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BOREL CANONIZATION OF ANALYTIC SETS WITH BOREL SECTIONS

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ABSTRACT. Kanovei, Sabok and Zapletal asked whether every proper σ-ideal satisfies the following property: given E an analytic equivalence relation with Borel classes, there exists a set B which is Borel and I-positive such that $E\restriction_B$ is Borel. We propose a related problem – does every proper σ-ideal satisfy: given A an analytic subset of the plane with Borel sections, there exists a set B which is Borel and I-positive such that $A \cap (B \times \omega^\omega)$ is Borel. We answer positively when a measurable cardinal exists, and negatively in L, where no proper σ ideal has that property. We show that a positive answer for all ccc σ-ideals implies that ω_1 is inaccessible to the reals and Mahlo in L.

1. INTRODUCTION

1.1. **Borel canonization of analytic equivalence relations.** Analytic equivalence relations are common in the world of mathematics, and given such an equivalence relation, one of the first questions traditionally asked is – "is it Borel"? A negative answer used to convince us that the equivalence relation is relatively complicated, but a new point of view proposed by Kanovei, Sabok and Zapletal has opened the way to a somewhat more optimistic conclusion. We all know that Lebesgue measurable functions are "almost continuous", analytic sets are Borel modulo meager sets and colorings of natural numbers are "almost" trivial. We can then hope that even the non-Borel analytic equivalence relations are Borel on a substantial set – which leads to the following question:

Problem 1.1. Given an analytic equivalence relation E on a Polish space X, does there exist a positive measure (or non-meager, or uncountable) Borel set B such that E restricted to B is Borel?

We can use the notion of a σ-ideal to state a more general problem. By 'σ-ideal' we will always refer to one that does not contain singletons. Given a σ-ideal I , we will say that A is an I-positive set if $A \notin I$, an I-small set if $A \in I$, and a co-I set if $X - A \in I$. The above mentioned problem involved the existence of an I-positive set for the null ideal, the meager ideal and the countable ideal. We restate it for all σ -ideals:

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Problem 1.2. Given an analytic equivalence relation E on a Polish space X and a σ-ideal I , does there exist an I-positive Borel set B such that E restricted to B is Borel?

Unfortunately, that problem has a negative answer, and further assumptions had to be made – both on the equivalence relation E and on the σ-ideal I. We recall that for a σ-ideal I, $\mathbb{P}_I$ is the partial order of Borel I-positive subsets, ordered by inclusion. We say that I is *proper* if the associated forcing notion $\mathbb{P}_I$ is proper. Then Kanovei, Sabok and Zapletal have asked the following:

Problem 1.3 ([12])**.** *Borel canonization of analytic equivalence relations with Borel classes:* Given an analytic equivalence relation E on a Polish space X, all of its classes Borel, and a proper σ-ideal I, does there exist an I-positive Borel set B such that E restricted to B is Borel?

They have shown the answer to be positive for two important classes of analytic equivalence relations with Borel classes: orbit equivalence relation, and countable equivalence relations (proofs are given in the next section). The problem in its full generality remained open.

1.2. **Borel canonization of analytic sets with Borel sections.** Analytic equivalence relations are only one example on which Borel canonization may apply. One can apply Borel canonization on any analytic subset of X^2 for X Polish. The property of "all classes are Borel" will be replaced by "all sections are Borel".

Definition 1.4. Let X be Polish, and I a σ-ideal on X.

(1) We say that I has *square Borel canonization of analytic sets with Borel sections* if for any $A \subseteq X^2$ an analytic set with vertical Borel sections, there exists an I-positive Borel set B such that $A \cap (B \times B)$ is Borel.
(2) We say that I has *cylindrical Borel canonization of analytic sets with Borel sections* if for any $A \subseteq X^2$ an analytic set with vertical Borel sections, there exists an I-positive Borel set B such that $A \cap (B \times X)$ is Borel.
(3) We say that I has *strong square Borel canonization of analytic sets with Borel sections* if for any $A \subseteq X^2$ an analytic set with vertical Borel sections, there exists a co-I Borel set B such that $A \cap (B \times B)$ is Borel.
(4) We say that I has *strong cylindrical Borel canonization of analytic sets with Borel sections* if for any $A \subseteq X^2$ an analytic set with vertical Borel sections, there exists a co-I Borel set B such that $A \cap (B \times X)$ is Borel.

In what follows, we will simply say: "I has square Borel canonization", etc. Cylindrical Borel canonization implies square Borel canonization, which implies Borel canonization of analytic equivalence relations with Borel classes. We do not know whether any of the inverse implications are true.

