FOURIER FRAMES FOR SINGULAR MEASURES
AND PURE TYPE PHENOMENA

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ABSTRACT. Let μ be a positive measure on $\mathbb{R}^d$. It is known that if the space $L^2(\mu)$ has a frame of exponentials, then the measure μ must be of "pure type": it is either discrete, absolutely continuous or singular continuous. It has been conjectured that a similar phenomenon should be true within the class of singular continuous measures, in the sense that μ cannot admit an exponential frame if it has components of different dimensions. We prove that this is not the case by showing that the sum of an arc length measure and a surface measure can have a frame of exponentials. On the other hand we prove that a measure of this form cannot have a frame of exponentials if the surface has a point of non-zero Gaussian curvature. This is in spite of the fact that each "pure" component of the measure separately may admit such a frame.

1. INTRODUCTION

1.1. Let μ be a positive and finite Borel measure on $\mathbb{R}^d$. By a *Fourier frame* for the space $L^2(\mu)$ one means a system of exponential functions

$$E(\Lambda) = \{e_\lambda\}_{\lambda \in \Lambda}, \quad e_\lambda(x) = e^{2\pi i \langle \lambda, x \rangle},$$

(where Λ is a countable subset of $\mathbb{R}^d$) which constitutes a *frame* in the space. The latter means that there are constants $0 < A, B < \infty$ such that the inequalities

$$A\|f\|_{L^2(\mu)}^2 \leqslant \sum_{\lambda \in \Lambda} \left| \langle f, e_\lambda \rangle_{L^2(\mu)} \right|^2 \leqslant B\|f\|_{L^2(\mu)}^2$$

hold for every $f \in L^2(\mu)$. The existence of a Fourier frame $E(\Lambda)$ for $L^2(\mu)$ allows one to decompose in a "stable" (but generally non-unique) way any function f from the space in a Fourier series $f = \sum_{\lambda \in \Lambda} c_\lambda e_\lambda$ with frequencies belonging to Λ (see [You01]).

For which measures μ does a Fourier frame exist? The origin of this question goes back to Fuglede [Fug74] who initiated a study of orthogonal bases of exponentials over domains in $\mathbb{R}^d$ endowed with Lebesgue measure, and to Jorgensen and Pedersen [JP98] who discovered the existence of fractal measures which admit such orthogonal bases. It is well known, however, that the existence of an orthogonal basis $E(\Lambda)$ for $L^2(\mu)$ is a strong requirement, satisfied by a relatively small class of measures μ. Hence it is of interest to understand whether measures which do not admit such a basis can at least have a frame of exponentials.

1.2. It is known [LW06, HLL13] that if a measure μ has a Fourier frame, then it must be of "pure type": μ is either discrete, absolutely continuous or singular continuous.

The case when μ is a discrete measure is well understood: μ has a Fourier frame if and only if it has finitely many atoms [HLL13]. It is also known precisely which absolutely continuous measures μ have a Fourier frame: this is the case if and only if μ is supported on a set of finite Lebesgue measure in $\mathbb{R}^d$, and its density function is bounded from above and from below almost everywhere on the support [Lai11, DL14, NOU16] (in the last paper the authors used the solution of the Kadison-Singer problem [MSS15] to prove the existence of a Fourier frame in the case when the support is unbounded).

The case when the measure μ is singular and continuous, on the other hand, is much less understood. In this context mainly self-similar measures have been studied (see e.g. [DLW16] and the references therein). However, even the following question [Str00] is still open: does the middle-third Cantor measure have a Fourier frame? (It is known [JP98] that this measure has no orthogonal basis of exponentials.)

1.3. It has been conjectured that a pure type phenomenon should also exist within the class of singular continuous measures. Namely, if such a measure has "components of different dimensions", then it cannot have a Fourier frame. A concrete formulation of such a conjecture was given explicitly in [DL14, Conjecture 5.2].

The paper [DHSW11] contains some results in this direction. These results establish a connection between the fractal dimension of some self-similar measures μ and the "Beurling dimension" of certain Fourier frames for these measures.

However, in the present paper we obtain some results which contradict the above conjecture. In particular, we prove the following:

Theorem 1.1. *There is a measure μ on $\mathbb{R}^d$ ($d \geqslant 3$) which is the sum of the arc length measure on a curve and the area measure on a hypersurface, such that $L^2(\mu)$ has a Fourier frame.*

The result thus shows that a measure with a Fourier frame can nevertheless have components of different dimensions. Actually, the measure μ in our example is simply the sum of the arc length measure on the segment $[0,1] \times \{0\}^{d-1}$ and the area measure on the hypersurface $\{0\} \times [0,1]^{d-1}$. We will obtain Theorem 1.1 as a special case of a more general result (Theorem 2.1) that allows one to construct many examples of "mixed type" measures which have a Fourier frame.

