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MAPPINGS OF EXPONENTIALLY INTEGRABLE DISTORTION: DECAY OF THE JACOBIAN

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ABSTRACT. We establish an integrability result on the reciprocal of the Jacobian determinant for a mapping of exponentially integrable distortion and thus answer a question raised by S. Hencl and P. Koskela.

1. Introduction

In this paper, we consider the class of mappings of finite distortion. Let $\Omega \subset \mathbb{R}^n$ be a domain. A mapping $f : \Omega \to \mathbb{R}^n$ is called a mapping of finite distortion if $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$, the Jacobian determinant J_f of the mapping f is locally integrable, and there exists a measurable almost everywhere finite function $K_f : \Omega \to \mathbb{R}$, called the distortion function of f, such that $K_f \geq 1$ and

$$|Df(x)|^n \leq K_f(x) J_f(x)$$

a.e. in Ω. If, in addition, there exists $\beta > 0$ such that $\exp(\beta K_f)$ is locally integrable in Ω, then the mapping f is called a mapping of exponentially integrable distortion. Alternatively, if the function K_f is (essentially) bounded, i.e. $\operatorname{ess\,sup}_{x \in \Omega} K_f(x) = K < \infty$, then the mapping f is called a K-quasiregular mapping. A K-quasiregular homeomorphism is called a K-quasiconformal mapping. For more on mappings of finite distortion, see the monographs by Hencl and Koskela [2] and by Iwaniec and Martin [5].

A non-constant quasiregular mapping is continuous, open and discrete, but a general mapping of finite distortion is none of these; see [2, 5] for counterexamples and comments. For any of these properties to hold, we need additional assumptions on the mapping f or on its distortion function K_f. A non-constant mapping of exponentially integrable distortion is continuous, open and discrete.

For a planar K-quasiregular mapping $f : \Omega \to \mathbb{R}^2$, where $\Omega \subset \mathbb{R}^2$ is a domain, it is well known that $J_f^{-q} \in L^1(E)$ for all $q < \frac{1}{Km-1}$, where $E \subset \Omega$ is compact and m is the maximal multiplicity of f in E. This follows from the optimal regularity result for quasiconformal mappings by Astala [1] and the Stoilow factorization result. In higher dimensions, an integrability result also holds, but the optimal degree of integrability is not known.

Recently, it has been of great interest to generalize the theory of quasiregular mappings to the class of mappings of finite distortion. Koskela and Malý [6] proved that under certain conditions, the Jacobian of a mapping of finite distortion is

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positive almost everywhere. Hence one may ask for conditions which guarantee that the reciprocal of the Jacobian is integrable, and if so, what is the optimal degree of integrability. The first integrability results on $\frac{1}{J_f}$ for a mapping of finite distortion f were obtained by Hencl, Koskela and Zhong [3], where they established the results in the cases $K_f \in L^{n-1}_{\text{loc}}(\Omega)$ and $\exp(\beta K_f) \in L^1_{\text{loc}}(\Omega)$, $\beta > 0$. The results in neither of these cases were optimal. The optimal result for mappings of L^p-integrable distortion was first given in the planar case by Koskela and Onninen [7], and later in all dimensions by Koskela, Onninen and Rajala [8], both of these with a weaker assumption $K_f \in L^{\frac{1}{n-1}}_{\text{loc}}(\Omega)$.

The result in [3] says that for a mapping of exponentially integrable distortion $f : \Omega \to \mathbb{R}^n$ we have $\exp\left(\log^s \log\left(1 + \frac{1}{J_f}\right)\right) \in L^1_{\text{loc}}(\Omega)$ for some $s > 1$. As said, this result is not optimal, and in [3] it was also shown that the optimal degree of integrability cannot be better than $\exp\left(C \log^{\frac{n-1}{n}}\left(e + \frac{1}{J_f}\right)\right) \in L^1_{\text{loc}}(\Omega)$ for some $C > 0$. Later, the question on the optimal degree of integrability of the reciprocal of the Jacobian of a mapping of exponentially integrable distortion was posed as an open problem in the monograph by Hencl and Koskela [2, Open Problem 18]. The main result of this article answers this question.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $f : \Omega \to \mathbb{R}^n$ be a non-constant mapping of finite distortion such that*

$$\exp(\beta K_f) \in L^1_{\text{loc}}(\Omega)$$

for some $\beta > 0$. Then

$$\exp\left(C \log^{\gamma}\left(e + \frac{1}{J_f}\right)\right) \in L^1_{\text{loc}}(\Omega), \tag{1}$$

for all $0 < \gamma < \frac{n-1}{n}$ and for all $C > 0$.