Remark 1.5. For ccc σ-ideals, the strong Borel canonization and the weak Borel canonization are equivalent. The strong Borel canonization of general proper σ-ideals is false – see [10] proposition 17.

When considering the square and cylindrical Borel canonizations, there is no difference between analytic and coanalytic sets:

Claim 1.6. I has square Borel canonization of analytic sets with Borel sections if and only if I has square Borel canonization of coanalytic sets with Borel sections (and the same for cylindrical, strong square and strong cylindrical Borel canonizations).

Proof. Consider the complement. □

Albeit being a new notion, strong cylindrical Borel canonization has been studied in the past by Fujita in [5] and by Ikegami in [9] and [10], culminating in the following result:

Theorem 1.7 (Ikegami [10]). *Let I be a σ-ideal with a Borel basis such that $\mathbb{P}_I$ is strongly arboreal, provably ccc and $\mathbf{\Sigma}^1_1$. Then the following are equivalent:*

(1) *I has strong cylindrical Borel canonization.*
(2) *$\mathbf{\Sigma}^1_2$ sets are measurable with respect to I , which is: For A $\mathbf{\Sigma}^1_2$ there is B Borel such that $A \triangle B \in I$.*

We say that I *has a Borel basis* if any $A \in I$ is contained in an I-small Borel set. We say that I is *provably ccc* if ZFC proves that I is ccc. The notions of "strongly arboreal" and "$\mathbf{\Sigma}^1_1$ forcing" are assumptions on the presentability and definability of $\mathbb{P}_I$, satisfied by, for example, the meager ideal and the null ideal. Hence, one learns from the theorem that the meager ideal has strong cylindrical Borel canonization if and only if $\mathbf{\Sigma}^1_2$ sets have the Baire property, and the null ideal has strong cylindrical Borel canonization if and only if $\mathbf{\Sigma}^1_2$ sets are Lebesgue measurable.

This paper will focus on general σ-ideals with minimal assumptions on definability and presentability. It is therefore interesting and illuminating to compare our results with Ikegami's results.

1.3. **The results of this paper.** The problem of Kanovei, Sabok and Zapletal can be restated as:

Problem 1.8. Do all proper σ-ideals I have Borel canonization of analytic equivalence relation with Borel classes?

We focus our paper on the following related problem:

Problem 1.9. Do all proper σ-ideals I have cylindrical Borel canonization of analytic sets with Borel sections?

Section 2 presents previous results about Borel canonization.

In section 3 we define a notion of ω_1-rank for analytic sets with Borel sections. We use the rank to prove:

Theorem 1.10. *Assume a measurable cardinal exists. Then proper σ-ideals have cylindrical Borel canonization and ccc σ-ideals have strong cylindrical Borel canonization.*

We say that ω_1 *is inaccessible to the reals* if for every z real, $\omega_1^{L[z]} < \omega_1$.

Theorem 1.11. *Assume ω_1 is inaccessible to the reals, and I is ccc in $L[z]$ for any real z. Then I has strong cylindrical Borel canonization of analytic sets all of whose sections are $\mathbf{\Pi}^0_\gamma$ for some $\gamma < \omega_1$.*

The last section presents counterexamples to cylindrical Borel canonization, both in L and in much larger universes:

Proposition 1.12. *In L, proper σ-ideals do not have cylindrical Borel canonization of analytic sets with Borel sections. The same is true for $L[z]$ where z is a real.*

Theorem 1.13. *If ω_1 is inaccessible to the reals and is not Mahlo in L, then there is a ccc σ-ideal not having cylindrical Borel canonization of analytic sets with Borel sections. Moreover, $\mathbb{P}_I \Vdash A_{x_G}$ non-Borel for some A analytic with Borel sections.*

Corollary 1.14. *Cylindrical Borel canonization for ccc σ-ideals implies that ω_1 is inaccessible to the reals and Mahlo in L.*

Non-absoluteness of "all sections / classes are Borel" is further demonstrated by the following proposition:

Proposition 1.15. *There is an analytic equivalence relation E such that:*

(1) *If ω_1 is inaccessible to the reals and is not Mahlo in L, then all E classes are Borel and there is a ccc σ-ideal I such that*

$$\mathbb{P}_I \Vdash [x_G] \text{ is non-Borel.}$$

(2) *If ω_1 is inaccessible to the reals, then all E classes are Borel, while in L there is a non-Borel class.*

The problem of square Borel canonization is sometimes discussed in this paper but the consistency of a negative answer remains open. The same applies for the problem of Borel canonization of equivalence relations.

