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NILPOTENT ELEMENTS OF OPERATOR IDEALS AS SINGLE COMMUTATORS

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ABSTRACT. For an arbitrary operator ideal $\mathcal{I}$, every nilpotent element of $\mathcal{I}$ is a single commutator of operators from $\mathcal{I}^t$ for an exponent t that depends on the degree of nilpotency.

1. Introduction

By *operator ideal* we mean a proper, nonzero, two-sided ideal of the algebra $B(\mathcal{H})$ of bounded operators on a separable, infinite Hilbert space $\mathcal{H}$. These ideals consist of compact operators. For a compact operator, A on $\mathcal{H}$, let $s(A) = (s_1(A), s_2(A), \ldots)$ be the sequence of singular numbers of A. This is the non-increasing sequence of nonzero eigenvalues of $|A| := (A^*A)^{1/2}$, listed in order of multiplicity, with a tail of zeros in case A has finite rank. As Calkin showed [4], an operator ideal $\mathcal{I}$ is characterized by $s(\mathcal{I}) = \{s(A) \mid A \in \mathcal{I}\}$. (See also, e.g., [7] or [5] for expositions.) For a positive real number t and an operator ideal $\mathcal{I}$, we let $\mathcal{I}^t$ denote the operator ideal generated by $\{|A|^t \mid A \in \mathcal{I}\}$.

Questions about additive commutators $[B, C] := BC - CB$ involving elements of operator ideals have been much studied. One of the questions asked in [8], by Pearcy and Topping, is whether every compact operator A is a single commutator $A = [B, C]$ of compact operators B and C. This question is still open. Important results about single commutators in operator ideals were obtained by Anderson [1]. Further results are found in Section 7 of [5]. More recently, Beltiţă, Patnaik, and Weiss [3] have made progress on the above-mentioned question.

Our purpose in this note is to show that every nilpotent compact operator is a single commutator of compact operators. In fact, we show (Theorem 3.2) that for a general operator ideal $\mathcal{I}$, every nilpotent element $A \in \mathcal{I}$ is a single commutator $A = [B, C]$ of $B, C \in \mathcal{I}^t$, where the value of $t > 0$ depends on the value of n for which $A^n = 0$. Except in the case $n \leq 4$, we don't know if we have found the optimal value of t.

2. Preliminaries

Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Everything in this section is known or is at least unsurprising, but we include proofs for convenience.

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Lemma 2.1. *Suppose $x, y \in B(\mathcal{H})$ and $t \in \mathbf{R}$, $t > 0$.*

(i) *If $t(x^*x) \geq y^*y$, then there exists $r \in B(\mathcal{H})$ such that $\|r\| \leq \sqrt{t}$ and $y = rx$.*
(ii) *If $t(xx^*) \geq yy^*$, then there exists $r \in B(\mathcal{H})$ such that $\|r\| \leq \sqrt{t}$ and $y = xr$.*

Proof. The assertion (ii) follows from (i) by taking adjoints. If we prove the assertion (i) when $t = 1$, then the case of arbitrary t follows by replacing x with $\sqrt{t}x$. So we will prove (i) in the case $t = 1$.

Suppose $x^*x \geq y^*y$. Given $\xi \in \mathcal{H}$, we have

$$\|y\xi\|^2 = \langle y^*y\xi, \xi\rangle \leq \langle x^*x\xi, \xi\rangle = \|x\xi\|^2.$$

Thus, we may define a contractive linear operator from $\mathrm{ran}(x)$ into $\mathcal{H}$ by

$$x\xi \mapsto y\xi.$$

This extends uniquely to a contractive linear operator, which we call r_0, from $\overline{\mathrm{ran}(x)}$ into $\mathcal{H}$. We have $r_0x = y$. Letting p be the orthogonal projection from $\mathcal{H}$ onto $\overline{\mathrm{ran}(x)}$, we set $r = r_0p$. Thus, $r \in B(\mathcal{H})$ is a contraction and $rx = y$. □

For $n \geq 1$, we make the natural identifications

$$B(\mathcal{H}^{\oplus n}) = M_n(B(\mathcal{H})) = B(\mathcal{H}) \otimes M_n(\mathbf{C}) \tag{1}$$

and we let $(e_{i,j})_{1\leq i,j\leq n}$ be the usual system of matrix units in $M_n(\mathbf{C})$.

The fact that nilpotent operators have an upper triangular form is well known (see, for instance, Section 2 of [2]). For our purposes, we require all the entries to act on the same space, so we provide a modified proof. Recall that $\mathcal{H}$ is assumed to be infinite dimensional (and here we do not need to assume it is separable).