1.4. On the other hand, we will see that under additional geometric assumptions on the components of the measure, one can establish pure type phenomena with respect to the dimension of these components. For example, we will prove the following result:

Theorem 1.2. *Let μ be a measure on $\mathbb{R}^d$ ($d \geqslant 3$) which is the sum of the arc length measure on an open subset of a smooth curve, and the area measure on an open subset of a smooth hypersurface. If the hypersurface has a point of non-zero Gaussian curvature, then $L^2(\mu)$ does not admit a Fourier frame.*

It is instructive to notice the contrast between the curvature requirement in this result and the "flatness" of the hypersurface in the example of Theorem 1.1. We

will deduce Theorem 1.2 from a more general result (Theorem 3.3) that establishes a certain pure type phenomenon within the class of singular continuous measures.

1.5. In order to interpret Theorem 1.2 as a genuine "pure type principle", it is desirable to verify that the assumptions on the measure μ in the theorem nevertheless permit the existence of a Fourier frame for each "pure" component of the measure separately. This would mean that the conclusion that μ admits no Fourier frame is really due to the combination of these components together in the measure μ.

That this is indeed the case can be seen from the following result, which provides many examples of "single dimensional" measures which do admit a Fourier frame:

Theorem 1.3. *Let μ be a measure on $\mathbb{R}^d$, which is the k-dimensional area measure on a compact subset of the graph $\{(x, \varphi(x)) : x \in \mathbb{R}^k\}$ of a smooth function $\varphi : \mathbb{R}^k \to \mathbb{R}^{d-k}$ $(1 \leqslant k \leqslant d-1)$. Then $L^2(\mu)$ has a Fourier frame.*

This result has a quite straightforward proof (see Section 4), but its main point is to clarify that the assumptions in Theorem 1.2 indeed do not prevent the existence of a Fourier frame for each "pure" component of the measure separately.

2. Mixed type measures with a Fourier frame

In this section we present a method for constructing examples of measures which have components of different dimensions, but nevertheless have a Fourier frame. As a special case we will obtain Theorem 1.1.

2.1. Let μ be a measure on $\mathbb{R}^n$, and ν be a measure on $\mathbb{R}^m$. We assume that both measures are positive and finite on their respective spaces. From the two measures μ, ν one can construct a new measure ρ on $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$, defined by

$$(2.1) \qquad \rho = \mu \times \delta_0 + \delta_0 \times \nu,$$

where δ_0 denotes the Dirac measure at the origin. Equivalently, the measure ρ may be defined by the requirement that

$$\int_{\mathbb{R}^n \times \mathbb{R}^m} f(x, y) d\rho(x, y) = \int_{\mathbb{R}^n} f(x, 0) d\mu(x) + \int_{\mathbb{R}^m} f(0, y) d\nu(y)$$

for every continuous, compactly supported function f on $\mathbb{R}^n \times \mathbb{R}^m$.

Notice that ρ is a singular measure, whose support is contained in the union of the two proper subspaces $\mathbb{R}^n \times \{0\}^m$ and $\{0\}^n \times \mathbb{R}^m$ of $\mathbb{R}^n \times \mathbb{R}^m$.

Theorem 2.1. *Assume that μ, ν are continuous measures, and that each one of them has a Fourier frame. Then also the measure ρ given by (2.1) has a Fourier frame.*

For example, if we take μ (respectively ν) to be the Lebesgue measure on the segment $[0, 1]$ (respectively on the cube $[0, 1]^{d-1}$), then the corresponding measure ρ is the sum of the arc length measure on the curve $[0, 1] \times \{0\}^{d-1}$ and the area measure on the hypersurface $\{0\} \times [0, 1]^{d-1}$. Since in this case both measures μ and ν have a Fourier frame (and, in fact, even an orthonormal basis of exponentials), it follows from Theorem 2.1 that also ρ has a Fourier frame. Thus we obtain Theorem 1.1 above.

Generally speaking, the dimensions of the components $\mu \times \delta_0$ and $\delta_0 \times \nu$ of the measure ρ in $\mathbb{R}^n \times \mathbb{R}^m$ (according to any reasonable definition of "dimension") coincide with the respective dimensions of μ in $\mathbb{R}^n$ and ν in $\mathbb{R}^m$. Hence Theorem 2.1

in fact provides a way to construct many examples (including fractal ones) of "mixed type" measures which nevertheless do admit a Fourier frame.

2.2. We now give the proof of Theorem 2.1.

Proof of Theorem 2.1. We assume that μ, ν are continuous measures (that is, both have no discrete part) and that each one of them has a Fourier frame. Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be countable sets such that the exponential system $E(U)$ is a frame in $L^2(\mu)$, and $E(V)$ a frame in $L^2(\nu)$. Hence there exist constants $0 < A, B < \infty$ such that

$$(2.2) \qquad A\|g\|_{L^2(\mu)}^2 \leqslant \sum_{u \in U} |\langle g, e_u \rangle_{L^2(\mu)}|^2 \leqslant B\|g\|_{L^2(\mu)}^2$$

and

$$(2.3) \qquad A\|h\|_{L^2(\nu)}^2 \leqslant \sum_{v \in V} |\langle h, e_v \rangle_{L^2(\nu)}|^2 \leqslant B\|h\|_{L^2(\nu)}^2$$

for every $g \in L^2(\mu)$, $h \in L^2(\nu)$.