Chan, in [3], has independently obtained much of the above results using similar techniques. He has been working with equivalence relations, but his proofs perfectly fit in the context of cylindrical Borel canonization. In particular, he has shown that all proper σ-ideals have cylindrical Borel canonization if there exist sharps for all reals and for a few more sets associated with the forcing notions of proper σ-ideals.

2. Preliminaries

For I a σ-ideal on a Polish space X, $\mathbb{P}_I$ is the partial order of Borel I-positive sets ordered by inclusion.

We say that I is ccc if $\mathbb{P}_I$ is ccc, and that I is proper if $\mathbb{P}_I$ is proper. Properness of $\mathbb{P}_I$ can be phrased in terms of the set of M-generics:

Proposition 2.1 ([16])**.** *$\mathbb{P}_I$ is proper if and only if for every M a countable elementary submodel of a large enough H_θ such that $\mathbb{P}_I \in M$ and for every $B \in \mathbb{P}_I \cap M$, the set of elements of B which are generic over M is I-positive.*

2.1. Borel canonization of orbit equivalence relations and countable equivalence relations.

In what follows, two Borel canonization results of Kanovei, Sabok and Zapletal [12] are presented. The first is rewritten using the notion of Hjorth rank, and the second is generalized so that it shows cylindrical Borel canonization of analytic sets with countable sections.

To present the first proof, we recall the principles of Hjorth analysis.

Definition 2.2. A *Scott analysis of Polish actions* is a method defining for a Polish group G acting continuously on a Polish space X a decreasing sequence of equivalence relations on X $\langle \equiv_\alpha : \alpha < \omega_1 \rangle$ and δ an ω_1 rank on X such that:

(1) $\equiv_\alpha$ are Borel and invariant under G.
(2) The orbit equivalence relation is exactly the intersection of all $\equiv_\alpha$.
(3) The function $\delta : X \to (\omega_1, <)$ is Borel and invariant under the action of G.
(4) There is an $\alpha < \omega_1$ such that for every $x, y \in X$, x and y are orbit equivalent if and only if $x \equiv_{\delta(x)+\alpha} y$.

We will say that a Scott analysis of Polish actions satisfies the *boundedness principle* if the Borel orbit equivalence relations are exactly those orbit equivalence relations on which δ is uniformly bounded.

Scott [13] presented such an analysis which is restricted for the logic actions – the actions of S_∞ on the collection of countable models of a countable theory. He proved his analysis satisfies the boundedness principle in the following sense: the Borel logic actions are exactly the logic actions on which Scott's rank δ is uniformly bounded.

Hjorth [8] extended Scott's construction to a Scott analysis of all Polish actions, which we call here *Hjorth analysis*. In [4] we have shown that Hjorth analysis satisfies the boundedness principle. By *Hjorth rank* we will refer to the rank associated with Hjorth analysis.

Theorem 2.3. *Proper σ-ideals have Borel canonization of orbit equivalence relations.*

Proof. Let G be a Polish group acting on a Polish space X, and I a proper σ-ideal. We find C Borel and I-positive such that $\left(E_G^X\right)\restriction_C$, the orbit equivalence relation restricted to C, is Borel. Let δ be the Hjorth rank associated with the action of G on X . Fix θ large enough and $M \preceq H_\theta$ an elementary submodel containing all the relevant information. Let C be the I-positive Borel set of M-generics, and let $x \in C$ be M-generic. Then

$$M[x] \models \delta(x) \leq \alpha$$

for some $\alpha < \omega_1^{M[x]} = \omega_1^M$. The rank δ has a Borel definition, hence $\mathbb{V} \models \delta(x) \leq \alpha$ as well. We have thus proved that the Hjorth rank on C is uniformly bounded below ω_1^M, hence $\left(E_G^X\right)\restriction_C$ is Borel. □

Theorem 2.4. *Proper σ-ideals have cylindrical Borel canonization of analytic sets with countable sections.*

Proof. Fix I proper and A an analytic subset of the plane with countable sections. Recall that a $\Sigma_1^1(x)$ set is countable if and only if all its elements are hyperarithmetic in x. One can then show that "all sections are countable" is still true in generic extensions. Use 2.3.1 of [16] to find $B \in \mathbb{P}_I$ and a Borel $f : B \to X^\omega$ such that $B \Vdash f(x_G)$ *enumerates* A_{x_G}.

