PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 7, July 2018, Pages 3049–3056
http://dx.doi.org/10.1090/proc/13859
Article electronically published on March 14, 2018

FOUR-DIMENSIONAL GRADIENT SHRINKING SOLITONS WITH PINCHED CURVATURE

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(Communicated by Guofang Wei)

ABSTRACT. We show that any four-dimensional gradient shrinking soliton with pinched Weyl curvature (∗) and satisfying $c_1 \leq R \leq c_2$ for some positive constant c_1 and c_2, will have nonnegative Ricci curvature. As a consequence, we prove that it must be a finite quotient of $\mathbb{S}^4$, $\mathbb{CP}^2$, or $\mathbb{S}^3 \times \mathbb{R}$. In particular, a compact four-dimensional gradient shrinking soliton with pinched Weyl curvature (∗) must be $\mathbb{S}^4$, RP^4 or $\mathbb{CP}^2$.

1. INTRODUCTION

A Riemannian manifold (M, g), couple with a smooth function f, is called a gradient Ricci soliton, if

$$R_{ij} + f_{ij} = \rho g_{ij}$$

holds for some constant ρ. The soliton is called shrinking, steady, or expanding, if $\rho > 0$, $\rho = 0$, or $\rho < 0$, respectively. Gradient shrinking solitons (GSS for short) play an important role in the singularity analysis of Ricci flow. We refer the readers to [2] for a quick overview and more information.

To understand singularity of Ricci flow, we should try to get a classification of GSS. In dimension $n \leq 3$, GSS are well understood by the work of Hamilton [9] for the two-dimensional case, and the work of Ivey [11], Perelman [17], Ni-Wallach [16], Naber [15] and Cao-Chen-Zhu [3] for the three-dimensional case.

Obviously, the Weyl tensor vanishes in dimension three. So it is natural to consider GSS with vanishing Weyl tensor in higher dimension. Indeed, by the work of Ni-Wallach [16], Petersen-Wylie [18], Cao-Wang-Zhang [4], and the author [22], we can give a complete classification of GSS with vanishing Weyl tensor.

However, our understanding of GSS in higher dimension is still very limited. Recently, there were some classification results on the GSS with some assumptions of the Weyl tensor. For example, Chen-Wang [7] could classify four-dimensional anti-self-dual GSS, and Wu-Wu-Wylie [20] dealt with GSS with half harmonic Weyl tensor. If the GSS satisfies a pinched Weyl curvature, Catino obtained a well classification result in a recent work [1].

In past work, a key fact to studying the GSS was that the soliton had some nonnegative curvature conditions. In this paper, we continue to consider solitons with a pinched Weyl curvature; then we can show that the Ricci curvature will be nonnegative, and then we can prove a classification result.

Received by the editors December 5, 2015, and, in revised form, June 9, 2017.
2010 *Mathematics Subject Classification.* 53C44.
Key words and phrases. Ricci flow, maximum principle, pinched Weyl tensor.
The author was supported in part by NSFC 11301191 and NSFC 11371377.

Before we state our results, we give some notation first. Let (M^n, g) be a complete Riemannian manifold. Denote by Ric the Ricci tensor with components R_{ij} and $R = g^{ij}R_{ij}$ will denote the scalar curvature. Then the Riemannian curvature tensor Rm can be decomposed into the orthogonal components:

$$Rm = W \oplus \frac{2}{n-2}\mathring{Ric} \wedge g \oplus \frac{R}{n(n-1)} g \wedge g,$$

where W and $\mathring{Ric} = Ric - \frac{1}{n}g$ denote, respectively, the Weyl tensor and the traceless Ricci tensor. Our first result is the following.

Theorem 1.1. *Let (M^4, g) be a complete four-dimensional GSS with $c_1 \leq R \leq c_2$ for some positive constant c_1 and c_2, and satisfy the pinched condition*

$$(*) \qquad |W| \leq \sqrt{2}\Big||\mathring{Ric}| - \frac{1}{2\sqrt{3}}R\Big|.$$

Then the Ricci curvature will be nonnegative.