Lemma 2.2. *Let $A \in B(\mathcal{H})$ satisfy $A^n = 0$. Then there exists a unitary $U : \mathcal{H} \to \mathcal{H}^{\oplus n}$ such that UAU^* is a strictly upper triangular element of $M_n(B(\mathcal{H}))$.*

Proof. We will first show that $\dim\ker A = \dim\mathcal{H}$, where dim is the cardinality of an orthonormal basis. Consider $B = A|_{\ker A^2}$. Note that $\ker A = \ker B$ and that B leaves $\ker A^2$ invariant.

Assume that $\dim\ker A^2 = \dim\mathcal{H}$. If $\dim\ker B < \dim\mathcal{H}$, then

$$\dim(\ker A^2 \ominus \ker B) = \dim\mathcal{H},$$

and B maps $(\ker A^2 \ominus \ker B)$ injectively to $\mathrm{ran}\, B$. Hence $\dim\overline{\mathrm{ran}\, B} = \dim\mathcal{H}$. But $\mathrm{ran}\, B \subset \ker B$, so $\dim\overline{\mathrm{ran}\, B} \leq \dim\ker B$. Hence $\dim\ker A^2 = \dim\mathcal{H}$ implies $\dim\ker A = \dim\mathcal{H}$. Since A is nilpotent, we have $\dim\ker A^{2^k} = \dim\mathcal{H}$, for some k. Arguing by induction on k, starting from $k = 1$, we must have $\dim\ker A = \dim\mathcal{H}$.

Let

$$\mathcal{V}_1 = \ker A,$$
$$\mathcal{V}_k = \ker A^k \ominus \ker A^{k-1} \quad (2 \leq k \leq n).$$

We will construct closed subspaces

$$\mathcal{W}_1 \subseteq \mathcal{W}_2 \subseteq \cdots \subseteq \mathcal{W}_n = \mathcal{H}$$

with

$$\mathcal{W}_k \subseteq \ker A^k$$

such that, letting $\mathcal{W}_0 = \{0\}$, we have, for every $1 \leq k \leq n$,

$$\dim(\mathcal{W}_k \ominus \mathcal{W}_{k-1}) = \dim\mathcal{H} \tag{2}$$

and for every $1 \le k \le n-1$,

$$\dim((\ker A^{k+1}) \ominus \mathcal{W}_k) = \dim \mathcal{H}, \tag{3}$$

$$A(\mathcal{V}_{k+1}) \subseteq \mathcal{W}_k. \tag{4}$$

Fixing $k = 1$, if $\dim \mathcal{V}_2 = \dim \mathcal{H}$, then let $\mathcal{W}_1 = \ker A$. We know $\dim \ker A = \dim \mathcal{H}$, so (2) holds. Moreover, $\ker A^2 \ominus \mathcal{W}_1 = \mathcal{V}_2$, so (3) holds and $A(\mathcal{V}_2) \subseteq A(\ker A^2) \subseteq \ker A$, so (4) holds. Otherwise, if $\dim \mathcal{V}_2 < \dim \mathcal{H}$, then choose $\mathcal{W}_1$ closed so that

$$A(\mathcal{V}_2) \subseteq \mathcal{W}_1 \subseteq \ker A$$

and

$$\dim \mathcal{W}_1 = \dim \mathcal{H} = \dim(\ker A \ominus \mathcal{W}_1).$$

This choice is possible because we know that $\dim \ker A = \dim \mathcal{H}$ and by hypothesis $\dim A(\mathcal{V}_2) \le \dim \mathcal{V}_2 < \dim \mathcal{H}$. Then (2) and (4) (for $k = 1$) hold by construction. We have

$$\dim \mathcal{H} \ge \dim((\ker A^2) \ominus \mathcal{W}_1) \ge \dim((\ker A) \ominus \mathcal{W}_1) = \dim \mathcal{H},$$

so (3) holds.

Now suppose $2 \le k \le n-1$ and $\mathcal{W}_1, \dots, \mathcal{W}_{k-1}$ have been constructed with the required properties. If $\dim \mathcal{V}_{k+1} = \dim \mathcal{H}$, then let $\mathcal{W}_k = \ker A^k$. Then (2) for k is just (3) for $k-1$, while (3) for k is just the hypothesis $\dim(\mathcal{V}_{k+1}) = \dim \mathcal{H}$. Moreover, $A(\mathcal{V}_{k+1}) \subseteq A(\ker A^{k+1}) \subseteq \ker A^k$, so (4) holds for this k as well.