Fix θ large enough and $M \preceq H_\theta$ an elementary submodel containing all the relevant information (including f and B). Let $C \subseteq B$ be the I-positive Borel set of M-generics, and let $x \in C$ be M-generic. Then

$$M[x] \models f(x) \text{ enumerates } A_x,$$

which is,

$$M[x] \models \forall y\ (y \in A_x) \Rightarrow \exists n \in \omega\ (f(x))(n) = y.$$

That statement is Π_1^1, so it must be true in $\mathbb{V}$ as well – which is, $(A_x)^{\mathbb{V}} \subseteq M[x]$. On the other hand, if

$$(f(x))(n) = y,$$

then $y \in M[x]$ and $M[x] \models y \in A_x$, hence $\mathbb{V}$ thinks the same.

The above results in a Borel definition of $A \cap (C \times X)$: For $x \in C$ and $y \in X$,

$$(x, y) \in A \Leftrightarrow \exists n \in \omega\ (f(x))(n) = y.$$ □

3. Ranks for analytic sets with Borel sections

Denote by WF the set of well-founded trees, and by WF_α the set of well-founded trees of rank less than α.

Let A be an analytic subset of $(\omega^\omega)^2$. There exists a tree $T \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ such that

$$(x,y) \in A \Leftrightarrow T_{xy} \notin WF.$$

For $\alpha < \omega_1$, define:

$$(x,y) \in A_\alpha \Leftrightarrow T_{xy} \notin WF_\alpha.$$

The sequence A_α is decreasing, $A_\delta = \bigcap_{\alpha<\delta} A_\alpha$ for δ limit, and

$$A = \bigcap_{\alpha<\omega_1} A_\alpha.$$

Definition 3.1. For $x \in \omega^\omega$, the *rank of* x , $\delta(x)$, is the least α such that $A_x = (A_\alpha)_x$, if such an α exists, and ∞ if there is no such α.

Proposition 3.2. *If A_x is Borel, then there is $\alpha < \omega_1$ such that $A_x = (A_\alpha)_x$.*

Proof. Since

$$(X - A)_x = \{y \ : \ (x,y) \notin A\} = \{y \ : \ T_{xy} \in WF\}$$

is a Borel set, its image under $y \to T_{xy}$ is an analytic subset of WF. By the boundedness theorem for WF, its image is contained in WF_α for some countable α, which is:

$$y \in A_x \Leftrightarrow T_{xy} \notin WF_\alpha \Leftrightarrow y \in (A_\alpha)_x$$

as we wanted to show. □

Proposition 3.3. *The set $\Delta = \{(x,f) \ : \ f \in WO,\ \delta(x) \le ot(f)\}$ is Π^1_2. The set $\{x \ : \ A_x \text{ is Borel}\}$ is Σ^1_3.*

Proof. $f \in WO$ is Π^1_1. The rank of x is less than the order type of f if and only if

$$\forall z : \ T_{xz} \in WF \Leftrightarrow T_{xz} \in WF_{ot(f)}$$

which is Π^1_2. For $x \in X$, A_x is Borel if and only if $\exists f$ such that $(x,f) \in \Delta$, which is Σ^1_3. □

Proposition 3.4. *Let $B \subseteq X$ be a Borel set. Then $A \cap (B \times X)$ is Borel if and only if there is an $\alpha < \omega_1$ such that for all $x \in B$, $\delta(x) < \alpha$.*

The proof uses the boundedness theorem for WF in the same way used in the proof of Proposition 3.2.

We remark that the rank is not canonical and depends on the choice of the tree T. However, all we will need for our results is the mere existence of such a rank.

3.1. **Cylindrical Borel canonization of proper σ-ideals.** Having those definitions in mind, one can try and prove cylindrical Borel canonization of proper σ-ideals in the following way:

- Fix a countable elementary submodel $M \preceq H_\theta$ for θ large enough, and force with $\mathbb{P}_I$ over M.

- Show that A_{x_G} is Borel in $M[x_G]$ and so
$$M[x_G] \models \delta(x_G) \leq \alpha$$
for some $\alpha < \omega_1^{M[x]} = \omega_1^M$ (recall that $\mathbb{P}_I$ preserves ω_1).
- Use absoluteness to show that $\mathbb{V} \models \delta(x_G) \leq \alpha$.
- Use properness to guarantee that the set of M-generics is I-positive, and the above arguments to conclude that all of them has rank less than $\omega_1^M < \omega_1$.