Remark 1.2. On round cylinder $\mathbb{S}^3 \times \mathbb{R}$ with scalar curvature $R \equiv 1$, the Weyl tensor vanishes, so the pinched condition $(*)$ holds. Because the round cylinder only has nonnegative Ricci curvature, our estimate is optimal in some sense.

As an application, we obtain the following classification of gradient shrinking solitons.

Theorem 1.3. *Under the same conditions as Theorem* 1.1*, the soliton must have nonnegative isotropy curvature.*

Furthermore, it must be a finite quotient of $\mathbb{S}^4$, $\mathbb{S}^3 \times \mathbb{R}$, or $\mathbb{CP}^2$.

Remark 1.4. In [1], Catino showed a classification result of n-dimensional GSS with nonnegative Ricci curvature satisfying a pinched condition

$$|W|R \leq \sqrt{\frac{2(n-1)}{n-2}}\Big(|\mathring{Ric}| - \frac{1}{\sqrt{n(n-1)}}R\Big)^2,$$

which implies that $|W| \leq \Big||\mathring{Ric}| - \frac{1}{2\sqrt{3}}R\Big|$ in dimension four. So our result can be considered as an extension of Catino's theorem on dimension four.

In particular, any nonflat compact GSS will have $R \geq c_1 > 0$. So we can obtain the following result.

Theorem 1.5. *Let (M^4, g) be a compact four-dimensional GSS satisfying the pinched condition $(*)$; then it must be $\mathbb{S}^4$, RP^4 or $\mathbb{CP}^2$.*

2. Preliminaries

Le (M^4, g_{ij}) be a complete Riemannian manifold with bounded curvature. We consider the Ricci flow equation on M^4

$$\begin{cases} \frac{\partial g_{ij}(x,t)}{\partial t} = -2R_{ij}(x,t), & x \in M^4, t > 0, \\ g_{ij}(x,0) = g_{ij}(x), & x \in M^4. \end{cases}$$

Since the curvature is bounded at the initial metric, it is well known [19] that there exists a complete solution $g(t)$ of the Ricci flow on a time interval $[0, T)$ with bounded curvature for each t. On the other hand, the Ricci curvature tensor R_{ij} and the scalar curvature R evolve by the (PDE) system (cf. Hamilton [8]):

$$\text{(PDE)} \qquad \begin{cases} \frac{\partial}{\partial t}R_{ij} &= \triangle R_{ij} + 2\sum_{k,l} R_{ikjl}R_{kl}, \\ \frac{\partial}{\partial t}R &= \triangle R + 2|Ric|^2. \end{cases}$$

Now we want to give a basic estimate of the least eigenvalue of Ricci tensor. Recall that a tensor evolved by a nonlinear heat equation may be controlled by a corresponding (ODE) system (cf. Hamilton [8]), while the (ODE) system corresponding to the above (PDE) is the following:

$$\text{(ODE)} \qquad \begin{cases} \frac{d}{dt}R_{ij} &= 2\sum_{k,l} R_{ikjl}R_{kl}, \\ \frac{d}{dt}R &= 2|Ric|^2. \end{cases}$$

If we diagonalize the Ricci tensor with the eigenvalue $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$, and let $\lambda = \frac{1}{3}(\lambda_2 + \lambda_3 + \lambda_4)$, $\delta_k = \lambda_k - \lambda$, then we have the following lemma:

Lemma 2.1. *Under the (ODE) system, we have*

$$\begin{cases} \frac{1}{2}\frac{d}{dt}R = & \lambda_1^2 + 3\lambda^2 + \sum_{k=2}^{4} \delta_k^2, \\ \frac{1}{2}\frac{d}{dt}\lambda_1 \geq & \lambda\lambda_1 + \frac{1}{2}\sum_k \delta_k^2 - \frac{1}{2\sqrt{2}}|W| \cdot \sqrt{\sum_k \delta_k^2}. \end{cases}$$

Proof. The fact $\sum_{k=2}^{4} \delta_k = 0$ implies that

$$\frac{1}{2}\frac{d}{dt}R = \sum_{i=1}^{4} \lambda_i^2 = \lambda_1^2 + \sum_{k=2}^{4}(\lambda + \delta_k)^2 = \lambda_1^2 + 3\lambda^2 + \sum_{k=2}^{4} \delta_k^2.$$

Thus the first equation holds.