This result is sharp in the following sense:

Theorem 1.2. *Let $\beta > 0$. There exist $C > 0$ and a mapping of finite distortion $f : B^n(0,1) \to \mathbb{R}^n$ such that*

$$\exp(\beta K_f) \in L^1_{\text{loc}}(B^n(0,1)),$$

but

$$\exp\left(C \log^{\frac{n-1}{n}}\left(e + \frac{1}{J_f}\right)\right) \notin L^1_{\text{loc}}(B^n(0,1)). \tag{2}$$

Proof. Let $a = \frac{n}{n-1}$, $k > 0$, $\rho(t) = \exp(-k|\log t|^a)$ for $t > 0$ and $f : \Omega \to \mathbb{R}^n$,

$$f(x) = \begin{cases} \frac{x}{|x|}\rho(|x|), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

By [2, Lemma 2.1],

$$|Df(x)| = |\rho'(|x|)| = ak\exp(-k\big|\log|x|\big|^a)\frac{\big|\log|x|\big|^{a-1}}{|x|},$$

$$J_f(x) = \rho'(|x|)\left(\frac{\rho(|x|)}{|x|}\right)^{n-1} = ak\exp(-nk\big|\log|x|\big|^a)\frac{\big|\log|x|\big|^{a-1}}{|x|^n}$$

and

$$K_f(x) = \left(ak\big|\log|x|\big|^{(a-1)}\right)^{(n-1)} = -(ak)^{n-1}\log|x|.$$

By choosing

$$k < \frac{n^{\frac{1}{n-1}}}{\beta^{\frac{1}{n-1}}a},$$

we have $\exp(\beta K_f) \in L^1(B(0,1))$. Now it is easy to compute that

$$\exp\left(C\log^{\frac{n-1}{n}}\left(e+\frac{1}{J_f(x)}\right)\right) \geq \frac{c_1}{|x|^{c_2C(nk)^{(n-1)/n}}}$$

for some constants $c_1, c_2 > 0$. Thus, by choosing C large enough, (2) holds. □

Theorem 1.1 is a corollary of the following more general theorem, which we will prove in the next section.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $f : \Omega \to \mathbb{R}^n$ be a non-constant, continuous, open and discrete mapping of finite distortion such that*

$$\exp\left(\beta K_f^\alpha\right) \in L^1_{\mathrm{loc}}(\Omega)$$

for some $\alpha, \beta > 0$. Then

$$\exp\left(C\log^\gamma\left(e+\frac{1}{J_f}\right)\right) \in L^1_{\mathrm{loc}}(\Omega) \tag{3}$$

for all $0 < \gamma < \frac{\alpha(n-1)}{\alpha(n-1)+1}$ and for all $C > 0$.

Notice that if $\alpha < 1$, then the mapping f is not necessarily continuous, open or discrete. In the case $\alpha \geq 1$ the additional topological assumptions are, of course, not needed.