However, the 2nd and 3rd steps are in general impossible. Although A has only Borel sections, that statement is Π^1_4 (see Proposition 3.3), hence one must work harder to show its preservation. The 3rd step provides us with another absoluteness challenge, since Π^1_2 absoluteness between a submodel N and the universe is guaranteed when N contains all countable ordinals, whereas $M[x_G]$ is countable.

The following proof follows the above lines and takes advantage of the measurable cardinal to overcome the above mentioned difficulties. We recall that by a theorem of Martin and Solovay (15.6 in [11]), when there is a measurable cardinal κ, forcing notions of cardinality less than κ preserve Σ^1_3 statements.

Theorem 3.5. *Assume a measurable cardinal exists. Then proper σ-ideals have cylindrical Borel canonization and ccc σ-ideals have strong cylindrical Borel canonization.*

Proof. The idea is as follows: given U a κ-complete ultrafilter on κ, one can form iterated ultrapowers of the universe, $\mathbb{V}_\alpha$, all well founded by a theorem of Gaifman. The same operation can be applied on M a countable elementary submodel of the universe such that $U \in M$. Since the sequence $j^{(\alpha)}(\kappa)$ is increasing and continuous, M_{ω_1}, the $\omega_1' th$ iterated ultrapower of M, contains all countable ordinals, so that M_{ω_1} and the universe agree on $\mathbf{\Pi}^1_2$ statements. On the other hand, M_{ω_1} is an iterated ultrapower of M, so they agree on all statements – there is an elementary embedding between them. We will then have enough absoluteness to conclude the proof.

So let $M \preceq H_\theta$ for θ large enough be a countable elementary submodel such that $\kappa \in M$ is measurable and M contains all the relevant information. Fix $U \in M$ a κ-complete ultrafilter on κ, and force with $\mathbb{P}_I$ over M. The Levy-Solovay theorem [14] guarantees that U remains a κ-complete ultrafilter in $M[x_G]$. For convenience, denote $M[x_G]$ by N, remembering that $\omega_1^N = \omega_1^M$ because $\mathbb{P}_I$ is proper. We can then use U to iterate ultrapowers of $\mathbb{V}$, N and M over all ordinals. Denote by $\mathbb{V}_\alpha$, N_α and M_α the $\alpha' th$ iterated ultrapowers of $\mathbb{V}$, N and M, respectively.

The $\mathbb{V}_\alpha' s$ are well founded, and so are the $M_\alpha' s$. We claim that the ordinals of N_α are the same as the ordinals of M_α, which is why N_α is well founded as well. This follows by induction on α: $M_0 = M$ and $N_0 = N$ clearly share the same ordinals. The limit case is immediate. For the successor case, note that an ordinal in $N_{\alpha+1}$ is equivalent, modulo the ultrafilter U, to a function from κ to the ordinals of N_α, hence by the induction hypothesis is equivalent to an element of $M_{\alpha+1}$. An analogous statement is true for the ordinals of $M_{\alpha+1}$.

We can now identify each N_α with its transitive collapse. Since $(j_\alpha)^N(\kappa)$ is a normal sequence, N_{ω_1} contains all countable ordinals. Hence, as stated above, N_{ω_1} and N are elementarily equivalent, and N_{ω_1} and $\mathbb{V}$ are $\mathbf{\Pi}^1_2$ equivalent.

By the assumption, $\mathbb{V} \models A_{x_G}$ *Borel*, and so there is a countable ordinal α such that $\mathbb{V} \models \delta(x_G) \leq \alpha$. We would like this statement to be true in N_{ω_1}, but it is meaningless there: Although α is an element of N_{ω_1}, it is not necessarily countable in N_{ω_1}. The natural solution will be collapsing α over N_{ω_1}. The resulting model, $N_{\omega_1}[Coll(\omega,\alpha)]$, still contains all ordinals countable in $\mathbb{V}$, and also knows that α is countable, so we can finally reflect the statement $\delta(x_G)\le\alpha$ to get that $N_{\omega_1}[coll(\omega,\alpha)]\models\delta(x_G)\le\alpha$ and

$$N_{\omega_1}[coll(\omega,\alpha)]\models A_{x_G}\ Borel.$$

Note that in N_{ω_1}, α is under a measurable cardinal, hence by Martin-Solovay's theorem, collapsing α over N_{ω_1} preserves Σ^1_3 statements. Proposition 3.3 then assures that $N_{\omega_1}\models A_{x_G}\ Borel$. Since N_{ω_1} is elementarily equivalent to N, we have so far shown that

$$N\models A_{x_G}\ Borel,$$

which means that $N\models\delta(x_G)\le\alpha$ for some $\alpha<\omega_1^N=\omega_1^M$. Another use of the elementary equivalence of N and N_{ω_1} proves that $N_{\omega_1}\models\delta(x_G)\le\alpha$, from which Σ^1_2 absoluteness guarantees

$$\mathbb{V}\models\delta(x_G)\le\alpha<\omega_1^M.$$

Taking B to be the set of M-generics concludes the proof. Notice that if I is ccc, B is co-I. □