Fix a positive integer q satisfying

$$(2.4) \qquad q > \frac{2B}{A}.$$

For each $u \in U$, choose a subset $F(u)$ of V consisting of exactly q elements, in such a way that the sets $F(u)$, $u \in U$, are disjoint subsets of V. This is possible since ν is a continuous measure, hence $L^2(\nu)$ is an infinite dimensional space and so V must be an infinite set. In a completely symmetric way we may choose, for each $v \in V$, a subset $G(v)$ of U consisting of exactly q elements, and such that the sets $G(v)$, $v \in V$, are disjoint subsets of U. Now define

$$\Lambda_\mu := \{(u,v) \,:\, u \in U, \, v \in F(u)\}, \quad \Lambda_\nu := \{(u,v) \,:\, v \in V, \, u \in G(v)\}.$$

It would be convenient to know that Λ_μ and Λ_ν are disjoint subsets of $U \times V$, which we may assume with no loss of generality by an appropriate choice of the sets $F(u)$, $G(v)$.

Finally, let

$$\Lambda := \Lambda_\mu \cup \Lambda_\nu.$$

We claim that the exponential system $E(\Lambda)$ is a frame in $L^2(\rho)$.

Indeed, let $f \in L^2(\rho)$. Then f has a unique decomposition as

$$f(x,y)d\rho = g(x)d(\mu \times \delta_0) + h(y)d(\delta_0 \times \nu),$$

where $g \in L^2(\mu)$, $h \in L^2(\nu)$. This is due to the definition (2.1) of the measure ρ (the uniqueness of this decomposition is true since μ, ν have no atom at the origin). Notice that for any $(u,v) \in \mathbb{R}^n \times \mathbb{R}^m$ we have

$$\langle f, e_{(u,v)} \rangle_{L^2(\rho)} = \langle g, e_u \rangle_{L^2(\mu)} + \langle h, e_v \rangle_{L^2(\nu)}.$$

For any two complex numbers a, b we have

$$(2.5) \qquad |a+b|^2 \geqslant \tfrac{1}{2}|a|^2 - |b|^2,$$

hence

$$\sum_{\lambda \in \Lambda_\mu} |\langle f, e_\lambda \rangle|^2 = \sum_{u \in U} \sum_{v \in F(u)} |\langle g, e_u \rangle + \langle h, e_v \rangle|^2 \geqslant \sum_{u \in U} \sum_{v \in F(u)} \left(\tfrac{1}{2}|\langle g, e_u \rangle|^2 - |\langle h, e_v \rangle|^2\right).$$

Using the fact that the sets $F(u)$ are disjoint subsets of V with exactly q elements each, this implies that

$$\sum_{\lambda \in \Lambda_\mu} |\langle f, e_\lambda \rangle|^2 \geqslant \frac{q}{2} \sum_{u \in U} |\langle g, e_u \rangle|^2 - \sum_{v \in V} |\langle h, e_v \rangle|^2 \geqslant \frac{q}{2} A \|g\|_{L^2(\mu)}^2 - B \|h\|_{L^2(\nu)}^2,$$

where we have used (2.2) and (2.3) in the last inequality. In a completely symmetric way one can also show that

$$\sum_{\lambda \in \Lambda_\nu} |\langle f, e_\lambda \rangle|^2 \geqslant \frac{q}{2} A \|h\|_{L^2(\nu)}^2 - B \|g\|_{L^2(\mu)}^2.$$

Summing the last two inequalities (and using the assumption that Λ_μ and Λ_ν are disjoint sets) we get

$$\sum_{\lambda \in \Lambda} |\langle f, e_\lambda \rangle|^2 = \sum_{\lambda \in \Lambda_\mu} |\langle f, e_\lambda \rangle|^2 + \sum_{\lambda \in \Lambda_\nu} |\langle f, e_\lambda \rangle|^2$$

$$\geqslant \left(\frac{q}{2} A - B \right) \left(\|g\|_{L^2(\mu)}^2 + \|h\|_{L^2(\nu)}^2 \right) = \left(\frac{q}{2} A - B \right) \|f\|_{L^2(\rho)}^2.$$

Due to (2.4) this provides a lower frame bound for the system $E(\Lambda)$ in $L^2(\rho)$.

