically conformal mapping $\phi_i : w^\mu(X_i) \to w^\nu(X_i)$ such that

$$w_i = (w^\nu|_{X_i})^{-1} \circ \phi_i \circ w^\mu|_{X_i} :\ X_i \to X_i$$

is isotopic to the identity mapping id_{X_i} modulo the boundary ∂X_i of X_i. By a result of Earle and McMullen [9], w_i can be extended to a self-homeomorphism of $\overline{X_i}$, which keeps the boundary points of X_i fixed.

Let

$$w(z) = \begin{cases} z, & z \in E; \\ w_i(z), & z \in X_i,\ i \in I. \end{cases}$$

Then $w(z)$ is a quasiconformal self-homeomorphism of $\hat{\mathbb{C}}$. Since for each $i \in I$, w_i is isotopic to the identity mapping id_{X_i} modulo the boundary ∂X_i of X_i and $\partial X_i \subset E$, it is clear that $w(z)$ is isotopic to the identity mapping $id_{\hat{\mathbb{C}}}$ modulo the closed subset E.

Let $\psi = w^\nu \circ w \circ (w^\mu)^{-1}$. Then, calculating the Beltrami coefficient and noting that $\mu_E = \nu_E$, ψ is a $w^\mu(E)$-asymptotically conformal self-homeomorphism of $\hat{\mathbb{C}}$. Thus, $(w^\nu)^{-1} \circ \psi \circ w^\mu(z) = w(z)$ is isotopic to $id_{\hat{\mathbb{C}}}$ modulo E. This implies $\mu \overset{E}{\approx} \nu$, that is, $[[\mu]]_E = [[\nu]]_E$.

Therefore, we have proved that for any $[[\mu]]_E$ and $[[\nu]]_E \in AT(E)$, $\tilde{\theta}([[\mu]]_E) = \tilde{\theta}([[\nu]]_E)$ implies $[[\mu]]_E = [[\nu]]_E$. Thus, $\tilde{\theta}$ is injective.

In the next, we prove that $\tilde{\theta}$ ia an isometry.

Let $[[\mu]]_E$ and $[[\nu]]_E \in AT(E)$. Then

$$d_{AT(E)}([[\mu]]_E, [[\nu]]_E) = \frac{1}{2} \inf\{\log H_{w^\mu(E)}(h)\},$$

where the infimum is taken over all quasiconformal self-homeomorphisms h of $\mathbb{C}$ which are isotopic to $w^\nu \circ (w^\mu)^{-1}$ modulo $w^\mu(E)$. By the definition of $H_{w^\mu(E)}(h)$, it is not hard to see that

$$H_{w^\mu(E)}(h) \geq \max\left\{ \frac{1 + \|\mu_h|_{w^\mu(E)}\|_\infty}{1 - \|\mu_h|_{w^\mu(E)}\|_\infty},\ H(h|_{w^\mu(X_i)}) \right\},\ \forall i \in I.$$

As $h = w^\nu \circ (w^\mu)^{-1}$ on $w^\mu(E)$, we have

$$H_{w^\mu(E)}(h) \geq \max\left\{ \frac{1 + \|\frac{\mu_E - \nu_E}{1 - \mu_E \overline{\nu_E}}\|_\infty}{1 - \|\frac{\mu_E - \nu_E}{1 - \mu_E \overline{\nu_E}}|\|_\infty},\ H(h|_{w^\mu(X_i)}) \right\},\ \forall i \in I.$$

