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CASTELNUOVO-MUMFORD REGULARITY OF KOSZUL CYCLES AND KOSZUL HOMOLOGIES

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Abstract. We extend to one dimensional quotients the result of A. Conca and S. Murai on the convexity of the regularity of Koszul cycles. By providing a relation between the regularity of Koszul cycles and Koszul homologies we prove a sharp regularity bound for the Koszul homologies of a homogeneous ideal in a polynomial ring under the same conditions.

1. Introduction

A classic way to describe projective variety and its properties is by means of its defining equations and syzygies among them. In this regard, M. Green and R. Lazarsfeld defined the property N_p which, roughly speaking, refers to the simplicity of syzygies of the homogeneous coordinate ring of a smooth projective variety embedded by a very ample line bundle. M. Green in [5] proved that the coordinate ring of the image of Veronese embedding of degree d satisfies the property N_d. W. Bruns, A. Conca, and T. Römer [1] improved this result so that the d-th Veronese subring of a polynomial ring has Green-Lazarsfeld index larger than or equal to $d + 1$. Their approach is based on investigation of the homological invariants of the Koszul cycles and Koszul homologies of d-th power of the maximal ideal.

With the aforementioned motivation A. Conca and S. Murai studied the Castelnuovo-Mumford regularity of the Koszul cycles $Z_t(I, S)$ of a homogeneous ideal in a polynomial ring S. Under mild assumptions on the base field A.Conca and S. Murai proved that regularity of Koszul cycles $Z_i(I, S)$ as a function of i is subadditive when $\dim S/I = 0$ as follows:

$$\operatorname{reg}(Z_{s+t}(I, S)) \leq \operatorname{reg}(Z_t(I, S)) + \operatorname{reg}(Z_s(I, S)).$$

We make a generalization showing that with the same assumptions on the base field the same formula holds when $\dim S/I \leq 1$.

From the convexity of the regularity of Koszul cycles in dimension 0, A. Conca and S. Murai [3, Corollary 3.3] obtained a bound on the regularity of Koszul homologies. Inspired by the remarkable result of M. Chardin and P. Symonds [4] on the regularity of cycles and homologies of a general complex, first we determine the regularity of Koszul cycles by the regularity of the previous Koszul homologies. Let S be a polynomial ring and let I be a homogeneous ideal of S. If $\dim S/I \leq 1$,

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then for all $0 < i < \mu(I)$

$$\operatorname{reg}(Z_i(I,S)) = \max_{0<j<n}\{\operatorname{reg}(H_{i-j}(I,S)) + j + 1\}.$$

Here $\mu(I)$ is the minimal number of generators of I.

As an application we state a sharp bound for the regularity of Koszul homologies in dimension 1 which is a refinement of the result of A. Conca and S. Murai in dimension 0. Let I be an ideal of S and $\dim S/I \leq 1$; then we have the following inequalities between Koszul homologies of I for all $i, j \geq 1$:

$$\operatorname{reg}(H_{i+j-1}(I,S)) \leq \max_{1\leq\alpha,\beta\leq n-1}\{\operatorname{reg}(H_{i-\alpha}(I,S)) + \operatorname{reg}(H_{j-\beta}(I,S)) + \alpha + \beta\}.$$

2. Preliminaries

Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and let M be a finitely generated graded S-module. A minimal free resolution of M is an exact sequence

$$0 \to F_p \to F_{p-1} \to \cdots \to F_0 \to M \to 0,$$

where each F_i is a graded S-free module of the form $F_i = \bigoplus S(-j)^{\beta_{i,j}(M)}$ such that the number of basis elements is minimal and each map is graded. The value $\beta_{i,j}(M)$ is called the i-th graded Betti numbers of M of degree j. Note that the minimal free resolution of M is unique up to isomorphism so the graded Betti numbers are uniquely determined.