The upper frame bound can be obtained in a similar way, by using the inequality

$$|a + b|^2 \leqslant 2 \left(|a|^2 + |b|^2 \right)$$

instead of (2.5) in the estimates above. Indeed, we have

$$\sum_{\lambda \in \Lambda_\mu} |\langle f, e_\lambda \rangle|^2 = \sum_{u \in U} \sum_{v \in F(u)} |\langle g, e_u \rangle + \langle h, e_v \rangle|^2 \leqslant 2 \sum_{u \in U} \sum_{v \in F(u)} \left(|\langle g, e_u \rangle|^2 + |\langle h, e_v \rangle|^2 \right)$$

$$\leqslant 2q \sum_{u \in U} |\langle g, e_u \rangle|^2 + 2 \sum_{v \in V} |\langle h, e_v \rangle|^2 \leqslant 2qB \|g\|_{L^2(\mu)}^2 + 2B \|h\|_{L^2(\nu)}^2.$$

Similarly one can verify that

$$\sum_{\lambda \in \Lambda_\nu} |\langle f, e_\lambda \rangle|^2 \leqslant 2qB \|h\|_{L^2(\nu)}^2 + 2B \|g\|_{L^2(\mu)}^2,$$

and summing the last two inequalities we get the estimate from above

$$\sum_{\lambda \in \Lambda} |\langle f, e_\lambda \rangle|^2 \leqslant 2(q + 1)B \|f\|_{L^2(\rho)},$$

which confirms that the exponential system $E(\Lambda)$ is indeed a frame for $L^2(\rho)$. $\square$

2.3. Although Theorem 2.1 allows us to construct many examples of "mixed type" measures with a Fourier frame, there exist certain limitations on the possible dimensions of the components of the measure in (2.1). For instance, the theorem does not tell us whether the measure in $\mathbb{R}^4$ which is the sum of the 2-dimensional area measure on $[0, 1]^2 \times \{0\}^2$ and the 3-dimensional area measure on $\{0\} \times [0, 1]^3$ has a Fourier frame.

However, more general examples are easy to construct by taking the product $\rho \times \sigma$ of the measure ρ in (2.1) with any measure σ in $\mathbb{R}^l$ ($l \geqslant 1$) such that also σ has a Fourier frame. For it is a general fact that if $E(\Lambda)$ is a frame in $L^2(\rho)$ and $E(\Gamma)$ a frame in $L^2(\sigma)$, then $E(\Lambda \times \Gamma)$ is a frame in $L^2(\rho \times \sigma)$, hence $\rho \times \sigma$ also has a Fourier frame.

For example, in this way one can obtain the following extension of Theorem 1.1 above:

Theorem 2.2. *Consider the measure on $\mathbb{R}^d$ which is the sum of the k-dimensional area measure on $[0,1]^k \times \{0\}^{d-k}$, and the j-dimensional area measure on $\{0\}^{d-j} \times [0,1]^j$ $(1 \leqslant j, k \leqslant d-1)$. Then this measure has a Fourier frame.*

Indeed, the measure in this result is (after permutation of the coordinates) of the form $\rho \times \sigma$, where ρ is given by (2.1) and μ, ν and σ are respectively the n, m and p-dimensional area measures on $[0,1]^n$, $[0,1]^m$ and $[0,1]^p \times \{0\}^q$ for appropriate values of n, m, p and q. Since in this case each one of μ, ν and σ has a Fourier frame, Theorem 2.2 follows. (In particular, for $d = 4$, $k = 2$ and $j = 3$ we obtain an affirmative answer to the question mentioned above.)

3. Pure type phenomena for singular continuous measures

The results in the previous section contradict the existence of a general pure type principle within the class of singular continuous measures. However, our goal in the present section is to demonstrate that such principles can nevertheless be established under some extra assumptions on the components of the measure. In particular we will prove Theorem 1.2 above.

3.1. Let μ be a positive, finite measure on $\mathbb{R}^d$. Given a real number α, $0 \leqslant \alpha \leqslant d$, we consider the following condition:

$$(3.1) \qquad \liminf_{r \to \infty} \frac{1}{r^{d-\alpha}} \int_{|t|<r} |\widehat{\mu}(t)|^2 dt > 0,$$

where

$$\widehat{\mu}(t) = \int_{\mathbb{R}^d} e^{-2\pi i \langle t, x \rangle} d\mu(x), \quad t \in \mathbb{R}^d,$$

is the Fourier transform of the measure μ.

Intuitively, we think of condition (3.1) in some sense as saying that μ has "at least one component of dimension not greater than α". For example, if μ is an absolutely continuous measure with an L^2 density, then (3.1) is satisfied only for $\alpha = d$, by the Plancherel identity. At the other extreme, by the classical Wiener's lemma, the condition (3.1) holds with $\alpha = 0$ if and only if μ has at least one non-zero atom (see [Str90a]).

An example of a singular continuous measure μ satisfying (3.1) for some $\alpha < d$ is given by the area measure on an open subset of a smooth k-dimensional submanifold in $\mathbb{R}^d$ $(1 \leqslant k \leqslant d-1)$. In this case the condition (3.1) holds for $\alpha = k$ [AH76, Str90a]. It was also proved in [Str90a] that (3.1) is true if μ is the α-dimensional Hausdorff measure on certain self-similar fractals of dimension α (in the latter case α need not be an integer). See also [Str90b, Str93a, Str93b] for further results in this connection.