On the other hand, since $R_{ijij} = W_{ijij} + \frac{\lambda_i + \lambda_j}{2} - \frac{R}{6}$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\lambda_1 \geq &\sum_{k=2}^{4} \lambda_k \Big(W_{1k1k} + \frac{\lambda_1 + \lambda_k}{2} - \frac{R}{6}\Big) \\ =&\frac{1}{2}\sum_k \lambda_k(\lambda_1 + \lambda_k - \frac{R}{3}) + \sum_k \lambda_k W_{1k1k}. \end{aligned}$$

Note that the scalar curvature $R = \lambda_1 + 3\lambda$, and $\sum_{k=2}^{4} W_{1k1k} = 0$, thus

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\lambda_1 &\geq \frac{1}{2}\sum_k(\lambda + \delta_k)\cdot(\frac{2}{3}\lambda_1 + \delta_k) + \sum_k(\lambda + \delta_k)W_{1k1k} \\ &= \lambda\lambda_1 + \frac{1}{2}\sum_k \delta_k^2 + \sum_k \delta_k W_{1k1k} \\ &\geq \lambda\lambda_1 + \frac{1}{2}\sum \delta_k^2 - \sqrt{\sum \delta_k^2}\cdot\sqrt{\sum W_{1k1k}^2}. \end{aligned}$$

So the second inequality follows by the fact that $|W|^2 \geq 8\sum W_{1k1k}^2$ on dimension $n = 4$. ∎

3. The Ricci curvature on GSS with pinched curvature

In this section, we will show a pinched estimate of the Ricci curvature, and then prove Theorem 1.1.

Lemma 3.1. *Suppose we have a complete solution of Ricci flow $g(t)_{t\in[0,T]}$ on a four-manifold with uniformly bounded curvature, satisfying the pinched condition $(*)$ for all $t \in [0,T]$.*

If $R \geq c_0$ and $\eta_0 = \inf\limits_{x\in M} \frac{\lambda_1(x)}{R(x)}$ holds for some constant $c_0 > 0$ and $\eta_0 < 0$ at time $t = 0$, λ_1 is the least eigenvalue of Ricci curvature tensor. Then by choosing a positive constant $\delta = \min\{1, \frac{1}{3}c_0(-\eta_0)^3\}$, the pinched estimate

$$\lambda_1 \geq (\eta_0 + \delta t)R$$

holds for all $t \in [0, T']$, where $T' = \min\{T, \frac{-\eta_0}{2}\}$.

Proof. Consider the set $\Omega(t)_{t\in[0,T']}$ of matrices defined by the inequalities

$$\Omega(t) : \begin{cases} R \geq c_0, \\ \lambda_1 \geq (\eta_0 + \delta t)R. \end{cases}$$

It is easy to see that Ω is closed, convex and $O(n)$-invariant. By using the advanced maximum principle, we only need to show the set Ω is preserved by the (ODE) system. Indeed, we only need to look at points on the boundary of the set.

From the (ODE) system, we have

$$\frac{d}{dt}R = 2|Ric|^2 \geq 0,$$

which implies that $R \geq c_0$ for all $t \geq 0$. Thus the first inequality is preserved. To prove the second inequality, we only need to show that

$$\frac{1}{2}\lambda_1' \geq (\eta_0 + \delta t)\frac{1}{2}R' + \frac{\delta}{2}R = \eta \cdot \frac{1}{2}R' + \frac{\delta}{2}R$$

where $\lambda_1 = (\eta_0 + \delta t)R = \eta R$.