Otherwise, if $\dim \mathcal{V}_{k+1} < \dim \mathcal{H}$, then choose $\mathcal{W}_k$ closed so that

$$A(\mathcal{V}_{k+1}) + \mathcal{W}_{k-1} \subseteq \mathcal{W}_k \subseteq \ker A^k$$

and

$$\dim(\mathcal{W}_k \ominus \mathcal{W}_{k-1}) = \dim \mathcal{H} = \dim((\ker A^k) \ominus \mathcal{W}_k).$$

This is possible because, by hypothesis (namely, (3) for $k-1$),

$$\dim(\ker A^k \ominus \mathcal{W}_{k-1}) = \dim \mathcal{H}$$

and $\dim(\overline{A(\mathcal{V}_{k+1})}) \le \dim \mathcal{V}_{k+1} < \dim \mathcal{H}$. Then (2) and (4) hold by construction, while for (3), we use

$$\dim \mathcal{H} \ge \dim((\ker A^{k+1}) \ominus \mathcal{W}_k) \ge \dim((\ker A^k) \ominus \mathcal{W}_k) = \dim \mathcal{H}.$$

Finally, set $\mathcal{W}_n = \mathcal{H} = \ker A^n$. Then (2) for $k = n$ follows from (3) for $k = n-1$. Using (4), we get

$$A(\mathcal{W}_k) \subseteq A(\ker A^k) = A(\mathcal{V}_1) + \cdots + A(\mathcal{V}_k) \subseteq \mathcal{W}_{k-1}.$$

Let $\mathcal{H}_1 = \mathcal{W}_1$ and $\mathcal{H}_k = \mathcal{W}_k \ominus \mathcal{W}_{k-1}$, $2 \le k \le n$. Then $\dim \mathcal{H}_k = \dim \mathcal{H}$ for all k and

$$A(\mathcal{H}_1) = \{0\}, \tag{5}$$

$$A(\mathcal{H}_k) \subseteq A(\mathcal{W}_k) \subseteq \mathcal{W}_{k-1} = \bigoplus_{j=1}^{k-1} \mathcal{H}_j \quad (2 \le k \le n). \tag{6}$$

Choosing unitaries $U_k : \mathcal{H}_k \to \mathcal{H}$ yields a unitary $U = \bigoplus_{k=1}^{n} U_j : \mathcal{H} \to \mathcal{H}^{\oplus n}$.

Let $e_i, 1 \leq i \leq n$ be the standard basis in $\mathbf{C}^n$. Identifying $\mathcal{H}^{\oplus n}$ with $\mathcal{H} \otimes \mathbf{C}^n$, consider $x \otimes e_k \in \mathcal{H}^{\oplus n}$. Then $U^*(x \otimes e_k) \in \mathcal{H}_k$. From (5) and (6), we have

$$\begin{aligned} UAU^*(x \otimes e_1) &= 0, \\ UAU^*(x \otimes e_k) &= \sum_{j=1}^{k-1} y_{jk} \otimes e_j \quad (2 \leq k \leq n), \end{aligned}$$

for some $y_{jk} \in \mathcal{H}$. Hence $UAU^* \in M_n(B(\mathcal{H}))$ is strictly upper triangular. □

Remark 2.3. We work in $B(\mathcal{H}) \otimes M_n(\mathbf{C})$ and suppose that

$$A = \sum_{1 \leq i < j \leq n} a_{i,j} \otimes e_{i,j},$$

for $a_{i,j} \in B(\mathcal{H})$, is a strictly upper triangular matrix of operators. Here, $e_{i,j}$ are matrix units in $M_n(\mathbf{C})$. We consider upper triangular matrices of the particular forms

$$B = \sum_{i=1}^{n-1} b_i \otimes e_{i,i+1}, \qquad C = \sum_{2 \leq i \leq j \leq n} c_{i,j} \otimes e_{i,j}$$

with $b_i, c_{i,j} \in B(\mathcal{H})$. Then the condition $A = BC - CB$ is equivalent to

$$\begin{aligned} a_{1,j} &= b_1 \, c_{2,j} && (2 \leq j \leq n), \\ a_{i,j} &= b_i \, c_{i+1,j} - c_{i,j-1} b_{j-1} && (2 \leq i < j \leq n) \end{aligned}$$

or, equivalently,

$$b_1 \, c_{2,j} = a_{1,j} \qquad (2 \leq j \leq n), \tag{7}$$

$$b_i \, c_{i+1,j} = a_{i,j} + c_{i,j-1} b_{j-1} \qquad (2 \leq i < j \leq n). \tag{8}$$

3. Nilpotents in operator ideals

Let $\mathcal{I} \subseteq B(\mathcal{H})$, with $\mathcal{H}$ separable, be an operator ideal. It is well known and easy to see that under any identification of $B(\mathcal{H})$ with $M_n(B(\mathcal{H}))$ as in (1), the ideal $\mathcal{I}$ is identified with $M_n(\mathcal{I})$.