It can be seen that $h|_{w^\mu(X_i)} : w^\mu(X_i) \to w^\nu(X_i)$ is isotopic to $w^\nu \circ (w^\mu)^{-1}|_{w^\mu(X_i)}$ modulo $\partial w^\mu(X_i)$ for every $i \in I$, whenever h is a quasiconformal self-homeomorphism of $\mathbb{C}$ which is isotopic to $w^\nu \circ (w^\mu)^{-1}$ modulo $w^\mu(E)$. Thus,

$$d_{AT(E)}([[\mu]]_E, [[\nu]]_E) \geq \max\left\{\log \frac{1 + \|\frac{\mu_E - \nu_E}{1-\mu_E\overline{\nu_E}}\|_\infty}{1 - \|\frac{\mu_E-\nu_E}{1-\mu_E\overline{\nu_E}}\|\|_\infty}, d_{AT(X_i)}([[\mu_i]], [[\nu_i]])\right\}$$

for every $i \in I$. Consequently,

$$d_{AT(E)}([[\mu]]_E, [[\nu]]_E) \geq d(\tilde{\theta}([[\mu]]_E), \tilde{\theta}([[\nu]]_E)). \tag{2.1}$$

To prove the converse inequality, we need the following lemma which can be found as Proposition 2 in Chapter 15 of [12].

Lemma. *For every $[\mu] \in T(X)$ and every $\epsilon > 0$, there is a representation $\eta \in [\mu]$ such that $\|\eta\|_\infty < k_0([\mu]) + \epsilon$ and $h^*(\eta) = h([\mu])$.*

By the definition of $d_{AT(X_i)}([[\mu_i]], [[\nu_i]])$ and the above lemma, there is a quasiconformal mapping $h_i : w^\mu(X_i) \to w^\nu(X_i)$ which is isotopic to $w^\nu \circ (w^\mu)^{-1}|_{w^\mu(X_i)}$ modulo $\partial w^\mu(X_i)$, such that $K(h_i) < K(w^\nu \circ (w^\mu)^{-1}|_{w^\mu(X_i)}) + 1$ and

$$H^*(h_i) = \inf\{H^*(\varphi_i)|\ \varphi_i : w^\mu(X_i) \to w^\nu(X_i) \text{ is isotopic to } w^\nu \circ (w^\mu)^{-1}|_{w^\mu(X_i)} \text{ modulo } \partial w^\mu(X_i)\}.$$

Then

$$G(z) = \begin{cases} h_i(z), & z \in w^\mu(X_i),\ i \in I, \\ w^\nu \circ (w^\mu)^{-1}(z), & z \in w^\mu(E), \end{cases}$$

is a quasiconformal self-homeomorphism of $\mathbb{C}$ which is isotopic to $w^\nu \circ (w^\mu)^{-1}$ modulo $w^\mu(E)$ and

$$d(\tilde{\theta}([[\mu]]_E), \tilde{\theta}([[\nu]]_E)) = \frac{1}{2} \log H^*_{w^\mu(E)}(G).$$

Consequently,

$$d_{AT(E)}([[\mu]]_E, [[\nu]]_E) \leq d(\tilde{\theta}([[\mu]]_E), \tilde{\theta}([[\nu]]_E)). \tag{2.2}$$

Therefore, by (2.1) and (2.2), $\tilde{\theta}$ is an isometry. The proof of Theorem 2.1 is completed.

3. The complex structure on $AT(E)$

In this section, we give a complex Banach manifold structure on $AT(E)$ via Theorem 2.1. First, we introduce a complex Banach manifold structure on $AT(\hat{\mathbb{C}} \setminus E)$. The method used here is somewhat similar to the one in [10, 15].

For each $i \in I$, a Beltrami differential $\mu \in M(X_i)$ is said to be vanishing at the infinity of X_i if, for every $\epsilon > 0$, there exists a compact subset C_i of X_i such that $\|\mu|_{X_i - C_i}\|_\infty < \epsilon$. Denote by $M_0(X_i)$ the set of all vanishing Beltrami differentials in $M(X_i)$. Let X_i^* be the conjugate Riemann surface of X_i. A quadratic differential $\phi \in B(X_i^*)$ is said to be vanishing at the infinity of X_i^* if the corresponding harmonic Beltrami differential $\rho_{X_i^*}^{-2}\overline{\phi}$ is vanishing at the infinity of X_i^*. Denote by $B_0(X_i^*)$ the closed subspace of $B(X_i^*)$ consisting of all vanished quadratic differentials.