From Lemma 2.1, it suffices to show that

$$I = \lambda\lambda_1 + \frac{1}{2}\sum_k \delta_k^2 - \frac{1}{2\sqrt{2}}|W| \cdot \sqrt{\sum_k \delta_k^2} - \eta \cdot \left(\lambda_1^2 + 3\lambda^2 + \sum_k \delta_k^2\right) \geq \frac{\delta}{2}R.$$

Since $\lambda_1 = \eta R$ and $\lambda = \frac{1-\eta}{3}R$, we have

$$\begin{aligned} I =& \eta R^2\Big[\frac{1-\eta}{3} - \eta^2 - \frac{(1-\eta)^2}{3}\Big] + \Big(\frac{1}{2} - \eta\Big)\sum \delta_k^2 - \frac{1}{2\sqrt{2}}|W| \cdot \sqrt{\sum_k \delta_k^2} \\ =& \frac{4}{3}\eta^2 R^2(\frac{1}{4} - \eta) + \Big(\frac{1}{4} + \frac{1}{4} - \eta\Big)\sum \delta_k^2 - \frac{1}{2\sqrt{2}}|W| \cdot \sqrt{\sum_k \delta_k^2} \\ =& \left(\frac{4}{3}\eta^2 R^2 + \sum \delta_k^2\right)(\frac{1}{4} - \eta) + \frac{1}{4}\sum \delta_k^2 - \frac{1}{2\sqrt{2}}|W| \cdot \sqrt{\sum_k \delta_k^2} \\ =& II + III, \end{aligned}$$

where

$$II = \left(\frac{4}{3}\eta^2 R^2 + \sum \delta_k^2\right) \cdot (-\eta) \geq \frac{4}{3}(-\eta)^3 R^2 \geq \frac{4}{3}(\frac{-\eta_0}{2})^3 \cdot c_0 R \geq \frac{\delta}{2}R,$$

and

$$III = \frac{1}{4}\Big[\Big(\frac{4}{3}\eta^2 R^2 + \sum \delta_k^2\Big) + \sum \delta_k^2 - \sqrt{2}|W| \cdot \sqrt{\sum_k \delta_k^2}\Big].$$

Claim 3.2.

$$\frac{4}{3}\eta^2 R^2 + \sum \delta_k^2 \geq \Big(|\overset{\circ}{Ric}| - \frac{1}{2\sqrt{3}}R\Big)^2.$$

Indeed, by defining three vectors on R^4 as follows:

$$v = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix}, \ v_s = \begin{pmatrix} \frac{R}{4} \\ \frac{R}{4} \\ \frac{R}{4} \\ \frac{R}{4} \end{pmatrix}, \ v_c = \begin{pmatrix} 0 \\ \frac{R}{3} \\ \frac{R}{3} \\ \frac{R}{3} \end{pmatrix};$$

then we have

$$|v - v_s| = |\overset{\circ}{Ric}|, \text{ and } |v_c - v_s| = \frac{1}{2\sqrt{3}}R.$$

Furthermore, since $\lambda_1 = \eta R$, we have

$$\begin{aligned} |v - v_c|^2 =& \lambda_1^2 + \sum_k \Big(\lambda_k - \frac{R}{3}\Big)^2 \\ =& \eta^2 R^2 + \sum_k \Big(\lambda + \delta_k - \frac{\eta R + 3\lambda}{3}\Big)^2 \\ =& \eta^2 R^2 + \sum_k (\delta_k - \frac{\eta R}{3})^2 \\ =& \frac{4}{3}\eta^2 R^2 + \sum \delta_k^2. \end{aligned}$$

The last equality follows by the fact that $\sum_k \delta_k = 0$. And then the assertion follows by the triangle inequality.

By using the above claim and the pinched condition $(*)$, we have

$$III \geq \frac{1}{4}\Big[\frac{|W|^2}{2} + \sum_k \delta_k^2 - \sqrt{2}|W| \cdot \sqrt{\sum_k \delta_k^2}\Big] \geq 0.$$

Thus $I \geq \frac{\delta}{2}R$, and we complete the proof of Lemma 3.1. □

Now we can prove Theorem 1.1. Let (M^4, g) be a complete GSS, which implies that there are a smooth function f and a positive constant ρ, such that

$$R_{ij} + f_{ij} = \rho g_{ij}.$$

It is well known (cf. [21]) that there exists a self-similar solution of Ricci flow (even if the soliton has unbounded curvature) as follows:

$$g(t) = \tau(t)\varphi_t^*(g), \ t \in (-\infty, \frac{1}{2\rho}),$$

where $\tau(t) = 1 - 2\rho t$, and φ_t is a family of diffeomorphisms.