3.2. Recall that a system of vectors $\{f_n\}$ in a Hilbert space H is called a *Bessel system* if there is a positive constant C such that the inequality

$$(3.2) \qquad \sum_n |\langle f, f_n \rangle|^2 \leqslant C \|f\|^2$$

(Bessel's inequality) is satisfied for every $f \in H$.

Lemma 3.1. *Suppose that a system of exponentials $E(\Lambda)$ constitutes a Bessel system in $L^2(\mu)$, where μ is a measure satisfying (3.1) for some α, $0 \leqslant \alpha \leqslant d$. Then*

$$\sup_{x \in \mathbb{R}^d} \# \big(\Lambda \cap B(x, r) \big) \leqslant C r^\alpha \quad (r \geqslant 1) \tag{3.3}$$

for a certain constant C which does not depend on r.

Here and below $B(x, r)$ denotes the open ball of radius r centered at the point x.

Lemma 3.1 should be compared to [DHSW11, Theorem 3.5(a)] where the estimate (3.3) was proved for exponential Bessel systems $E(\Lambda)$ on certain self-similar measures μ of fractal dimension α. For such measures which also satisfy condition (3.1) we obtain the latter result as a special case of Lemma 3.1.

Proof of Lemma 3.1. Assume that $E(\Lambda)$ is a Bessel system in $L^2(\mu)$. Then applying Bessel's inequality (3.2) to the functions e_t $(t \in \mathbb{R}^d)$ we obtain

$$\sum_{\lambda \in \Lambda} |\widehat{\mu}(t - \lambda)|^2 \leqslant C_1, \quad t \in \mathbb{R}^d, \tag{3.4}$$

for a certain constant C_1. Integrating this inequality over the ball $B(x, 2r)$ yields

$$\int_{B(x,2r)} \sum_{\lambda \in \Lambda} |\widehat{\mu}(t - \lambda)|^2 \, dt \leqslant C_2 \, r^d. \tag{3.5}$$

On the other hand, we have

$$\int_{B(x,2r)} \sum_{\lambda \in \Lambda} |\widehat{\mu}(t - \lambda)|^2 \, dt$$

$$= \sum_{\lambda \in \Lambda} \int_{B(x-\lambda,2r)} |\widehat{\mu}(t)|^2 \, dt \geqslant \sum_{\lambda \in \Lambda \cap B(x,r)} \int_{B(x-\lambda,2r)} |\widehat{\mu}(t)|^2 \, dt.$$

Notice that for each $\lambda \in \Lambda \cap B(x, r)$, the ball $B(x - \lambda, 2r)$ contains $B(0, r)$. Hence

$$\int_{B(x,2r)} \sum_{\lambda \in \Lambda} |\widehat{\mu}(t - \lambda)|^2 \, dt \geqslant \# \big(\Lambda \cap B(x, r) \big) \int_{|t|<r} |\widehat{\mu}(t)|^2 \, dt.$$

Due to the assumption (3.1), the integral on the right hand side of the last inequality is not less than $C_3 \, r^{d-\alpha}$ for all sufficiently large r, for an appropriate constant $C_3 > 0$. Combining this with (3.5) we obtain the assertion of the lemma. $\qquad\square$

3.3. Given a set $\Lambda \subset \mathbb{R}^d$, one defines its *upper Beurling dimension* to be the infimum of the numbers α for which there exists a constant C such that (3.3) holds (see [CKS08]). Lemma 3.1 thus shows that if $E(\Lambda)$ is a Bessel system for a measure μ satisfying the condition (3.1) for some α, then the upper Beurling dimension of Λ cannot exceed α.

If the exponential system $E(\Lambda)$ is not just a Bessel system, but moreover is a frame in $L^2(\mu)$ for some "α-dimensional" measure μ, then in the spirit of the classical Landau's results [Lan67] one might expect that the upper Beurling dimension of Λ should in fact be equal to α. A result of this type was proved in [DHSW11, Theorem 3.5(b)], where the authors showed that for certain self-similar measures μ, if $E(\Lambda)$ is a frame in $L^2(\mu)$ and if the set Λ is assumed to enjoy a certain structure,

then the upper Beurling dimension of Λ indeed coincides with the fractal dimension of the measure.

However, in general this is not the case. As a simple example one may take μ to be the arc length measure on the segment $[0, 1] \times \{0\}$ in $\mathbb{R}^2$. Then μ is a 1-dimensional measure, and it is easy to verify that if $\Lambda = \{(n, 2^{|n|}) : n \in \mathbb{Z}\}$, then the system $E(\Lambda)$ constitutes an orthonormal basis in $L^2(\mu)$. Nevertheless, the estimate

$$\sup_{x \in \mathbb{R}^2} \#\big(\Lambda \cap B(x, r)\big) \leqslant C \log r$$

holds for all sufficiently large r, and in particular, the upper Beurling dimension of Λ is zero. In a similar way one can construct examples of this type in $\mathbb{R}^d$ for any $d \geqslant 2$. See also [DHL13] where such an example for fractal measures on $\mathbb{R}$ was given.