Let $I=(f_1, \ldots, f_r)$ be a graded S-ideal minimally generated in degrees $d_1, \cdots, d_r$. Define $K(I,S) = \bigoplus K_t(I,S)$ as the Koszul complex associated to the S-linear map $\phi : F_0 = \bigoplus S(-d_i) \to S$ in which $\phi(e_i) = f_i$. Let $K(I,M) = K(I,S) \otimes M$ and denote $Z_t(I,M)$, $B_t(I,M)$, and $H_t(I,M)$ the cycles, boundaries and homologies of $K(I,M)$, respectively, at the homological position t. We use $Z_t(I), B_t(I)$, and $H_t(I)$ whenever $M = S$. We set $K_t(I,S) = 0$ for $t < 0$.

Remark 2.1. The Koszul complex does depend on the choice of the generators, but it is unique up to isomorphism if we choose a minimal set of generators. Since we only deal with the case that the set of generators is minimal, we use $K(I)$ instead of $K(f_1, \ldots, f_r)$.

Let $\mathfrak{m} = (x_1 \ldots, x_n)$ be the graded maximal ideal of S. For a finitely generated module M define the Čech complex as follows:

$$\mathcal{C}^\bullet_{\mathfrak{m}}(M) : 0 \to M \to \bigoplus_{1\leq i\leq n} M_{x_i} \to \bigoplus_{1\leq i,j\leq n} M_{x_i x_j} \to \cdots \to M_{x_1\ldots x_n} \to 0.$$

The local cohomology modules of an S-module M are the homologies of the Čech complex. It is a well-known fact that each local cohomology module is artinian so we can identify the last nonzero degree of each of them. We define

$$a_i^{\mathfrak{m}}(M) := \operatorname{end}(H^i_{\mathfrak{m}}(M)) = \max\{j \mid H^i_{\mathfrak{m}}(M)_j \neq 0\};$$

then the Castelnuovo-Mumford regularity of module M is defined as follows:

Definition 2.2. Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring, let $\mathfrak{m} = (x_1, \ldots, x_n)$ be a unique graded maximal ideal of S, and let M be a finitely generated S-module; then

$$\operatorname{reg}(M) = \max\{a_i^{\mathfrak{m}}(M) + i\}.$$

3. Regularity of Koszul cycles

In this section we will present a generalization of a result about convexity of regularity of Koszul cycles of A. Conca and S. Murai.

The following result is due to W. Bruns, A. Conca and T. Römer in [2]

Lemma 3.1 ([2, Lemma 2.4])**.** *Let S be a polynomial ring, let I be a homogeneous ideal of S, and let M be a finitely generated graded S-module. Suppose that the element $\begin{pmatrix} s+t \\ s \end{pmatrix}$ is invertible in S. Then $Z_{s+t}(I,M)$ is a direct summand of $Z_s(I,Z_t(I,M))$*

The following lemma allows us to compare regularities of different terms of exact sequences and basically it plays the main role in the generalization of the result of A. Conca and S. Murai [3] on the convexity of regularity of Koszul cycles.

Lemma 3.2. *Let $L : 0 \longrightarrow L_4 \xrightarrow{d_4} L_3 \xrightarrow{d_3} L_2 \xrightarrow{d_2} L_1 \longrightarrow 0$ be an exact sequence of finitely generated graded S-modules such that L_1 and L_4 have dimension ≤ 1, and* $\operatorname{depth} L_2 \geq 2$*; then*

$$\operatorname{reg}(L_3) = \max\{\operatorname{reg}(L_4), \operatorname{reg}(L_2), \operatorname{reg}(L_1) - 1\},$$

in particular $\operatorname{reg}(L_2) \leq \operatorname{reg}(L_3)$.