It is shown that the asymptotic Bers mapping $\widetilde{\mathcal{B}}_i : AT(X_i) \to B(X_i^*)/B_0(X_i^*)$ is a biholomorphic mapping of $AT(X_i)$ onto a bounded open subset of $B(X_i^*)/B_0(X_i^*)$ and $AT(X_i)$ has a unique complex Banach manifold structure such that

$$\widetilde{\Psi}_i : M(X_i)/M_0(X_i) \to AT(X_i)$$

is a holomorphic split submersion and $\widetilde{\mathcal{B}}_i \circ \widetilde{\Psi}_i = \widetilde{S}_i$, where

$$\widetilde{S}_i : M(X_i)/M_0(X_i) \to B(X_i^*)/B_0(X_i^*)$$

is the asymptotically Schwarzian derivative mapping, which is a holomorphic split submersion too. For more details, we refer to [6, 8].

Let

$$L^\infty(\hat{\mathbb{C}} \setminus E) := \left\{ (\mu_i)_{i\in I} \in \prod_{i\in I} L^\infty(X_i) : \sup_{i\in I} \|\mu_i\|_\infty < 1 \right\}.$$

Then $L^\infty(\hat{\mathbb{C}} \setminus E)$ is a complex Banach space with norm

$$\|(\mu_i)_{i\in I}\|_\infty = \sup_{i\in I} \|\mu_i\|_\infty.$$

Denote by $M(\hat{\mathbb{C}} \setminus E)$ the open unit ball of $L^\infty(\hat{\mathbb{C}} \setminus E)$ centered at $\mu_i \equiv 0$ for all $i \in I$. An element $(\mu_i)_{i\in I} \in M(\hat{\mathbb{C}} \setminus E)$ is said to be vanishing at infinity if every $\mu_i \in M(X_i)$ $(i \in I)$ is vanishing at infinity. Denote by $M_0(\hat{\mathbb{C}} \setminus E)$ the set of all vanishing elements in $M(\hat{\mathbb{C}} \setminus E)$.

Let

$$B(\hat{\mathbb{C}} \setminus E) = \left\{ (\phi_i)_{i\in I} \in \prod_{i\in I} B(X_i^*) : \sup_{i\in I} \rho_{X_i^*}^{-2} |\phi_i| < \infty \right\}.$$

Then $B(\hat{\mathbb{C}} \setminus E)$ is a complex Banach space with norm

$$\|(\phi_i)_{i\in I}\| = \sup_{i\in I} \rho_{X_i^*}^{-2} |\phi_i|.$$

Denote by $B_0(\hat{\mathbb{C}} \setminus E)$ the set of all elements in $B(\hat{\mathbb{C}} \setminus E)$ vanishing at infinity. Here, we say that an element $(\phi_i)_{i\in I} \in B(\hat{\mathbb{C}} \setminus E)$ is vanishing at infinity if every $\rho_i^{-2}\overline{\phi_i}$ is vanishing at infinity.

Let

$$\widetilde{S} = (\widetilde{S}_i)_{i\in I} : M(\hat{\mathbb{C}} \setminus E)/M_0(\hat{\mathbb{C}} \setminus E) \to B(\hat{\mathbb{C}} \setminus E)/B_0(\hat{\mathbb{C}} \setminus E).$$

Then the following diagram:

$$\begin{array}{ccc} M(\hat{\mathbb{C}} \setminus E) & \xrightarrow{S} & B(\hat{\mathbb{C}} \setminus E) \\ \big\downarrow \widetilde{P}_M & & \big\downarrow \widetilde{P}_B \\ M(\hat{\mathbb{C}} \setminus E)/M_0(\hat{\mathbb{C}} \setminus E) & \xrightarrow{\widetilde{S}} & B(\hat{\mathbb{C}} \setminus E)/B_0(\hat{\mathbb{C}} \setminus E) \end{array}$$

is commutative. Here, $\widetilde{P}_M$ and $\widetilde{P}_B$ are the canonical quotient mappings and $S = (S_i)_{i\in I}$, where $S_i : M(X_i) \to B(X_i^*)$ is the usual Schwarzian derivative mapping defining the Bers embedding.