Let us close the introduction with a few remarks. In Theorem 1.2, we only show that for every $\beta > 0$ these exists some $C > 0$ such that the integrability condition on the reciprocal of the Jacobian fails. However, it is still open if there is a constant $C_{n,\beta} > 0$, depending on the dimension n and β, such that $\exp\left(C_{n,\beta}\log^{\frac{n-1}{n}}\left(e+\frac{1}{J_f}\right)\right) \in L^1_{\mathrm{loc}}(\Omega)$ for all mappings of finite distortion f such that $\exp\left(\beta K_f\right) \in L^1_{\mathrm{loc}}(\Omega)$. Also, in Theorem 1.2 the same argument works for $a < \frac{n}{n-1}$ by choosing the exponent in the integrability condition to be $\gamma = \frac{1}{a}$ instead of $\gamma = \frac{n-1}{n}$. On the other hand, the counterexample fails for $a > \frac{n}{n-1}$ since in that case the mapping f will not be a mapping of exponentially integrable distortion; this is where the critical exponent $\gamma = \frac{n-1}{n}$ shows up. Notice that for quasiregular mappings, the degree of integrability of the reciprocal of the Jacobian determinant is only known in the plane. Surprisingly, in the case of mappings of finite distortion, the dimension does not play any role in the sense that the same proof for the optimal degree of integrability works in all dimensions.

2. Proof of Theorem 1.3

Throughout this section, the set $\Omega \subset \mathbb{R}^n$ is a domain and the mapping $f : \Omega \to \mathbb{R}^n$ satisfies the conditions of Theorem 1.3. The constant $C > 0$ is fixed, but the letter C with various subindices indicates a positive constant depending on the subindices. The value of the constants may change from line to line; we are not interested in the exact value of the constants.

The local multiplicity function for the mapping f and a set $A \subset \Omega$ is defined as follows:

$$N(y, f, A) = \operatorname{card} f^{-1}(y) \cap A.$$

Since the mapping f is discrete, $N(y, f, A)$ is bounded for all $A \Subset \Omega$, that is, for all sets A compactly contained in Ω. Next, recall the area formula:

$$\int_A \eta(f(x))|J_f(x)|\, dx \leq \int_{\mathbb{R}^n} \eta(y)N(y, f, A)\, dy \tag{4}$$

for all Borel sets $A \subset \Omega$ and for all non-negative Borel functions $\eta : \mathbb{R}^n \to \mathbb{R}$. Moreover, we can choose a Borel set $\Omega' \subset \Omega$ of full measure such that the equality in (4) holds for all sets $A \subset \Omega'$; see the proof of Lemma A.35 in [2].

A domain $D \Subset \Omega$ is a normal domain if $f(\partial D) = \partial f(D)$. By [9, Lemma I.4.9] there exists $s_x > 0$ such that the x-component of $f^{-1}(B(f(x), s))$ is a normal domain for all $0 < s \leq s_x$. Fix $x \in \Omega$, choose such s, and denote $B_s = B(f(x), s)$ and $U = f^{-1}(B_s)$. Now $N(y, f, U)$ is bounded in B_s, denote $m = \max_{y \in B_s} N(y, f, U)$. By [6, Theorem 1.2], $J_f > 0$ a.e. in U.

The mapping f is continuous, open and discrete, but not a homeomorphism, so the inverse mapping, or in particular, the Jacobian of its inverse mapping doesn't necessarily exist. To tackle this problem, we follow the approach from [8] and define the functions $j, g : \mathbb{R}^n \to \mathbb{R}$ as follows: Let $\Omega' \subset \Omega$ be a Borel set such that the equality in (4) holds for all $A \subset \Omega'$. Now we define

$$\varphi(x) = \begin{cases} \frac{1}{J_f(x)}, & \text{if } x \in \Omega' \text{ and } 0 < J_f(x) < \infty, \\ 0, & \text{otherwise,} \end{cases}$$

and then

$$j(y) = \max_{x \in f|_U^{-1}(y)} \varphi(x).$$

Similarly, we define

$$\psi(x) = \begin{cases} \frac{|Df(x)|}{J_f(x)}, & \text{if } x \in \Omega' \text{ and } 0 < J_f(x) < \infty, \\ 0, & \text{otherwise,} \end{cases}$$

and then

$$g(y) = \max_{x \in f|_U^{-1}(y)} \psi(x).$$

Note that if f is a homeomorphism, then $j(y) = J_{f^{-1}}(y)$ and $g(y) = |\operatorname{adj} Df^{-1}(y)|$ for a.e. $y \in f(\Omega)$.