Proof. First we decompose the complex L into the following short exact sequences:

$$0 \longrightarrow L_4 \xrightarrow{d_4} L_3 \xrightarrow{can.} \operatorname{coker}(d_4) \longrightarrow 0,$$

$$0 \longrightarrow \operatorname{coker}(d_4) \xrightarrow{\bar{d_3}} L_2 \xrightarrow{d_2} L_1 \longrightarrow 0.$$

Given the above short exact sequences, one can obtain the following induced long exact sequences on local cohomology:

$$\text{(I)} \qquad \cdots \longrightarrow H^i_{\mathfrak{m}}(L_4) \to H^i_{\mathfrak{m}}(L_3) \to H^i_{\mathfrak{m}}(\operatorname{coker}(d_4)) \to H^{i+1}_{\mathfrak{m}}(L_4) \longrightarrow \cdots$$

$$\text{(II)} \qquad \cdots \longrightarrow H^i_{\mathfrak{m}}(L_2) \to H^i_{\mathfrak{m}}(L_1) \to H^{i+1}_{\mathfrak{m}}(\operatorname{coker}(d_4)) \to H^{i+1}_{\mathfrak{m}}(L_2) \longrightarrow \cdots$$

$H^i_{\mathfrak{m}}(L_4) = 0$ for $i \geq 2$ as $\dim L_4 \leq 1$, thus (I) gives

$$H^2_{\mathfrak{m}}(L_3) \cong H^2_{\mathfrak{m}}(\operatorname{coker}(d_4)). \tag{3.1}$$

As $\dim L_1 \leq 1$, by (II) we have

$$H^i_{\mathfrak{m}}(\operatorname{coker}(d_4)) \cong H^i_{\mathfrak{m}}(L_2) \ \forall i \geq 3.$$

As $\operatorname{depth} L_2 \geq 2$ and $H^i_{\mathfrak{m}}(L_1) = 0$ for $i = 0, 1$, by (II)

$$H^0_{\mathfrak{m}}(L_1) \cong H^1_{\mathfrak{m}}(\operatorname{coker}(d_4)) \quad \text{and} \quad H^0_{\mathfrak{m}}(\operatorname{coker}(d_4)) = 0. \tag{3.2}$$

From the exact sequences (I) and (3.2) we get the following short exact sequences:

$$0 \to H^1_{\mathfrak{m}}(L_4) \to H^1_{\mathfrak{m}}(L_3) \to H^0_{\mathfrak{m}}(L_1) \to 0.$$

Also the exact sequences (II) and (3.1) give

$$0 \to H^1_{\mathfrak{m}}(L_1) \to H^2_{\mathfrak{m}}(L_3) \to H^2_{\mathfrak{m}}(L_2) \to 0.$$

As a result we have

$$a^i_{\mathfrak{m}}(L_3) = \begin{cases} a^0_{\mathfrak{m}}(L_4) & \text{if } i = 0, \\ \max\{a^1_{\mathfrak{m}}(L_4), a^0_{\mathfrak{m}}(L_1)\} & \text{if } i = 1, \\ \max\{a^2_{\mathfrak{m}}(L_2), a^1_{\mathfrak{m}}(L_1)\} & \text{if } i = 2, \\ a^i_{\mathfrak{m}}(L_2) & \text{if } i \geq 3, \end{cases}$$

which proves the statement. □

Proposition 3.3. *Let $S = k[x_1, \dots, x_n]$ be a polynomial ring, let M be a finitely generated graded S-module with* $\operatorname{depth} M \geq 2$*, and let I be a graded ideal of S such that* $\dim S/I \leqslant 1$*; then*

$$\operatorname{reg}(Z_t(Z_s(I,M))) \leqslant \operatorname{reg}(Z_t(I)) + \operatorname{reg}(Z_s(I,M)).$$

Proof. By definition one has the following exact sequences:

$$(\dagger) \qquad 0 \to Z_s(I,M)) \to K_s(I,M) \xrightarrow{d_t} K_{s-1}(I,M),$$

$$(\ddagger) \qquad 0 \to Z_t(I, Z_s(I,M)) \to K_t(I, Z_s(I,M)) \xrightarrow{d_t} K_{t-1}(I, Z_s(I,M)).$$

Note that $K_s(I,M)$ and $K_{s-1}(I,M)$ are direct sums of copies of M; (†) then implies that $\operatorname{depth} Z_s(I,M) \geqslant \min\{2, \operatorname{depth} M\} = 2$. Using (‡), $\operatorname{depth} Z_t(I, Z_s(I,M)) \geq \min\{2, \operatorname{depth} Z_s(I,M)\} = 2$.