Let

$$\widetilde{\mathcal{B}} = (\widetilde{\mathcal{B}}_i)_{i\in I} : AT(\hat{\mathbb{C}} \setminus E) \to B(\hat{\mathbb{C}} \setminus E)/B_0(\hat{\mathbb{C}} \setminus E).$$

It is clear that $\widetilde{\mathcal{B}}$ is a well-defined injective mapping and the following diagram:

$$\begin{array}{ccc} M(\hat{\mathbb{C}} \setminus E)/M_0(\hat{\mathbb{C}} \setminus E) & \xrightarrow{\widetilde{S}} & B(\hat{\mathbb{C}} \setminus E)/B_0(\hat{\mathbb{C}} \setminus E) \\ & \searrow^{\widetilde{\Psi}} & \uparrow \widetilde{\mathcal{B}} \\ & & AT(\hat{\mathbb{C}} \setminus E) \end{array}$$

is commutative. Then we have the following theorem.

Theorem 3.1. *There is a unique complex Banach manifold structure on $AT(\hat{\mathbb{C}} \setminus E)$ such that the projection mapping*

$$\widetilde{\Psi} = (\widetilde{\Psi}_i)_{i \in I} : M(\hat{\mathbb{C}} \setminus E)/M_0(\hat{\mathbb{C}} \setminus E) \to AT(\hat{\mathbb{C}} \setminus E)$$

is a holomorphic split submersion.

Proof. It is proved in [15] that $S : M(\hat{\mathbb{C}} \setminus E) \to B(\hat{\mathbb{C}} \setminus E)$ is a holomorphic split submersion. So the holomorphy of $\widetilde{S}$ follows from the one of S directly. To see that $\widetilde{S}$ is a split submersion, we consider a given $(\mu_i)_{i \in I} \in M(\hat{\mathbb{C}} \setminus E)$ and its image in $M(\hat{\mathbb{C}} \setminus E)/M_0(\hat{\mathbb{C}} \setminus E)$ under the mapping $\widetilde{P}_M$. The right inverse of $S'((\mu_i)_{i \in I})$ is composed of the right inverses of $S_i'(\mu_i)$. Since every right inverse of $S_i'(\mu_i)$ sends vanishing differentials in $B(X_i^*)$ to vanishing differentials in $M(X_i)$, the right inverse of $S'((\mu_i)_{i \in I})$ sends vanishing elements in $B(\hat{\mathbb{C}} \setminus E)$ to vanishing elements in $M(\hat{\mathbb{C}} \setminus E)$, which induces a mapping from $B(\hat{\mathbb{C}} \setminus E)/B_0(\hat{\mathbb{C}} \setminus E)$ to $L^\infty(\hat{\mathbb{C}} \setminus E)/L_0^\infty(\hat{\mathbb{C}} \setminus E)$. This induced mapping is the right inverse of $\widetilde{S}'(\widetilde{P}_M((\mu_i)_{i \in I}))$. So $\widetilde{S}$ is a holomorphic split submersion.