For the next lemma, we need to define the distance function $\mu : \mathbb{R}^n \to \mathbb{R}$,

$$\mu(y) = \operatorname{dist}(y, \mathbb{R}^n \setminus B_s).$$

Lemma 2.1. *Let j, g and μ be as above and let $t > 1$. Then*

$$\int_{\{\mu^n j > t^n\}} \mu(y)^n j(y)\, dy \leq A_n t \int_{\{A_n \mu^{n-1} g > t^{n-1}\}} \mu(y)^{n-1} g(y)\, dy, \tag{5}$$

where the constant A_n depends only on the dimension n.

The proof of Lemma 2.1 can be found in [8]. The proof is based on the isoperimetric inequality, a Gehring's lemma type estimate (see [4]) and the basic properties of the Hardy-Littlewood maximal function. Note that the normality of U is needed in the proof.

The next lemma on higher integrability of the function j is crucial on the proof of Theorem 1.3.

Lemma 2.2. *Let* $C > 0$ *and* $0 < \gamma < 1$ *and let* j, g *and* μ *be as above. Then*

$$\int_{\{1<\mu^n j<T^n\}} \mu(y)^n j(y) \exp\left(C\log^\gamma\left(\mu(y)j(y)^{\frac{1}{n}}\right)\right)\, dy$$

$$\leq C_n\Bigg(\int_{\{1<A_n\mu^{n-1}g<T^{n-1}\}} \mu(y)^n g(y)^{\frac{n}{n-1}} \frac{\exp\left(C\log^\gamma\left(A_n\mu(y)g(y)^{\frac{1}{n-1}}\right)\right)}{\log^{1-\gamma}\left(A_n\mu(y)g(y)^{\frac{1}{n-1}}\right)}\, dy + |B_s||U|\Bigg) \tag{6}$$

for all $T > 1$.

Proof. First, we define the function

$$\Phi(t) = \exp\left(C\log^\gamma(t)\right)\left(\frac{1}{C\gamma} + \frac{1}{\log^{1-\gamma}(t)}\right). \tag{7}$$

An easy computation shows us that

$$t\Phi'(t) = \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{t\exp\left(C\log^\gamma(t)\right)}{\log^{1-\gamma}(t)}\right).$$

Now, we multiply (5) by $\Phi'(t)$ and integrate from 1 to T with respect to t and use Fubini's theorem to obtain

$$\int_{\{1<\mu^n j<T^n\}}\int_1^{\mu(y)j(y)^{\frac{1}{n}}} \Phi'(t)\mu(y)^n j(y)\, dtdy$$

$$\leq C_n \int_{\{1<A_n\mu^{n-1}g<T^{n-1}\}}\int_1^{A_n\mu(y)g(y)^{\frac{1}{n-1}}} t\Phi'(t)\mu(y)^{n-1}g(y)\, dtdy. \tag{8}$$

By computing the inner integrals in (8) we obtain

$$\int_{\{1<\mu^n j<T^n\}} \mu(y)^n j(y) \exp\left(C\log^\gamma\left(\mu(y)j(y)^{\frac{1}{n}}\right)\right)\, dy$$

$$\leq C_n\Bigg(\int_{\{1<A_n\mu^{n-1}g<T^{n-1}\}} \mu(y)^n g(y)^{\frac{n}{n-1}} \frac{\exp\left(C\log^\gamma\left(A_n\mu(y)g(y)^{\frac{1}{n-1}}\right)\right)}{\log^{1-\gamma}\left(A_n\mu(y)g(y)^{\frac{1}{n-1}}\right)}\, dy + \int_{B_s}\mu(y)^n j(y)\, dy\Bigg), \tag{9}$$

By the definition of the function μ,

$$\mu(y)^n \leq C_n|B_s| \tag{10}$$

for all $y \in \mathbb{R}^n$. Let $M = \{x \in U : \frac{1}{J_f(x)} = j(f(x))\} \cap \Omega'$. By the definition of the function j, the area formula (4) and the definition of the set M

$$\int_{B_s} j(y)\, dy = \int_{f(M)} N(y,f,U)j(y)\, dy = \int_M j(f(x))J_f(x)\, dx = \int_M 1\, dx \leq |U|. \tag{11}$$