For the canonical map in [2, section 5]

$$u_{s,t} : Z_t(I) \otimes Z_s(I,M) \to Z_t(I, Z_s(I,M)).$$

Proposition 5.1 in [2] gives an exact sequence,

$$\begin{aligned} 0 &\to \ker(u_{s,t}) \to Z_t(I) \otimes Z_s(I,M) \to Z_t(Z_s(I,M)) \\ &\to \operatorname{Tor}_1^S\Big(\frac{K_{s-1}(I,M)}{B_{s-1}(I,M)}, Z_t(I)\Big) \to 0. \end{aligned}$$

Notice that after localization at prime ideals not in the support of S/I all the Koszul cycles become a direct sum of copies of M and the map $u_{s,t}$ becomes an isomorphism. Therefore, $\operatorname{Tor}_1^S(\frac{K_{s-1}(I,M)}{B_{s-1}(I,M)}, Z_t(I))$ and $\ker(u_{s,t})$ are supported in S/I, hence have a dimension at most 1.

Thus the conditions of Lemma 3.2 are fulfilled, and this lemma gives:

$$\operatorname{reg}(Z_t(Z_s(I,M))) \leqslant \operatorname{reg}(Z_t(I,M) \otimes Z_s(I,M)).$$

Notice that $\operatorname{Tor}_1^R(Z_t(I), Z_s(I,M))$ has Krull dimension at most 1 because $Z_t(I)$ is free when we localize at prime ideals not in the support of S/I. So we apply Corollary 3.1 in [6] to get

$$\operatorname{reg}(Z_t(I,M) \otimes Z_s(I,M)) \leqslant \operatorname{reg}(Z_t(I)) + \operatorname{reg}(Z_s(I,M)).$$

As a result we get

$$\operatorname{reg}(Z_t(Z_s(I,M))) \leqslant \operatorname{reg}(Z_t(I)) + \operatorname{reg}(Z_s(I,M)).$$

□

Theorem 3.4. *Let $S = k[x_1, \dots, x_n]$ and let I be a graded ideal of S. If* $\dim S/I \leqslant 1$ *and the characteristic of k is 0 or larger than $s+t$, then*

$$\operatorname{reg}(Z_{s+t}(I)) \leqslant \operatorname{reg}(Z_t(I)) + \operatorname{reg}(Z_s(I)).$$

Proof. The theorem follows from Proposition 3.3 and Lemma 3.1. □

4. Regularity of Koszul homologies

We start this section with a fact which is likely part of folklore but we did not find in the classical references.

Proposition 4.1. *Let $S = k[x_1, \dots, x_n]$ be a polynomial ring and I be an ideal of S minimaly generated by $f_1 \dots, f_r$. Then $Z_i(I) \subset \mathfrak{m}K_i(I)$ for all i.*

Proof. Suppose it is not; then there exists $z \in Z_i(I)$ that is not in $\mathfrak{m}K_i(I)$. By symmetry we may assume it has the form:

$$z = e_1 \wedge \cdots \wedge e_i + \sum_{j>i} c_j e_1 \wedge \cdots \wedge e_{i-1} \wedge e_j + \text{terms without } e_1 \wedge \cdots \wedge e_{i-1}.$$

Since $\partial(z) = 0$ it follows that $(-1)^i f_i + \sum_{j>i}(-1)^j c_j f_j = 0$, as it is the coefficient of $e_1 \wedge \cdots \wedge e_{i-1}$ in the expression of $\partial(z)$, which is a contradiction with the fact that $f_1, \dots, f_r$ is a minimal set of generators for I. □

Corollary 4.2. *Let $S = k[x_1, \dots, x_n]$ be a polynomial ring and let $I = (f_1, \dots, f_r)$ be a homogeneous ideal of S. Let $f_1, \dots, f_r$ be a minimal generating set of I and $\deg(f_i) = d_i$ where $d_1 \geq d_2 \geq \cdots \geq d_r$. Then $\operatorname{reg}(Z_i(I)) > d_1 + \cdots + d_i$ for $i \leq r$.*