Now that $\widetilde{S}$ is a split submersion, its image $\widetilde{S}(M(\hat{\mathbb{C}} \setminus E)/M_0(\hat{\mathbb{C}} \setminus E))$ is open in $B(\hat{\mathbb{C}} \setminus E)/B_0(\hat{\mathbb{C}} \setminus E)$. So $\widetilde{\mathcal{B}}$ is a bijection between $AT(\hat{\mathbb{C}} \setminus E)$ and the image $\widetilde{S}(M(\hat{\mathbb{C}} \setminus E)/M_0(\hat{\mathbb{C}} \setminus E))$. Therefore, $AT(\hat{\mathbb{C}} \setminus E)$ inherits a complex Banach manifold structure from $B(\hat{\mathbb{C}} \setminus E)/B_0(\hat{\mathbb{C}} \setminus E)$. Under this complex structure $\widetilde{\mathcal{B}}$ is biholomorphic and $\widetilde{\Psi}$ is a holomorphic split submersion.

Let $M(E)$ be the open unit ball in $L^\infty(E)$. Then, by Theorem 3.1, the product $AT(\hat{\mathbb{C}} \setminus E) \times M(E)$ is a complex Banach manifold modeled on the complex Banach space $B(\hat{\mathbb{C}} \setminus E)/B_0(\hat{\mathbb{C}} \setminus E) \times M(E)$. Therefore, by Theorem 2.1, there is a complex Banach manifold structure on $AT(E)$ induced by $\tilde{\theta}$ naturally.

Theorem 3.2. *There is a unique complex Banach manifold structure on the asymptotic Teichmüller space $AT(E)$ such that $\tilde{\theta}$ is biholomorphic.*

It is proved in [15] that

$$\begin{array}{rccc} \theta: & T(E) & \longrightarrow & T(\hat{\mathbb{C}} \setminus E) \times M(E) \\ & [\mu]_E & \longmapsto & (([\mu|_{X_i}])_{i \in I}, \mu|_E) \end{array}$$

is a bijective mapping and there is a complex Banach manifold structure on $T(\hat{\mathbb{C}} \setminus E) \times M(E)$ induced by the following mapping:

$$(\mathcal{B}, id_{M(E)}) : T(\hat{\mathbb{C}} \setminus E) \times M(E) \to B(\hat{\mathbb{C}} \setminus E) \times M(E),$$

where $\mathcal{B} = (\mathcal{B}_i)_{i \in I}$ and $\mathcal{B}_i : T(X_i) \to B(X_i^*)\,(i \in I)$ is the Bers embedding. Therefore, there is a natural complex Banach manifold structure on $T(E)$.

It is not hard to verify that the following diagram:

$$\begin{array}{ccccc} T(E) & \xrightarrow{\theta} & T(\hat{\mathbb{C}} \setminus E) \times M(E) & \xrightarrow{(\mathcal{B}, id_{M(E)})} & B(\hat{\mathbb{C}} \setminus E) \times M(E) \\ \downarrow \pi & & & & \downarrow (\tilde{P}_B, id_{M(E)}) \\ AT(E) & \xrightarrow{\tilde{\theta}} & AT(\hat{\mathbb{C}} \setminus E) \times M(E) & \xrightarrow{(\widetilde{\mathcal{B}}, id_{M(E)})} & B(\hat{\mathbb{C}} \setminus E)/B_0(\hat{\mathbb{C}} \setminus E)) \times M(E) \end{array}$$

is commutative. Thus, $(\tilde{P}_B, id_{M(E)})$ is the local representation of π. Consequently, we have the following theorem.

Theorem 3.3. *The mappings* $\pi : T(E) \to AT(E)$ *and* $\Psi_E : M(\mathbb{C}) \to AT(E)$ *are both holomorphic.*

The holomorphy of Ψ_E is deduced from the holomorphy of $\Phi_E : M(\mathbb{C}) \to T(E)$.

References

[1] L. V. Ahlfors, *Some remarks on Teichüller's space of Riemann surfaces,* Ann. Math., **74** (1961), 171–191.

[2] L. V. Ahlfors, *Lectures on quasiconformal mappings,* D.Van Nostrand, 1966.

[3] Lars Ahlfors and Lipman Bers, *Riemann's mapping theorem for variable metrics*, Ann. of Math. (2) **72** (1960), 385–404. MR0115006

[4] C. J. Earle, F. P. Gardiner, and N. Lakic, *Teichmüller spaces with asymptotic conformal equivalence*, No.IHES-M-95-60. SCAN-9507105. 1994, 1–20.