The claim follows from (9), (10) and (11). □

Lemma 2.3 (Change of variables for Lemma 2.2). *Let*

$$E = \left\{ x \in U : 0 < \mu(f(x))^n j(f(x)) < \frac{T^n}{A_n} \right\}.$$

Then

$$\begin{aligned} &\int_E \mu(f(x))^n \exp\left(C \log^\gamma \left(e + \frac{\mu(f(x))}{J_f(x)^{\frac{1}{n}}} \right) \right) \, dx \\ &\leq C_{m,n} \Bigg(\int_E \mu(f(x))^n K_f(x)^{\frac{1}{n-1}} \frac{\exp\left(C \log^\gamma \left(e + \mu(f(x)) \left(\frac{|Df(x)|}{J_f(x)} \right)^{\frac{1}{n-1}} \right) \right)}{\log^{1-\gamma} \left(e + \mu(f(x)) \left(\frac{|Df(x)|}{J_f(x)} \right)^{\frac{1}{n-1}} \right)} \, dx \\ &\qquad + |B_s||U| \Bigg). \end{aligned} \tag{12}$$

Proof. Recall that the set $\{x \in E : J_f(x) = 0 \text{ or } J_f(x) = \infty\}$ has measure zero. Also, $\frac{1}{J_f(x)} \leq j(f(x))$ for all $x \in U$. Using the area formula (4) we get the following estimate for the integral on the left hand side of (6):

$$\begin{aligned} &\int_E \mu(f(x))^n \exp\left(C \log^\gamma \left(e + \frac{\mu(f(x))}{J_f(x)^{\frac{1}{n}}} \right) \right) \, dx \\ &\leq \int_{\{x \in E : 0 < J_f(x) < \infty\}} \mu(f(x))^n \exp\left(C \log^\gamma \left(e + \frac{\mu(f(x))}{J_f(x)^{\frac{1}{n}}} \right) \right) \frac{J_f(x)}{J_f(x)} \, dx \\ &\leq \int_E \mu(f(x))^n \exp\left(C \log^\gamma \left(e + \mu(f(x)) j(f(x))^{\frac{1}{n}} \right) \right) j(f(x)) J_f(x) \, dx \\ &\leq m \int_{\{0 < \mu^n j < \frac{T^n}{A_n}\}} \mu(y)^n \exp\left(C \log^\gamma \left(e + \mu(y) j(y)^{\frac{1}{n}} \right) \right) j(y) \, dy \\ &\leq m \Bigg(\int_{\{1 < \mu^n j < T^n\}} \mu(y)^n j(y) \exp\left(C \log^\gamma \left(e + \mu(y) j(y)^{\frac{1}{n}} \right) \right) \, dy \\ &\qquad + \int_{\{0 < \mu^n j < 1\}} \mu(y)^n j(y) \exp\left(C \log^\gamma \left(e + \mu(y) j(y)^{\frac{1}{n}} \right) \right) \, dy \Bigg) \\ &\leq C_{m,n} \left(\int_{\{1 < \mu^n j < T^n\}} \mu(y)^n j(y) \exp\left(C \log^\gamma \left(\mu(y) j(y)^{\frac{1}{n}} \right) \right) \, dy + |B_s| \right). \end{aligned} \tag{13}$$

Here we assumed that $A_n \geq 1$. The last inequality follows from the fact that the integrand in the second integral is bounded and $\{0 < \mu^n j < 1\} \subset B_s$.

Notice that $j(y)^{\frac{n-1}{n}} \leq g(y)$ and that $j(y) = 0$ implies $g(y) = 0$. Therefore

$$\{1 < A_n \mu^{n-1} g < T^{n-1}\} \subset \left\{ 0 < \mu^n j < \frac{T^n}{A_n} \right\}.$$