Proof. Fix a basis element $e_1 \wedge \cdots \wedge e_i \in K_i$. Since $K_\bullet(I)$ is a complex,

$$\begin{aligned}&\partial(e_1 \wedge \cdots \wedge e_i \wedge e_{i+1})\\ &\quad = (-1)^{i+1} f_{i+1} e_1 \wedge \cdots \wedge e_i + \sum_{0<j<i+1} (-1)^j f_j e_1 \wedge \cdots \wedge \hat{e_j} \wedge \cdots \wedge e_{i+1} \in Z_i(I).\end{aligned}$$

Therefore an element of the form $g e_1 \wedge \cdots \wedge e_i$ should appear as a summand in a minimal generating element of $Z_i(I)$. By Proposition 4.1, $g \in \mathfrak{m}$. So there exists a minimal generator of degree at least $d_1 + \cdots + d_i + 1$. Hence $\operatorname{reg}(Z_i(I)) > d_1 + \cdots + d_i$. □

M. Chardin and P. Symonds in [4] presented a new approach to the study of the regularity of cycles of a general complex by the regularity of previous homologies. Here we determine a concrete relation between regularity of cycles and homologies of a Koszul complex.

Theorem 4.3. *Let $S = k[x_1, \dots, x_n]$ be a polynomial ring and let I be a homogeneous ideal of S minimally generated by $f_1, \dots, f_r$. If $\dim S/I \leq 1$, then for $0 < i < r$:*

$$\operatorname{reg}(Z_i(I)) = \max_{0<j\leq \min\{n-1,i\}} \{\operatorname{reg}(H_{i-j}(I)) + j + 1\}. \tag{4.1}$$

Proof. Let $I = (f_1, \dots, f_r)$ and $\deg(f_i) = d_i$ where $d_1 \geq d_2 \geq \cdots \geq d_r$. Let $K^i_\bullet(I)$ be the i-th truncated Koszul complex of I as follows:

$$K^i_\bullet(I) : 0 \to Z_i(I) \xrightarrow{\partial'_i} K_i(I) \xrightarrow{\partial_i} K_{i-1}(I) \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_1} K_0(I) \to 0$$

and $C^\bullet$ be the Čech complex. Consider double complex $X = C^\bullet \otimes K^r_\bullet(I)$ where $X_{p,q} = C^{\bullet-p} \otimes K^r_\bullet(I)_q$, and its associated spectral sequence. We first compute homology vertically and we get

$$\begin{array}{ccccc}
H^0_{\mathfrak{m}}(Z_i(I)) & 0 & 0 & \cdots & 0 \\
H^1_{\mathfrak{m}}(Z_i(I)) & 0 & 0 & \cdots & 0 \\
H^2_{\mathfrak{m}}(Z_i(I)) & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
H^n_{\mathfrak{m}}(Z_i(I)) \xrightarrow{H^n_{\mathfrak{m}}(\partial'_i)} & H^n_{\mathfrak{m}}(K_i(I)) \xrightarrow{H^n_{\mathfrak{m}}(\partial_i)} & H^n_{\mathfrak{m}}(K_{i-1}(I)) \xrightarrow{H^n_{\mathfrak{m}}(\partial_{i-1})} & \cdots \xrightarrow{H^n_{\mathfrak{m}}(\partial_1)} & H^n_{\mathfrak{m}}(K_0(I)).
\end{array}$$

By continuing the process we have:

$$E^\infty_{p,q} = E^2_{p,q} \begin{cases} H^p_{\mathfrak{m}}(Z_q(I)) & \text{if } q = i+1, p < n, \\ H_q(H^n_{\mathfrak{m}}(K^i_\bullet(I))) & \text{if } p = n, q \le i, \\ \ker(H^n_{\mathfrak{m}}(\partial'_i)) & \text{if } (p,q) = (n, i+1), \\ 0 & \text{otherwise.} \end{cases}$$

Notice that since $a^n_{\mathfrak{m}}(K_j(I)) = d_1 + \cdots + d_j - n$, it follows that for all $0 \le q \le i$ we have $\operatorname{end}(E^\infty_{n,q}) \le \operatorname{end}(E^1_{n,q}) = d_1 + \cdots + d_q - n$.