3.2. **Cylindrical Borel canonization of provably ccc σ-ideals.** We follow Stern's definitions and results from [15] . By an α-Borel code, for α a not necessarily countable ordinal, we mean a well-founded tree on α whose maximal points are associated with basic open sets, and all other points are labeled by union or intersection. An α-Borel code naturally codes a set generated from basic open sets by unions and intersections of length at most α. If α is countable, the set coded by an α-Borel code is Borel.

For a countable ordinal $\gamma<\omega_1$, $L[\gamma]$ stands for $L[a]$ where a codes a well order of ω of order type γ.

Theorem 3.6 (Stern [15])**.** *If A is $\mathbf{\Pi}^0_\gamma\cap\Pi^1_1(z)$, then $L[z,\gamma]$ has an $\omega_\gamma^{L[z,\gamma]}$-Borel code for A.*

Proposition 3.7. *Let A be a $\Sigma^1_1(z)$ subset of the plane with $\mathbf{\Pi}^0_\gamma$ sections. Let I be a σ-ideal proper in $L[z,\gamma]$, and x generic over $L[z,\gamma]$. Then*

$$\delta(x)<\omega_{\gamma+1}^{L[z,\gamma]}.$$

Proof. Since $\mathbb{V}\models A_x\ is\ \mathbf{\Pi}^0_\gamma$, using Stern's theorem we know that $L[z,x,\gamma]\models A_x\ is\ \omega_\gamma^{L[z,x,\gamma]}-Borel$. Collapsing $\omega_\gamma^{L[z,x,\gamma]}$ over $L[z,x,\gamma]$, we have:

$$L[z,x,\gamma][Coll(\omega,\omega_\gamma^{L[z,x,\gamma]}]\models A_x\ Borel.$$

ω_1 of the new model is $\omega_{\gamma+1}^{L[z,x,\gamma]}$. Since x is assumed to be $L[z,\gamma]$-generic and $L[z,\gamma]\models CH$, $\mathbb{P}_I$ doesn't collapse cardinals in $L[z,\gamma]$:

$$\omega_{\gamma+1}^{L[z,\gamma]}=\omega_{\gamma+1}^{L[z,\gamma][x]}.$$

Hence there must be an $\alpha<\omega_{\gamma+1}^{L[z,\gamma]}$ such that

$$L[z,x,\gamma][Coll(\omega,\omega_\gamma^{L[z,x,\gamma]}]\models\delta(x)\le\alpha.$$

Shoenfield's absoluteness concludes the proof. □

Theorem 3.8. *Assume ω_1 is inaccessible to the reals, and I is ccc in $L[z]$ for any real z. Then I has strong cylindrical Borel canonization of analytic sets all of whose sections are $\mathbf{\Pi}^0_\gamma$ for some $\gamma < \omega_1$.*

Note that part of the assumption here is that I is defined and a σ-ideal in $L[z]$ for any real z.

Proof. Let A be a $\Sigma^1_1(z)$ set with $\mathbf{\Pi}^0_\gamma$ sections. Since I is ccc in $L[z,\gamma]$, the set of generics over $L[z,\gamma]$ is co-I. ω_1 is inaccessible in $L[z,\gamma]$, so that in particular $\omega_{\gamma+1}^{L[z,\gamma]} < \omega_1$. The previous proposition then concludes the proof. □

4. Counterexamples to cylindrical Borel canonization

Counterexamples are implicit in [10]:

Example 4.1 ([10])**.** Consider the meager ideal and Theorem 1.7. For that ideal, cylindrical Borel canonization and strong cylindrical Borel canonization are equivalent (see Remark 1.5). Hence Theorem 1.7 provides counterexamples when not all $\mathbf{\Sigma}^1_2$ sets have the Baire property. The same is true for the null ideal.