[5] Clifford J. Earle, Frederick P. Gardiner, and Nikola Lakic, *Vector fields for holomorphic motions of closed sets*, Lipa's legacy (New York, 1995), Contemp. Math., vol. 211, Amer. Math. Soc., Providence, RI, 1997, pp. 193–225, DOI 10.1090/conm/211/02821. MR1476988

[6] C. J. Earle, F. P. Gardiner, and N. Lakic, *Asymptotic Teichmüller space. I. The complex structure*, In the tradition of Ahlfors and Bers (Stony Brook, NY, 1998), Contemp. Math., vol. 256, Amer. Math. Soc., Providence, RI, 2000, pp. 17–38, DOI 10.1090/conm/256/03995. MR1759668

[7] Clifford J. Earle, Frederick P. Gardiner, and Nikola Lakic, *Asymptotic Teichmüller space. II. The metric structure*, In the tradition of Ahlfors and Bers, III, Contemp. Math., vol. 355, Amer. Math. Soc., Providence, RI, 2004, pp. 187–219, DOI 10.1090/conm/355/06452. MR2145063

[8] Clifford J. Earle, Vladimir Markovic, and Dragomir Saric, *Barycentric extension and the Bers embedding for asymptotic Teichmüller space*, Complex manifolds and hyperbolic geometry (Guanajuato, 2001), Contemp. Math., vol. 311, Amer. Math. Soc., Providence, RI, 2002, pp. 87–105, DOI 10.1090/conm/311/05448. MR1940165

[9] C. J. Earle and C. Mcmullen, *Quasiconformal isotopies,* In Holomorphic Functions and Moduli I, MSRI, Springer-Verlag, 1988.

[10] Clifford J. Earle and Sudeb Mitra, *Variation of moduli under holomorphic motions*, In the tradition of Ahlfors and Bers (Stony Brook, NY, 1998), Contemp. Math., vol. 256, Amer. Math. Soc., Providence, RI, 2000, pp. 39–67, DOI 10.1090/conm/256/03996. MR1759669

[11] Frederick P. Gardiner, *Teichmüller theory and quadratic differentials*, Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, 1987. A Wiley-Interscience Publication. MR903027

[12] Frederick P. Gardiner and Nikola Lakic, *Quasiconformal Teichmüller theory*, Mathematical Surveys and Monographs, vol. 76, American Mathematical Society, Providence, RI, 2000. MR1730906

[13] Frederick P. Gardiner and Dennis P. Sullivan, *Symmetric structures on a closed curve*, Amer. J. Math. **114** (1992), no. 4, 683–736, DOI 10.2307/2374795. MR1175689

[14] O. Lehto, *Unibalent functions and Teichmüller space*, Springer-Verlag, New York, 1986.

[15] Gregory Stephen Lieb, *Holomorphic motions and Teichmuller space*, ProQuest LLC, Ann Arbor, MI, 1989. Thesis (Ph.D.)–Cornell University. MR2638376

[16] Oswald Teichmüller, *Beweis der analytischen Abhängigkeit des konformen Moduls einer analytischen Ringflächenschar von den Parametern* (German), Deutsche Math. **7** (1944), 309–336. MR0018760

Key Laboratory of Ministry of Education – "Mathematics, Informatics and Behavioral Semantics", School of Mathematics and Systems Science, Beihang University, Beijing 100191, People's Republic of China

Email address: yiqi@buaa.edu.cn

Key Laboratory of Ministry of Education – "Mathematics, Informatics and Behavioral Semantics", School of Mathematics and Systems Science, Beihang University, Beijing 100191, People's Republic of China

Current address: School of Science, Linyi University, Shandong 276005, People's Republic of China

Email address: by1209113@buaa.edu.cn