On the other hand, if we start taking homology horizontally we have ${E'}^2_{p,q} = H^p_{\mathfrak{m}}(H_q(I))$ for all p and $q < i$ and ${E'}^2_{p,q} = 0$ for $q = i, i+1$. Notice that $\dim H_i(I) \le \dim S/I \le 1$, therefore the spectral sequence collapses in the second page and we have:

$${E'}^\infty_{p,q} = {E'}^2_{p,q} \begin{cases} H^p_{\mathfrak{m}}(H_q(I)) & \text{if } p = 0, 1 \text{ and } q < i, \\ 0 & \text{otherwise.} \end{cases}$$

The comparison of two spectral sequences gives

$$\begin{aligned} & H^0_{\mathfrak{m}}(Z_i(I)) = H^1_{\mathfrak{m}}(Z_i(I)) = 0, \\ & a^2_{\mathfrak{m}}(Z_i(I)) = a^0_{\mathfrak{m}}(H_{i-1}(I)), \\ & a^j_{\mathfrak{m}}(Z_i(I)) = \max\{a^1_{\mathfrak{m}}(H_{i-j+2}(I)), a^0_{\mathfrak{m}}(H_{i-j+1}(I))\} \ \forall\ 2 < j < n. \end{aligned}$$

In addition, for the last local cohomology we have

$$a^n_{\mathfrak{m}}(Z_i(I)) \le \max\{a^1_{\mathfrak{m}}(H_{i-n+2}(I)), a^0_{\mathfrak{m}}(H_{i-n+1}(I)), d_1 + \cdots d_i - n\}.$$

Furthermore

$$a^n_{\mathfrak{m}}(Z_i(I)) = \max\{a^1_{\mathfrak{m}}(H_{i-n+2}(I)), a^0_{\mathfrak{m}}(H_{i-n+1}(I))\}$$

if $a^n_{\mathfrak{m}}(Z_i(I)) > d_1 + \cdots d_i - n$. By Corollary 4.2, we can deduce that

$$a^n_{\mathfrak{m}}(Z_i(I)) = \max\{a^1_{\mathfrak{m}}(H_{i-n+2}(I)), a^0_{\mathfrak{m}}(H_{i-n+1}(I))\} \text{ or } a^n_{\mathfrak{m}}(Z_i(I)) + n < \operatorname{reg}(Z_i(I))$$

In addition, the comparison of the two spectral sequences and Corollary 4.2 gives

$$a^1_{\mathfrak{m}}(H_{i-n+1}(I)) \le \operatorname{end}(E^\infty_{n,i-1}) \le d_1 + \cdots + d_{i-1} - n < \operatorname{reg}(Z_i(I)) - n.$$

As a result we have:

$$\begin{aligned}\operatorname{reg}(Z_i(I)) &= \max_{0\le j\le n}\{a^j_{\mathfrak{m}}(Z_i(I))+j\}\\ &= \max_{3\le j\le \max\{n,i+2\}}\{a^0_{\mathfrak{m}}(H_{i-1}(I))+2, a^1_{\mathfrak{m}}(H_{i-j+2}(I))\\ &\qquad\qquad + j, a^0_{\mathfrak{m}}(H_{i-j+1}(I))+j, d_1+\cdots d_i\}\\ &= \max_{2\le j\le \max\{n,i+1\}}\{\operatorname{reg}(H_{i-j+1}(I))+j\}.\end{aligned}$$

□

Remark 4.4. From the proof of Theorem 4.3, the following equality also holds:

$$\operatorname{reg}(Z_i(I)) = \max_{j>0}\{\operatorname{reg}(H_{i-j}(I))+j+1\}.$$

As a consequence of Theorems 4.3 and 3.4 we give a regularity bound for Koszul homologies in dimension at most 1.