Proof of Theorem 1.1. We will argue by contradiction. Suppose the Ricci curvature is not nonnegative somewhere. Note that the work of Munteanu-Wang [14] implies that the soliton will have bounded curvature. Then we have a self-similar solution $g(t)_{t\in[0,\frac{1}{10\rho}]}$ with uniformly bounded curvature with $g(0)=g$.

Since $R \geq c_0$, there exists some constant

$$\eta_0 = \inf_{x\in(M^4,g)} \frac{\lambda_1(x)}{R(x)} < 0.$$

Then from Lemma 3.1, there exists a positive constant $\delta = \delta(c_0,\eta_0) \in (0,1]$, such that

$$\lambda \geq (\eta_0 + \delta t)R.$$

is preserved under the Ricci flow for all small $t \in [0, T']$, where $T' = \min\{\frac{1}{10\rho}, \frac{-\eta_0}{2}\}$.

Hence we have

$$\lambda_1 \geq (\eta_0 + \delta T')R$$

at every point. But this is impossible since there exists some point $p \in M$, such that $\lambda_1(p) \leq (\eta_0 + \frac{\delta}{2}T')R(p)$ at time $t = 0$. Note that $g(t)$ only changes by scaling and a diffeomorphism on M^4, so at time $t = T'$, there is some point $q \in M$ with

$$\lambda_1(q,T') = \frac{1}{1-2\rho T'}\lambda_1(p) \leq \frac{1}{1-2\rho T'}(\eta_0 + \frac{\delta}{2}T')R(p) = (\eta_0 + \frac{\delta}{2}T')R(q,T'),$$

which is contradictive with $\lambda_1(q,T') \geq (\eta_0 + \delta T')R(q,T')$.

And we complete the proof of Theorem 1.1. □

4. Four-dimensional GSS with positive isotropy curvature

Recall that a Riemannian manifold is said to have positive isotropy curvature, if for any orthonormal four-frame $\{e_1,e_2,e_3,e_4\}$, we have

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} + 2R_{1234} > 0.$$

It has nonnegative isotropy curvature if the LHS is only nonnegative. Manifolds with positive isotropy curvature were introduced by Micallef and Moore [13], and had become an important object in Riemannian geometry. Actually, compact four-manifolds with positive isotropy curvature had been completely classified due to the work of Hamilton [10], Chen-Zhu [6], and Chen-Tang-Zhu [5].

In an oriented four-dimensional manifold, it is natural to decompose two-forms $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$. And this decomposition induces a block decomposition of the curvature operator matrix as

$$M_{\alpha\beta} = \begin{pmatrix} A^+ & B \\ {}^tB & A^- \end{pmatrix}.$$

It is well known that $A^+ = \frac{R}{6}I + W^+$, $A^- = \frac{R}{6}I + W^-$, where $W^\pm$ are self-dual and anti-self-dual parts of Weyl tensor. By a direct computation, an oriented four-dimensional manifold has nonnegative isotropy curvature if and only if both A^+ and A^- are two-nonnegative.

Lemma 4.1. *Let (M^4,g) be a four-dimensional Riemannian manifold satisfying*

$$|W| \leq \frac{1}{\sqrt{6}}R;$$

then it will have nonnegative isotropy curvature.

Proof. If M is not orientable, we can lift the metric onto an oriented two-cover of M. So we assume the manifold is an oriented four-dimensional manifold satisfying the pinched Weyl curvature. Now we diagonalize W^+ with the eigenvalue $\lambda_1 \le \lambda_2 \le \lambda_3$; then we have

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 0, \\ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = |W^+|^2 \le \frac{1}{6}R^2. \end{cases}$$

Obviously, $\lambda_1 \le 0$, and $\lambda_3 \ge 0$. If $\lambda_2 \le 0$; then

$$\lambda_3^2 = (\lambda_1 + \lambda_2)^2 \le 2(\lambda_1^2 + \lambda_2^2) \le 2(\frac{1}{6}R^2 - \lambda_3^2).$$

So $\lambda_3 \le \frac{1}{3}R$, and then $\lambda_1 + \lambda_2 \ge -\frac{1}{3}R$.