3.4. On the other hand, if we impose extra assumptions on the measure μ, then we can show that if $E(\Lambda)$ is a frame in $L^2(\mu)$, then (3.3) cannot hold for arbitrarily small α.

For a real number β, $0 < \beta \leqslant d$, consider the following condition on the measure μ:

(3.6)   there exists a function $\varphi \in L^2(\mu)$, $\|\varphi\|_{L^2(\mu)} > 0$, such that
$$|(\varphi\mu)^\wedge(t)| \leqslant C\,|t|^{-\beta/2}, \quad t \in \mathbb{R}^d,$$

for some positive constant C. Notice that $\varphi\mu$ is a (non-zero) finite, complex measure on $\mathbb{R}^d$, since $\varphi \in L^2(\mu)$ and μ is a finite measure.

It is known that (at least, for non-negative φ) the estimate in (3.6) implies that the measure $\varphi\mu$ cannot charge any compact set of Hausdorff dimension smaller than β (see for example [Mat15, Sections 2.5, 3.5, 3.6]). This means, intuitively, that μ has "at least one component of dimension not less than β".

A well-known example of a singular measure μ satisfying (3.6) for some $\beta > 0$ is the area measure on the unit sphere in $\mathbb{R}^d$, or, more generally, on an open subset Ω of a smooth hypersurface in $\mathbb{R}^d$ with a point of non-zero Gaussian curvature. In this case one may take φ to be any smooth function supported by a sufficiently small neighborhood of a point $x_0 \in \Omega$ where the Gaussian curvature does not vanish, and the estimate in (3.6) then holds with $\beta = d - 1$ (see e.g. [Ste93, Section VIII.3]).

It should be mentioned that the theory of *Salem sets* is concerned with various constructions of sets in $\mathbb{R}^d$ of Hausdorff dimension β which support a measure μ satisfying (3.6). See [Mat15, Section 3.6] and the references mentioned there.

Lemma 3.2. *Suppose that a system of exponentials $E(\Lambda)$ is a frame in $L^2(\mu)$, where μ is a measure satisfying (3.6) for some β, $0 < \beta \leqslant d$. If there is α such that*

$$(3.7) \qquad\qquad \#\big(\Lambda \cap B(0, r)\big) \leqslant Cr^\alpha \quad (r \geqslant 1)$$

for some constant C, then we must have

$$(3.8) \qquad\qquad \alpha \geqslant \left(\frac{1}{\beta} + \frac{1}{d}\right)^{-1}.$$

Proof. By the assumption (3.6) there exists a function $\varphi \in L^2(\mu)$, $\|\varphi\|_{L^2(\mu)} > 0$, such that the (non-zero, finite, complex) measure $\nu := \varphi\mu$ satisfies

$$(3.9) \qquad\qquad |\widehat{\nu}(t)| \leqslant C_1\,|t|^{-\beta/2}, \quad t \in \mathbb{R}^d.$$

We also assume that the system $E(\Lambda)$ is a frame in $L^2(\mu)$. Then applying the lower frame inequality to the functions $\overline{\varphi} \cdot e_t$ $(t \in \mathbb{R}^d)$ yields

$$(3.10) \qquad \sum_{\lambda \in \Lambda} |\widehat{\nu}(t - \lambda)|^2 \geqslant C_2 > 0, \quad t \in \mathbb{R}^d.$$

Suppose now to the contrary that (3.7) does hold for some α such that

$$(3.11) \qquad \alpha < \left(\frac{1}{\beta} + \frac{1}{d} \right)^{-1}.$$

Then this allows us to choose a real number γ satisfying

$$(3.12) \qquad \frac{\alpha}{\beta} < \gamma < 1 - \frac{\alpha}{d}.$$

Given a large number R we define $T := R^\gamma$ and consider the union of balls $B(\lambda, T)$ of radius T centered at the points $\lambda \in \Lambda \cap B(0, R)$. Then the d-dimensional volume of the union of these balls is not greater than

$$\#\big(\Lambda \cap B(0, R)\big) \cdot |B(0, T)| \leqslant C_3 \, R^\alpha \, T^d = C_3 \, R^{\alpha + \gamma d} = o(R^d),$$

due to (3.7) and (3.12). Hence if R is sufficiently large, then these balls cannot cover $B(0, R/2)$. So we may find a point $t_0 \in B(0, R/2)$ such that $|t_0 - \lambda| \geqslant T$, $\lambda \in \Lambda \cap B(0, R)$. We will show that the inequality (3.10) is then violated at this point t_0.