We first prove the following easy result, whose proof is similar to that of Proposition 3.2 of [6]. It serves as a precursor to and easier version of Theorem 3.2, though we won't need it in the proof of that theorem.

Proposition 3.1. *Let $\mathcal{I}$ be an operator ideal and suppose $A \in \mathcal{I}$ is nilpotent. Then there exist $B \in B(\mathcal{H})$ and $C \in \mathcal{I}$ such that $A = BC - CB$.*

Proof. Let $n \geq 2$ be such that $A^n = 0$. By Lemma 2.2, we may work in $B(\mathcal{H}) \otimes M_n(\mathbf{C})$ and suppose that

$$A = \sum_{1 \leq i < j \leq n} a_{i,j} \otimes e_{i,j},$$

for $a_{i,j} \in \mathcal{I}$. We need only find elements $b_i \in B(\mathcal{H})$ and $c_{i,j} \in \mathcal{I}$, as in Remark 2.3, so that (7) and (8) hold. This is easily done by setting $b_i = 1$ for all $i \in \{1, \ldots, n\}$ and recursively assigning

$$\begin{aligned} c_{2,j} &= a_{1,j} && (2 \leq j \leq n), \\ c_{i+1,j} &= a_{i,j} + c_{i,j-1} && (2 \leq i < j \leq n). \end{aligned}$$

□

Theorem 3.2. *Let $\mathcal{I}$ be an operator ideal and suppose $A \in \mathcal{I}$ satisfies $A^n = 0$, for some integer $n \geq 4$. Then there exist $B, C \in \mathcal{I}^{1/2^{n-3}}$ such that $A = BC - CB$.*

Proof. By Lemma 2.2, we may work in $B(\mathcal{H}) \otimes M_n(\mathbf{C})$ and suppose that

$$A = \sum_{1 \leq i < j \leq n} a_{i,j} \otimes e_{i,j},$$

for $a_{i,j} \in \mathcal{I}$. We will find elements b_i and $c_{i,j}$ of $\mathcal{I}^{1/2^{n-3}}$, as in Remark 2.3, so that (7) and (8) hold.

Step 1 (Assign values to $b_1, \dots, b_{n-2}$). Let

$$\begin{aligned} b_1 &= \left(\sum_{j=2}^{n} |a_{1,j}^*|^2 \right)^{1/4} \in \mathcal{I}^{1/2}, \\ b_i &= \left(b_{i-1}^2 + \sum_{j=i+1}^{n} |a_{i,j}^*|^2 \right)^{1/4} \in \mathcal{I}^{1/2^i} \qquad (2 \leq i \leq n-3), \\ b_{n-2} &= \left(b_{n-3}^4 + \sum_{j=i+1}^{n} |a_{i,j}^*|^2 \right)^{1/4} \in \mathcal{I}^{1/2^{n-3}}. \end{aligned}$$

Since for every $1 \leq i \leq n-2$ and every $i < j \leq n$, we have $b_i^4 \geq |a_{i,j}^*|^2$, by Lemma 2.1 there exists $r_{i,j} \in B(\mathcal{H})$ such that

$$b_i^2 r_{i,j} = a_{i,j} \qquad (1 \leq i \leq n-2,\ i < j \leq n).$$

Moreover, for every $2 \leq i \leq n-3$, since $b_i^4 \geq b_{i-1}^2$, by the same lemma there exists $x_i \in B(\mathcal{H})$ such that

$$b_i^2 x_i = b_{i-1} \qquad (2 \leq i \leq n-3).$$

Furthermore, since $b_{n-2}^4 \geq b_{n-3}^4$ and the square root function is operator monotone, we have $b_{n-2}^2 \geq b_{n-3}^2$. Thus, by Lemma 2.1 there exists $z \in B(\mathcal{H})$ so that

$$b_{n-2}\, z = b_{n-3}.$$

Step 2 (Assign values to $c_{2,j}$ and auxiliary variables $y_{2,j}$ for $2 \leq j \leq n$ and verify (7)). Let

$$y_{2,j} = r_{1,j}, \qquad c_{2,j} = b_1 y_{2,j} \qquad (2 \leq j \leq n).$$

Thus, $c_{2,j} \in \mathcal{I}^{1/2}$. Then we have

$$b_1 c_{2,j} = b_1^2 r_{1,j} = a_{1,j} \qquad (2 \leq j \leq n);$$

namely, (7) holds.