On the other hand, if $\lambda_2 > 0$, then $\lambda_1 \ge -\frac{1}{3}R$, so $\lambda_1 + \lambda_2 > -\frac{1}{3}R$.

Altogether, we have $a_1^+ + a_2^+ = (\lambda_1 + \frac{1}{6}R) + (\lambda_2 + \frac{1}{6}R) \ge 0$. Similarly, we have $a_1^- + a_2^- \ge 0$. And we complete the proof. $\square$

For the proof of Theorem 1.3, we need the following lemma.

Lemma 4.2. *The traceless Ricci curvature satisfies* $|\overset{\circ}{Ric}| \le 2\sqrt{3}(\frac{R}{4} - \frac{\lambda_1+\lambda_2}{2})$, *where* λ_1 *and* λ_2 *are the least two eigenvalues of the Ricci curvature.*

Proof. We diagonalize the Ricci tensor with the eigenvalue $\lambda_1 \le \lambda_2 \le \lambda_3 \le \lambda_4$; then the scalar curvature $R = \sum_{k=1}^{4} \lambda_k$ and $|\overset{\circ}{Ric}|^2 = \sum_{k=1}^{4} (\lambda_k - \frac{R}{4})^2 = |Ric|^2 - \frac{R^2}{4}$.

Take $\tilde{\lambda}_4 = \lambda_4 + (\lambda_3 - \lambda_2)$; then $R = \lambda_1 + 2\lambda_2 + \tilde{\lambda}_4$, and $|Ric|^2 \le \lambda_1^2 + 2\lambda_2^2 + \tilde{\lambda}_4^2$. Denote by $\lambda = \frac{\lambda_1+\lambda_2}{2}$, $\delta = \lambda - \lambda_1 = \lambda_2 - \lambda \ge 0$. Then

$$\begin{aligned} |Ric|^2 \le& (\lambda - \delta)^2 + 2(\lambda + \delta)^2 + (R - 3\lambda - \delta)^2 \\ =& 3\lambda^2 + (R - 3\lambda)^2 - 2\delta(R - 4\lambda - 2\delta). \end{aligned}$$

Now $R - 4\lambda - 2\delta = R - (\lambda_1 + \lambda_2 + 2\lambda_2) \ge 0$, so $|Ric|^2 \le 3\lambda^2 + (R - 3\lambda)^2$. Hence $|\overset{\circ}{Ric}|^2 \le 12(\frac{R}{4} - \lambda)^2$. $\square$

Proof of Theorem 1.3. Since the soliton has nonnegative Ricci curvature, and the Weyl curvature satisfies the pinched condition $(*)$, we have$\lambda_1 + \lambda_2 \ge \frac{1}{3}R$ (Theorem 1.5 in [23]). Then $|\overset{\circ}{Ric}| \le \frac{1}{2\sqrt{3}}R$ by Lemma 4.2, and $(*)$ becomes

$$|W| \le \frac{1}{\sqrt{6}}R.$$

Hence the soliton will have nonnegative isotropy curvature by Lemma 4.1. Then Theorem 1.3 follows by Corollary 3.1 in [12]. $\square$

References

[1] Giovanni Catino, *Complete gradient shrinking Ricci solitons with pinched curvature*, Math. Ann. **355** (2013), no. 2, 629–635, DOI 10.1007/s00208-012-0800-6. MR3010141

[2] Huai-Dong Cao, *Recent progress on Ricci solitons*, Recent advances in geometric analysis, Adv. Lect. Math. (ALM), vol. 11, Int. Press, Somerville, MA, 2010, pp. 1–38. MR2648937

[3] Huai-Dong Cao, Bing-Long Chen, and Xi-Ping Zhu, *Recent developments on Hamilton's Ricci flow*, Surveys in differential geometry. Vol. XII. Geometric flows, Surv. Differ. Geom., vol. 12, Int. Press, Somerville, MA, 2008, pp. 47–112, DOI 10.4310/SDG.2007.v12.n1.a3. MR2488948