Proposition 4.2. *In L, proper σ-ideals do not have cylindrical Borel canonization of analytic sets with Borel sections. The same is true for $L[z]$ where z is a real.*

Proof. The argument is based on example 2.3.5 of [16]. Working in L, let

$$(x,y) \in A \Leftrightarrow x \in L_{\omega_1^{ck(y)}}.$$

The set A is coanalytic with Borel vertical sections, since given $x \in L_\alpha$ and α minimal with that property,

$$A_x = \{y \ : \ x \in L_{\omega_1^{ck(y)}}\} = \{y \ : \ \omega_1^{ck(y)} \geq \alpha\},$$

which is Borel. By way of contradiction, fix B Borel I-positive such that $A \cap (B \times \omega^\omega)$ is Borel. Using $\mathbb{P}_I$ -uniformization (2.3.4 of [16]) there exists $C \subseteq B$ Borel I-positive and $f : C \to \omega^\omega$ Borel such that $f \in L$ and $f \subseteq A$. Let $x \in C$ be any new real added by forcing over L. By analytic absoluteness, $L[x] \models f \subseteq A$, and in particular, $(x, f(x)) \in A$, contradicting the fact that x is not constructible. □

Let A be an analytic subset of the plane and $B \subseteq \omega^\omega$ Borel I-positive subset of reals such that $A \cap (B \times \omega^\omega)$ is Borel. Then using Shoenfield's absoluteness, $\mathbb{P}_I \Vdash A \cap (B \times \omega^\omega)$ *Borel*, and in particular

$$B \Vdash A_{x_G} \textit{ Borel}.$$

Hence a σ-ideal I such that $\mathbb{P}_I$ adds a non-Borel section is a counterexample to cylindrical Borel canonization. We now show that even under mild large cardinal assumptions, there might exist such a σ-ideal which is ccc:

Fact 4.3. *If ω_1 is inaccessible to the reals and is not Mahlo in L, then there is a ccc forcing adding a real x such that $\omega_1^{L[x]} = \omega_1$.*

For the proof, see theorem 6 of [1].

Proposition 4.4. *If $\mathbb{P}$ is a ccc forcing adding a real x, then there is a ccc σ-ideal I such that $\mathbb{V}[x] \subseteq \mathbb{V}^{\mathbb{P}}$ is a $\mathbb{P}_I$ extension and x is the $\mathbb{P}_I$ generic real.*

Proof. Fix τ a $\mathbb{P}$-name for the real x. For B Borel, define

$$B \in I \Leftrightarrow \mathbb{P} \Vdash \tau \notin B.$$

I is a σ-ideal (in fact, a σ-ideal on Borel sets which generates a σ-ideal). We claim that it is ccc. Let $\langle B_\alpha \ : \ \alpha < \omega_1 \rangle$ be an antichain of I-positive sets, which is, for $\alpha_1 \neq \alpha_2$,

$$\mathbb{P} \Vdash \tau \notin (B_{\alpha_1} \cap B_{\alpha_2}) .$$

Fix $p_\alpha \in \mathbb{P}$ such that $p_\alpha \Vdash \tau \in B_\alpha$. Then $\langle p_\alpha \ : \ \alpha < \omega_1 \rangle$ must be an antichain, hence countable, as we have hoped.

In $\mathbb{V}^{\mathbb{P}}$, the generic x, as a realization of τ, avoids all Borel I-small sets of the ground model, hence it is $\mathbb{P}_I$ generic over $\mathbb{V}$. Thus $\mathbb{V}[x]$ is the promised $\mathbb{P}_I$ extension. □

Theorem 4.5. *If ω_1 is inaccessible to the reals and is not Mahlo in L, then there is a ccc σ-ideal I not having cylindrical Borel canonization of analytic sets with Borel sections. Moreover, $\mathbb{P}_I \Vdash A_{x_G}$ non-Borel for some A analytic with Borel sections.*

Proof. For every x, in $L[x]$ there exists a $\Pi^1_1(x)$ uncountable set with no perfect subset. Fix $\Phi(x)$ a $\Pi^1_1(x)$ formula defining that set. Then in any universe $\mathbb{V}$, $\Phi(x)$ defines a subset of $L[x]$ of size $\omega_1^{L[x]}$ with no perfect subset. Moreover, the definition is uniform – there is a Π^1_1 formula $\Psi(x, y)$ such that for every x,

$$\{y \ : \ \Psi(x, y)\}$$

is a subset of $L[x]$ of size $\omega_1^{L[x]}$ with no perfect subset.