Theorem 4.5. *Let $S=k[x_1,\dots,x_n]$ be a polynomial ring and I be a homogeneous ideal of S. If $\dim S/I\le 1$, then for all $i,j\ge 1$ we have the following regularity bound for the Koszul homologies of I:*

$$\operatorname{reg}(H_{i+j-1}(I)) \le \max_{0<\alpha,\beta<n}\{\operatorname{reg}(H_{i-\alpha}(I))+\operatorname{reg}(H_{j-\beta}(I))+\alpha+\beta\}. \tag{4.2}$$

Proof. By Theorem 3.4 we have the following inequality for all i,j:

$$\operatorname{reg}(Z_{i+j}(I)) \le \operatorname{reg}(Z_i(I))+\operatorname{reg}(Z_j(I)).$$

By using Theorem 4.3 we have

$$\begin{aligned}&\operatorname{reg}(H_{i+j-1}(I))+2\\ &\quad\le \operatorname{reg}(Z_{i+j}(I))\\ &\quad\le \operatorname{reg}(Z_i(I))+\operatorname{reg}(Z_j(I))\\ &\quad= \max_{0<\alpha<n}\{\operatorname{reg}(H_{i-\alpha}(I))+\alpha+1\}+\max_{0<\beta<n}\{\operatorname{reg}(H_{j-\beta}(I))+\beta+1\}\\ &\quad= \max_{0<\alpha,\beta<n}\{\operatorname{reg}(H_{i-\alpha}(I))+\operatorname{reg}(H_{j-\beta}(I))+\alpha+\beta+2\}.\end{aligned}$$

□

The following example shows the deviation degree of our bound compared to the bound provided by A. Conca and S. Murai in dimension 0.

Example 4.6. Let $S=k[x,y,z]$ be a polynomial ring and $I=(x,y,z)^4$. We compare our bound for the regularity of $H_{12}(I)$ for different i,j by the bound in [3]. By using MACAULAY2 [7] one can see that the $\operatorname{reg}(H_{12}(I))=57$. For bounding regularity of $H_{12}(I)$ we should choose i,j such that $i+j=13$. By choosing $(i,j)=(1,12)$ (respectively $(2,11),(3,10),(4,9),(5,8),(6,7)$) the right hand side of (4.2) is 57 (respectively $58,58,59,59,58$). On the other hand in the bound proposed by A. Conca and S. Murai the best possible estimate is 61.

Corollary 4.7. *Let $S=k[x_1,\dots,x_n]$ be a polynomial ideal and let I be an ideal of S. If $\dim S/I\le 1$, then*

$$\operatorname{reg}(H_i(I)) \le (i+1)\operatorname{reg}(H_0(I))+2i.$$

In particular, $\operatorname{reg}(H_i(I)) \le (i+1)(\operatorname{reg}(I)-1)+2i.$

Proof. We prove by induction. For $i = 1$ by Theorem 4.5 we have

$$\operatorname{reg}(H_1(I)) \leq \{\operatorname{reg}(H_0(I)) + 1 + \operatorname{reg}(H_0(I)) + 1\} = 2\operatorname{reg}(H_0(I)) + 2.$$

Let $\operatorname{reg}(H_i(I)) \leq (i+1)\operatorname{reg}(H_0(I)) + 2i$ for all $i \leq r$, by choosing $i = 1$ and $j = r+1$ in Corollary 4.2 we have

$$\operatorname{reg}(H_{r+1}(I)) \leq \max_{0<\beta<n} \{\operatorname{reg}(H_0(I)) + \operatorname{reg}(H_{r+1-\beta}(I)) + \beta + 1\}.$$

For all $0 < \beta < n$ we have

$$\begin{aligned}\operatorname{reg}(H_{r+1-\beta}(I)) + \beta + 1 &\leq (r - \beta + 2)\operatorname{reg}(H_0(I)) + 2(r + 1 - \beta) + \beta + 1 \\ &\leq (r + 1)\operatorname{reg}(H_0(I)) + 2(r + 1).\end{aligned}$$

Therefore, $\operatorname{reg}(H_{r+1}(I)) \leq (r + 2)\operatorname{reg}(H_0(I)) + 2(r + 1)$. □

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