[4] Xiaodong Cao, Biao Wang, and Zhou Zhang, *On locally conformally flat gradient shrinking Ricci solitons*, Commun. Contemp. Math. **13** (2011), no. 2, 269–282, DOI 10.1142/S0219199711004191. MR2794486

[5] Bing-Long Chen, Siu-Hung Tang, and Xi-Ping Zhu, *Complete classification of compact four-manifolds with positive isotropic curvature*, J. Differential Geom. **91** (2012), no. 1, 41–80. MR2944961

[6] Bing-Long Chen and Xi-Ping Zhu, *Ricci flow with surgery on four-manifolds with positive isotropic curvature*, J. Differential Geom. **74** (2006), no. 2, 177–264. MR2258799

[7] Xiuxiong Chen and Yuanqi Wang, *On four-dimensional anti-self-dual gradient Ricci solitons*, J. Geom. Anal. **25** (2015), no. 2, 1335–1343, DOI 10.1007/s12220-014-9471-8. MR3319974

[8] Richard S. Hamilton, *Four-manifolds with positive curvature operator*, J. Differential Geom. **24** (1986), no. 2, 153–179. MR862046

[9] Richard S. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), Int. Press, Cambridge, MA, 1995, pp. 7–136. MR1375255

[10] Richard S. Hamilton, *Four-manifolds with positive isotropic curvature*, Comm. Anal. Geom. **5** (1997), no. 1, 1–92, DOI 10.4310/CAG.1997.v5.n1.a1. MR1456308

[11] Thomas Ivey, *Ricci solitons on compact three-manifolds*, Differential Geom. Appl. **3** (1993), no. 4, 301–307, DOI 10.1016/0926-2245(93)90008-O. MR1249376

[12] X. L. Li, L. Ni, and K. Wang, *Four-dimensional gradient shrinking solitons with positive isotropy curvature*, arXiv: 1603.05264.

[13] Mario J. Micallef and John Douglas Moore, *Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes*, Ann. of Math. (2) **127** (1988), no. 1, 199–227, DOI 10.2307/1971420. MR924677

[14] Ovidiu Munteanu and Jiaping Wang, *Geometry of shrinking Ricci solitons*, Compos. Math. **151** (2015), no. 12, 2273–2300, DOI 10.1112/S0010437X15007496. MR3433887

[15] Aaron Naber, *Noncompact shrinking four solitons with nonnegative curvature*, J. Reine Angew. Math. **645** (2010), 125–153, DOI 10.1515/CRELLE.2010.062. MR2673425

[16] Lei Ni and Nolan Wallach, *On a classification of gradient shrinking solitons*, Math. Res. Lett. **15** (2008), no. 5, 941–955, DOI 10.4310/MRL.2008.v15.n5.a9. MR2443993

[17] G. Perelman, *Ricci flow with surgery on three manifolds*, arXiv: 0303109v1.

[18] Peter Petersen and William Wylie, *On the classification of gradient Ricci solitons*, Geom. Topol. **14** (2010), no. 4, 2277–2300, DOI 10.2140/gt.2010.14.2277. MR2740647

[19] Wan-Xiong Shi, *Deforming the metric on complete Riemannian manifolds*, J. Differential Geom. **30** (1989), no. 1, 223–301. MR1001277

[20] J. Y. Wu, P. Wu, and W. Wylie, *Gradient shrinking Ricci solitons of half harmonic Weyl curvature*, arXiv: 1410.7303.

[21] Zhu-Hong Zhang, *On the completeness of gradient Ricci solitons*, Proc. Amer. Math. Soc. **137** (2009), no. 8, 2755–2759, DOI 10.1090/S0002-9939-09-09866-9. MR2497489

[22] Zhu-Hong Zhang, *Gradient shrinking solitons with vanishing Weyl tensor*, Pacific J. Math. **242** (2009), no. 1, 189–200, DOI 10.2140/pjm.2009.242.189. MR2525510

[23] Z. H. Zhang, *A gap theorem of four-dimensional gradient shrinking solitons*, arXiv:1606.01154.

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