Applying the change of variables formula and using the definition of the function g yields the following approximation for the integral on the right hand side of (6):

$$\int_{\{1<A_n\mu^{n-1}g<T^{n-1}\}} \mu(y)^n g(y)^{\frac{n}{n-1}} \frac{\exp\left(C\log^\gamma\left(A_n\mu(y)g(y)^{\frac{1}{n-1}}\right)\right)}{\log^{1-\gamma}\left(A_n\mu(y)g(y)^{\frac{1}{n-1}}\right)}\,dy$$

$$\le C_n \int_{\left\{0<\mu^n j<\frac{T^n}{A_n}\right\}} \mu(y)^n g(y)^{\frac{n}{n-1}} \frac{\exp\left(C\log^\gamma\left(e+\mu(y)g(y)^{\frac{1}{n-1}}\right)\right)}{\log^{1-\gamma}\left(e+\mu(y)g(y)^{\frac{1}{n-1}}\right)}\,dy$$

$$\le C_n \int_E \mu(f(x))^n K_f(x)^{\frac{1}{n-1}} \frac{\exp\left(C\log^\gamma\left(e+\mu(f(x))\left(\frac{|Df(x)|}{J_f(x)}\right)^{\frac{1}{n-1}}\right)\right)}{\log^{1-\gamma}\left(e+\mu(f(x))\left(\frac{|Df(x)|}{J_f(x)}\right)^{\frac{1}{n-1}}\right)}\,dx. \tag{14}$$

Now the claim follows by combining (13) and (14) with Lemma 2.2. □

Our final lemma is the following inequality:

Lemma 2.4. *Let $\theta, \lambda, \varepsilon > 0$. There exists a constant $C_{\theta,\lambda} > 0$ such that*

$$ab \le \exp\left(\frac{\lambda}{\varepsilon}a^\theta\right) + \varepsilon^{\frac{1}{\theta}} C_{\theta,\lambda}\, b\log^{\frac{1}{\theta}}\left(e+\left(\frac{\varepsilon}{\lambda}\right)^{\frac{1}{\theta}} b\right) \tag{15}$$

for all $a \ge 1$ and $b > 0$.

Proof. We may assume that

$$ab > \exp\left(\frac{\lambda}{\varepsilon}a^\theta\right).$$

Let $k = \frac{2}{\theta}$. There exists $M \ge 0$, depending on θ, such that $x^k \le e^x$ for all $x \ge M$. First, let us assume that $\frac{\lambda}{\varepsilon}a^\theta \ge M$. Then

$$\left(\frac{\lambda}{\varepsilon}\right)^k a^{k\theta} \le \exp\left(\frac{\lambda}{\varepsilon}a^\theta\right) < ab,$$

and thus

$$\exp\left(\frac{\lambda}{\varepsilon}a^\theta\right) < ab = a^{k\theta-1}b < \left(\frac{\varepsilon}{\lambda}\right)^{\frac{2}{\theta}} b^2.$$

Now

$$\frac{\lambda}{\varepsilon}a^\theta < 2\log\left(e+\left(\frac{\varepsilon}{\lambda}\right)^{\frac{1}{\theta}} b\right),$$

and from that we get

$$ab \le \varepsilon^{\frac{1}{\theta}}\left(\frac{2}{\lambda}\right)^{\frac{1}{\theta}} b\log^{\frac{1}{\theta}}\left(e+\left(\frac{\varepsilon}{\lambda}\right)^{\frac{1}{\theta}} b\right).$$

In the second case we assume that

$$\frac{\lambda}{\varepsilon}a^\theta < M. \tag{16}$$

Since $\log^{\frac{1}{\theta}}\left(e+\left(\frac{\varepsilon}{\lambda}\right)^{\frac{1}{\theta}} b\right) \ge 1$ for all $b > 0$ and by the assumption (16) we have that

$$ab \le \varepsilon^{\frac{1}{\theta}}\left(\frac{M}{\lambda}\right)^{\frac{1}{\theta}} b\log^{\frac{1}{\theta}}\left(e+\left(\frac{\varepsilon}{\lambda}\right)^{\frac{1}{\theta}} b\right).$$

The claim follows by choosing $C_{\theta,\lambda} = \max\left\{\left(\frac{2}{\lambda}\right)^{\frac{1}{\theta}}, \left(\frac{M}{\lambda}\right)^{\frac{1}{\theta}}\right\}$. □