Consider the subset of the plane defined by Ψ. The vertical sections of Ψ are either countable or strictly coanalytic – since we assume ω_1 is inaccessible to the reals, they are all countable and in particular Borel. Use the forcing of fact 4.3 to obtain a ccc extension $\mathbb{V}^{\mathbb{P}}$ with $x \in \mathbb{V}^{\mathbb{P}}$ such that $\omega_1^{L[x]} = \omega_1$. Use the previous proposition to construct a ccc σ-ideal I such that

$$\mathbb{V}[x] = \mathbb{V}^{\mathbb{P}_I}.$$

Obviously, $\mathbb{P}_I \Vdash \omega_1^{L[x_G]} = \omega_1$ for x_G its generic real. In particular, Ψ has a new section which is non-Borel, and cylindrical Borel canonization fails. □

Remark 4.6. The reader is encouraged to compare the above example with the positive results of previous sections. When doing so, note that I is not even defined in L – its definition requires a club in ω_1 of ordinals which are singular cardinals in L.

Corollary 4.7. *Cylindrical Borel canonization for ccc σ-ideals implies that ω_1 is inaccessible to the reals and Mahlo in L.*

Proof. Recall that Hechler forcing is the standard ccc forcing adding a dominating real. By Hechler ideal we refer to the σ-ideal associated to it, in the sense of Proposition 4.4. Theorem 1.7 shows that Hechler ideal has cylindrical Borel canonization if and only if $\mathbf{\Sigma}^1_2$ sets are Hechler measurable. In [2] it is shown that measurability of $\mathbf{\Sigma}^1_2$ sets with respect to the Hechler ideal is equivalent to ω_1 being inaccessible to the reals.

To see that ω_1 is Mahlo in L, use the previous theorem. □

The case of square Borel canonization is different – for A analytic and B Borel, if $A \cap (B \times B)$ is Borel, then $\mathbb{P}_I \Vdash A \cap (B \times B)$ *is Borel* , hence

$$B \Vdash (A_{x_G} \cap B) \text{ is Borel}.$$

In order to construct a counterexample, we can try and find A and I such that no B Borel I-positive forces the Borelness of $A_x \cap B$:

Problem 4.8. Let Ψ and I be as in Theorem 4.5, and let A be the coanalytic subset of the plane defined by Ψ. Can we find $B \in \mathbb{P}_I$ such that $B \Vdash (A_{x_G} \cap B)$ *is Borel*?

4.1. **Non-absoluteness of "all classes are Borel".** The previous example shows that for A an analytic subset of the plane, the property "all vertical sections of A are Borel" can be forced false by a ccc σ-ideal. The same applies for analytic equivalence relations:

Proposition 4.9. *There is an analytic equivalence relation E such that:*

(1) *If ω_1 is inaccessible to the reals and is not Mahlo in L, then all E classes are Borel and there is a ccc σ-ideal I such that*

$$\mathbb{P}_I \Vdash [x_G] \text{ is non-Borel.}$$

(2) *If ω_1 is inaccessible to the reals, then all E classes are Borel, while in L there is a non-Borel class.*

Proof. We use a variation of the example introduced in Theorem 4.5.

Let $\Psi(x, y)$ be as in Theorem 4.5 – a Π^1_1 formula whose vertical sections are subsets of $L[x]$ of size $\omega_1^{L[x]}$ with no perfect subset. Let

$$(x_1, y_1)E(x_2, y_2) \Leftrightarrow (x_1 = x_2) \wedge (((\neg\Psi(x_1, y_1) \wedge \neg\Psi(x_2, y_2)) \vee (y_1 = y_2))).$$

E is an analytic equivalence relation, and the equivalence class of (x_0, y_0) is either a singleton or

$$\{(x_0, y) \ : \ \neg\Psi(x_0, y)\}.$$

Hence if $\neg\Psi(x_0, y_0)$, $[(x_0, y_0)]_E$ is Borel if and only if $\omega_1^{L[x_0]} < \omega_1$.

The 1st clause then follows using the forcing notion introduced in the previous subsection, while the 2nd clause is obvious. □

Remark 4.10. Failure of downward absoluteness of "all classes are Borel" follows from ZFC alone: In L, fix A a coanalytic uncountable set without a perfect subset, and let

$$xEy \Leftrightarrow (x = y) \vee (x, y \notin A).$$

The analytic equivalence relation E has a non-Borel class, but after collapsing ω_1 over L, all its classes become Borel.

Problem 4.11. The nature of the above examples raises the following questions:

(1) Is there an analytic equivalence relation with Borel classes in L but non-Borel classes under large cardinal assumptions?
(2) Can we prove the failure of upward absoluteness of "all classes are Borel" without using the consistency of an inaccessible cardinal?

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