Indeed, we have

$$\sum_{\lambda \in \Lambda, \, |\lambda| < R} |\widehat{\nu}(t_0 - \lambda)|^2 \leqslant \#\big(\Lambda \cap B(0, R)\big) \cdot \sup_{|t| \geqslant T} |\widehat{\nu}(t)|^2 \leqslant C_4 \, R^\alpha \, T^{-\beta} = C_4 \, R^{\alpha - \gamma \beta},$$

according to (3.7) and (3.9). We also have

$$\sum_{\lambda \in \Lambda, \, |\lambda| \geqslant R} |\widehat{\nu}(t_0 - \lambda)|^2 = \sum_{k=0}^{\infty} \sum_{2^k R \leqslant |\lambda| < 2^{k+1} R} |\widehat{\nu}(t_0 - \lambda)|^2$$

$$\leqslant \sum_{k=0}^{\infty} \#\big(\Lambda \cap B(0, 2^{k+1} R)\big) \cdot \sup_{|\lambda| \geqslant 2^k R} |\widehat{\nu}(t_0 - \lambda)|^2$$

$$\leqslant C_5 \sum_{k=0}^{\infty} \big(2^{k+1} R\big)^\alpha \big(2^{k-1} R\big)^{-\beta} = C_6 \, R^{\alpha - \beta},$$

where we have used (3.7), (3.9) and the fact that (3.11) implies that $\alpha < \beta$. It follows that

$$(3.13) \qquad \sum_{\lambda \in \Lambda} |\widehat{\nu}(t_0 - \lambda)|^2 \leqslant C_4 \, R^{\alpha - \gamma \beta} + C_6 \, R^{\alpha - \beta}.$$

Due to (3.12) we have $\alpha - \gamma \beta < 0$, hence the right hand side of (3.13) can be made as small as we wish provided that R is chosen sufficiently large. But this contradicts (3.10), and so the lemma is proved. $\qquad \square$

3.5. We can now combine the previous lemmas to obtain the main result of this section, which establishes a pure type principle for singular continuous measures.

Theorem 3.3. *Let μ be a measure on $\mathbb{R}^d$ which satisfies both conditions* (3.1) *and* (3.6) *for some $0 < \alpha, \beta \leqslant d$. If*

$$(3.14) \qquad \frac{1}{\alpha} - \frac{1}{\beta} > \frac{1}{d},$$

then $L^2(\mu)$ does not admit a Fourier frame.

We view this result as a pure type principle, since the assumptions mean, intuitively, that μ has both a component of dimension not greater than α, and (in a certain strong sense) a component of dimension not less than β, and the conclusion is that μ cannot have a Fourier frame if α and β are sufficiently far apart.

Proof of Theorem 3.3. Suppose to the contrary that $L^2(\mu)$ does have a frame $E(\Lambda)$. Then by Lemma 3.1 the set Λ must satisfy (3.3). In turn, Lemma 3.2 implies that the numbers α, β must satisfy the relation (3.8). However this contradicts (3.14). □

As an application of Theorem 3.3, we can conclude the following result:

Theorem 3.4. *Let μ be a measure on $\mathbb{R}^d$ $(d \geqslant 3)$ which is the sum of the area measure on an open subset of a smooth k-dimensional submanifold, where*

$$(3.15) \qquad 1 \leqslant k \leqslant \left\lfloor \frac{d-1}{2} \right\rfloor,$$

and the area measure on an open subset of a smooth hypersurface with a point of non-zero Gaussian curvature. Then $L^2(\mu)$ does not admit a Fourier frame.

Here $\lfloor x \rfloor$ denotes the greatest integer which is less than or equal to a real number x.

Proof of Theorem 3.4. The assumptions imply that the measure μ satisfies condition (3.1) with $\alpha = k$ (see [Str90a, Theorem 5.5]) and condition (3.6) with $\beta = d-1$ (see [Ste93, Section VIII.3.1]). Moreover, it follows from (3.15) that condition (3.14) is satisfied for these values of α and β. Hence the assertion follows from Theorem 3.3. □

In the special case when $k = 1$ in Theorem 3.4 we obtain Theorem 1.2.

4. Fourier frames for surface measures

In the previous section we proved that if a measure μ in $\mathbb{R}^d$ is the sum of the area measure on a k-dimensional submanifold (at least, for k not too large) and the area measure on a hypersurface with a point of non-zero Gaussian curvature, then μ does not have a Fourier frame. Our main goal in the present section is to show that in many examples, a Fourier frame does exist for each "pure" component of such a measure separately.

4.1. We will consider a measure μ in $\mathbb{R}^d$ which is the area measure on a k-dimensional submanifold defined as a *graph*. Let $\varphi : U \to \mathbb{R}^{d-k}$ $(1 \leqslant k \leqslant d-1)$ be a smooth function defined on an open set $U \subset \mathbb{R}^k$. Let E be a compact subset of U, and let

$$(4.1) \qquad \Gamma = \Gamma(\varphi, E) = \{(x, \varphi(x)) : x \in E\}$$

be the *graph of φ over E*. Then Γ is a compact subset of a k-dimensional submanifold in $\mathbb{R}^d = \mathbb{R}^k \times \mathbb{R}^{d-k}$, and so it admits an area measure μ induced from its embedding in $\mathbb{R}^k \times \mathbb{R}^{d-k}$. The measure μ can be defined by the requirement that

$$(4.2) \qquad \int_\Gamma f(x, y) d\mu(x, y) = \int_E f(x, \varphi(x)) w(x) dx$$

for every continuous function f on Γ, where $w(x) > 0$ is a certain smooth weight function which depends on the function φ. Actually, it will be sufficient for us to assume that $w(x)$ is an arbitrary continuous, strictly positive function on E.