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let

$$E = \left\{x \in U : 0 < \mu(f(x))^n j(f(x)) < \frac{T^n}{A_n}\right\}.$$

Since

$$\left(\frac{|Df(x)|}{J_f(x)}\right)^{\frac{1}{n-1}} = \frac{K_f(x)^{\frac{1}{n(n-1)}}}{J_f(x)^{\frac{1}{n}}},$$

we can replace the integral on the right hand side of (12) with

$$\int_E \mu(f(x))^n K_f(x)^{\frac{1}{n-1}} \frac{\exp\left(C\log^\gamma\left(e+K_f(x)^{\frac{1}{n(n-1)}}\right)\right)\exp\left(C\log^\gamma\left(e+\frac{\mu(f(x))}{J_f(x)^{\frac{1}{n}}}\right)\right)}{\log^{1-\gamma}\left(e+\frac{\mu(f(x))}{J_f(x)^{\frac{1}{n}}}\right)}\,dx,$$

which, in turn, due to the trivial inequality $\exp(\log^a(e+t)) \le t^\varepsilon$ for all $0 < a, \varepsilon < 1$ when t is large enough, we can replace with

$$\int_E \mu(f(x))^n K_f(x)^{\frac{1}{n-1-\eta}} \frac{\exp\left(C\log^\gamma\left(e+\frac{\mu(f(x))}{J_f(x)^{\frac{1}{n}}}\right)\right)}{\log^{1-\gamma}\left(e+\frac{\mu(f(x))}{J_f(x)^{\frac{1}{n}}}\right)}\,dx \tag{17}$$

by fixing arbitrarily small $\eta > 0$.

Let

$$\gamma = \frac{\alpha(n-1-\eta)}{\alpha(n-1-\eta)+1} = \frac{\alpha(n-1)}{\alpha(n-1)+1} - \hat{\eta},$$

where $\hat{\eta} > 0$. Note that $\hat{\eta} \to 0$ as $\eta \to 0$. Also, let $\varepsilon > 0$ be a small number which will be chosen later. Next, we will use Lemma 2.4 for

$$a = K_f(x)^{\frac{1}{n-1-\eta}},$$

$$b = \mu(f(x))^n \frac{\exp\left(C\log^\gamma\left(e+\frac{\mu(f(x))}{J_f(x)^{\frac{1}{n}}}\right)\right)}{\log^{1-\gamma}\left(e+\frac{\mu(f(x))}{J_f(x)^{\frac{1}{n}}}\right)},$$

$$\theta = \alpha(n-1-\eta),$$

$$\lambda = \varepsilon\beta,$$

but before doing so, observe that

$$\log^{\frac{1}{\theta}}(e + \left(\frac{\varepsilon}{\lambda}\right)^{\frac{1}{\theta}} b)$$

$$= \log^{\frac{1}{\theta}} \left(e + \left(\frac{1}{\beta}\right)^{\frac{1}{\theta}} \mu(f(x))^n \frac{\exp\left(C \log^\gamma\left(e + \frac{\mu(f(x))}{J_f(x)^{\frac{1}{n}}}\right)\right)}{\log^{1-\gamma}\left(e + \frac{\mu(f(x))}{J_f(x)^{\frac{1}{n}}}\right)} \right)$$

$$\le \left(\log\left(e + \left(\frac{1}{\beta}\right)^{\frac{1}{\theta}} \mu(f(x))^n\right) + \log\left(e + \exp\left(C \log^\gamma\left(e + \frac{\mu(f(x))}{J_f(x)^{\frac{1}{n}}}\right)\right)\right)\right)^{\frac{1}{\theta}} \tag{18}$$

$$\le \left(C_{n,\beta,\theta} + \log\left(e + \exp\left(C \log^\gamma\left(e + \frac{\mu(f(x))}{J_f(x)^{\frac{1}{n}}}\right)\right)\right)\right)^{\frac{1}{\theta}}, \tag{19}$$

where (18) follows from a simple estimate for logarithms and from the fact that $\log^{1-\gamma}\left(e + \frac{\mu(f(x))}{J_f(x)^{\frac{1}{n}}}\right) \ge 1$.

Next, notice that

$$\log\left(e + \exp\left(C \log^\gamma\left(e + \frac{\mu(f(x))}{J_f(x)^{\frac{1}{n}}}\right)\right)\right) \ge 1.$$

Therefore we may approximate (19) from above by

$$C_{n,\beta,\theta} \log^{\frac{1}{\theta}}\left(e + \exp\left(C \log^\gamma\left(e + \frac{\mu(f(x))}{J_f(x)^{\frac{1}{n}}}\right)\right)\right)$$

$$\le C_{n,\beta,\theta} \log^{\frac{\gamma}{\theta}}\left(e + \frac{\mu(f(x))}{J_f(x)^{\frac{1}{n}}}\right)$$

$$= C_{n,\beta,\theta} \log^{1-\gamma}\left(e + \frac{\mu(f(x))}{J_f(x)^{\frac{1}{n}}}\right). \tag{20}$$

Now it is easy to see that by using Lemma 2.4 we obtain

$$\mu(f(x))^n K_f(x)^{\frac{1}{n-1-\eta}} \frac{\exp\left(C \log^\gamma\left(e + \frac{\mu(f(x))}{J_f(x)^{\frac{1}{n}}}\right)\right)}{\log^{1-\gamma}\left(e + \frac{\mu(f(x))}{J_f(x)^{\frac{1}{n}}}\right)}$$

$$\le \exp\left(\beta K_f^\alpha\right) + \varepsilon^{\frac{1}{\theta}} C_{n,\alpha,\beta} \mu(f(x))^n \exp\left(C \log^\gamma\left(e + \frac{\mu(f(x))}{J_f(x)^{\frac{1}{n}}}\right)\right)$$

and therefore we can replace (17) with

$$\int_U \exp\left(\beta K_f^\alpha\right) dx + \varepsilon^{\frac{1}{\theta}} C_{n,\alpha,\beta} \int_E \mu(f(x))^n \exp\left(C \log^\gamma\left(e + \frac{\mu(f(x))}{J_f(x)^{\frac{1}{n}}}\right)\right)\, dx. \tag{21}$$

By choosing ε small enough, we can subtract the last integral of (21) from the left hand side of (12) and thus obtain
(22)
$$\int_E \exp\left(C\log^\gamma\left(e+\frac{\mu(f(x))}{J_f(x)^{\frac{1}{n}}}\right)\right)\,dx \le C_{m,n,\alpha,\beta}|B_s|\left(\int_U \exp\left(\beta K_f^\alpha\right)+|U|\right).$$
Let $T\to\infty$. By the monotone convergence theorem,
(23)
$$\int_U \exp\left(C\log^\gamma\left(e+\frac{\mu(f(x))}{J_f(x)^{\frac{1}{n}}}\right)\right)\,dx \le C_{m,n,\alpha,\beta}|B_s|\left(\int_U \exp\left(\beta K_f^\alpha\right)+|U|\right).$$

By choosing $0<\delta<1$ we can approximate the distance function μ from below:
$$\mu(y)\ge(1-\delta)s$$
for all $y\in\delta B_s=B(f(x),\delta s)$. Applying the inequalities $\log(e+a)+\log(e+b)\le C_b\log(e+ab)$ for fixed $b\ge 0$ and for all $a\ge 0$ and $a^\gamma+b^\gamma\le C_\gamma(a+b)^\gamma$ for all $a,b\ge 0$ and $0<\gamma<1$ we obtain
(24)
$$\int_{U_\delta} \exp\left(\hat{C}\log^\gamma\left(e+\frac{1}{J_f(x)}\right)\right)\,dx \le C_{m,n,s,\alpha,\beta,\delta}\left(\int_U \exp\left(\beta K_f^\alpha\right)+|U|\right)<\infty,$$
where $U_\delta=f^{-1}(\delta B_s)$ and $\hat{C}=CC_n$. By a standard covering argument, it is enough to show that every point $x\in\Omega$ has a neighborhood such that the claim holds. Thus (24) implies (3). □

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