Theorem 4.1. *The measure μ on $\mathbb{R}^d$ defined by (4.2) has a Fourier frame.*

Proof. In fact, it will be easy to verify that if $\delta > 0$ is sufficiently small and if

$$(4.3) \qquad \Lambda = (\delta\mathbb{Z})^k \times \{0\}^{d-k},$$

then the system $E(\Lambda)$ is a frame in $L^2(\mu)$. Indeed, fix $\delta > 0$ small enough such that E is contained in the k-dimensional cube

$$I_\delta^k := \left[-\frac{1}{2\delta}, \frac{1}{2\delta} \right]^k,$$

and let Λ be defined by (4.3). Given $f \in L^2(\mu)$, we have

$$\sum_{\lambda \in \Lambda} |\langle f, e_\lambda \rangle_{L^2(\mu)}|^2 = \sum_{m \in \mathbb{Z}^k} \left| \int_E f(x, \varphi(x)) e^{-2\pi i \langle \delta m, x \rangle} w(x) dx \right|^2$$

$$(4.4) \qquad\qquad = C \int_E |f(x, \varphi(x))|^2 w(x)^2 dx,$$

where the last equality is true with a certain constant $C = C(k, \delta)$ since the system of exponentials $E(\delta\mathbb{Z}^k)$ forms an orthogonal basis in the space $L^2(I_\delta^k)$ with respect to the Lebesgue measure, and since E is contained in I_δ^k. On the other hand, we have

$$(4.5) \qquad \|f\|_{L^2(\mu)}^2 = \int_E |f(x, \varphi(x))|^2 w(x) dx.$$

Since $w(x)$ is a continuous, strictly positive function on the compact set E, it is bounded from above and from below on E. Hence the ratio between (4.4) and (4.5) is also bounded from above and from below by certain positive constants not depending on f. This confirms that the system $E(\Lambda)$ is indeed a frame in $L^2(\mu)$. $\square$

As a consequence, we obtain Theorem 1.3 above.

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4.2. In the case when $k = d - 1$, the function φ is scalar-valued, and Γ is then a subset of a hypersurface in $\mathbb{R}^d$. We remark that the requirement in Theorem 3.4 that the hypersurface has a point of non-zero Gaussian curvature is easy to express in terms of the function φ: this is the case if and only if the $(d-1) \times (d-1)$ matrix given by

$$\left(\frac{\partial^2 \varphi}{\partial x_j \partial x_k} \right)(x)$$

is invertible at some point $x \in U$.

5. Open problems

Finally we mention some questions concerning possible extensions of our results.

5.1. Notice that Theorem 2.1 allows us to construct "mixed type" measures with a Fourier frame only in dimensions 2 and higher. It would be interesting to know whether examples of this type can be found on $\mathbb{R}$ as well. Specifically: can one construct examples of two self-similar Cantor measures μ and ν on $\mathbb{R}$ of different dimensions, such that their sum $\mu + \nu$ is a measure with a Fourier frame?

Here the main point in the proof of Theorem 2.1 may still be useful: it would be enough to show that given any positive number q there is a set Λ_μ, such that the exponential system $E(\Lambda_\mu)$ is a frame in $L^2(\mu)$ with lower frame bound not less than q, and at the same time $E(\Lambda_\mu)$ constitutes a Bessel system in $L^2(\nu)$ with Bessel constant bounded from above independently of q; and similarly, to show that there is also a set Λ_ν such that the system $E(\Lambda_\nu)$ has the same properties but with the roles of μ and ν interchanged. Indeed, in this case the proof of Theorem 2.1 shows that a frame $E(\Lambda)$ for $L^2(\mu + \nu)$ can be obtained by choosing q sufficiently large and taking $\Lambda = \Lambda_\mu \cup \Lambda_\nu$.

5.2. It would also be interesting to understand to what extent Theorem 3.3 is sharp. Can the requirement (3.14) be relaxed to $\alpha < \beta$? Analogously, can one relax the requirement (3.15) in Theorem 3.4 to $1 \leqslant k \leqslant d - 2$?

In the same spirit, one may ask whether it is possible to get a version of Theorem 3.4 for two submanifolds in $\mathbb{R}^d$ of different dimensions, but such that not necessarily one of the dimensions is equal to $d - 1$. What geometric condition would replace the curvature requirement in such a result? (We know that some condition of this sort is necessary, due to Theorem 2.2.)

5.3. Another question is concerned with the assumption in Theorem 4.1 that the submanifold is globally contained in the graph of some function. Can this assumption be removed? In particular, we do not know whether the area measure on the unit sphere

$$S^{d-1} = \{x \in \mathbb{R}^d \,:\, |x| = 1\}$$

has a Fourier frame ($d \geqslant 2$). (Theorem 4.1 only tells us that the restriction of the area measure to a sufficiently small *spherical cap* admits such a frame.)

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