Step 3 (Assign values to $c_{p,j}$ and auxiliary variables $y_{p,j}$ for $3 \leq p \leq n-2$ and $p \leq j \leq n-1$ and verify the equality in (8) for $2 \leq i \leq n-3$ and $i < j \leq n-1$). We let p increase from 3 to $n-2$ and for each such p we define (recursively in p) for every $j \in \{p, p+1, \dots, n-1\}$,

$$y_{p,j} = r_{p-1,j} + x_{p-1} y_{p-1,j-1} b_{j-1}, \qquad c_{p,j} = b_{p-1} y_{p,j}.$$

Thus, $c_{p,j} \in \mathcal{I}^{1/2^{p-1}}$, and we have

$$\begin{aligned} b_i c_{i+1,j} &= b_i^2 r_{i,j} + b_i^2 x_i y_{i,j-1} b_{j-1} \\ &= a_{i,j} + b_{i-1} y_{i,j-1} b_{j-1} \\ &= a_{i,j} + c_{i,j-1} b_{j-1} \qquad (2 \le i \le n-3,\ i < j \le n-1), \end{aligned}$$

and the equality in (8) holds for these values of i and j.

Step 4 (Assign a value to $c_{n-1,n-1}$ and verify the equality in (8) for $i = n-2$ and $j = n-1$). Let

$$c_{n-1,n-1} = b_{n-2} r_{n-2,n-1} + z y_{n-2,n-2} b_{n-2}.$$

Then $c_{n-1,n-1} \in \mathcal{I}^{1/2^{n-3}}$ and

$$\begin{aligned} b_{n-2} c_{n-1,n-1} &= b_{n-2}^2 r_{n-2,n-1} + b_{n-2} z y_{n-2,n-2} b_{n-2} \\ &= a_{n-2,n-1} + b_{n-3} y_{n-2,n-2} b_{n-2} \\ &= a_{n-2,n-1} + c_{n-2,n-2} b_{n-2}. \end{aligned}$$

Thus, the equality in (8) holds for $i = n-2$ and $j = n-1$.

Step 5 (Assign a value to b_{n-1}). Let

$$b_{n-1} = \left(|a_{n-1,n}^*|^2 + |c_{n-1,n-1}^*|^4\right)^{1/4}.$$

Then $b_{n-1} \in \mathcal{I}^{1/2^{n-3}}$. Since $b_{n-1}^4 \ge |a_{n-1,n}^*|^2$, by Lemma 2.1 there is $r_{n-1,n} \in B(\mathcal{H})$ so that

$$b_{n-1}^2 r_{n-1,n} = a_{n-1,n}.$$

Since $b_{n-1}^4 \ge |c_{n-1,n-1}^*|^4$ and the square root function is operator monotone, we have $b_{n-1}^2 \ge |c_{n-1,n-1}^*|^2$ and, from Lemma 2.1, we have $s \in B(\mathcal{H})$ so that

$$b_{n-1} s = c_{n-1,n-1}.$$

Step 6 (Assign values to $c_{p,n}$ for all $3 \le p \le n-2$ and verify the equality in (8) for all $2 \le i \le n-3$ and $j = n$). Let

$$c_{p,n} = b_{p-1} r_{p-1,n} + b_{p-1} x_{p-1} y_{p-1,n-1} b_{n-1}.$$

Then $c_{p,n} \in \mathcal{I}^{1/2^{p-1}}$ and

$$\begin{aligned} b_i c_{i+1,n} &= b_i^2 r_{i,n} + b_i^2 x_i y_{i,n-1} b_{n-1} \\ &= a_{i,n} + b_{i-1} y_{i,n-1} b_{n-1} \\ &= a_{i,n} + c_{i,n-1} b_{n-1} \qquad (2 \le i \le n-3); \end{aligned}$$

namely, the equality in (8) holds for these values of i and for $j = n$.

Step 7 (Assign a value to $c_{n-1,n}$ and verify the equality in (8) for $i = n-2$ and $j = n$). Let

$$c_{n-1,n} = b_{n-2} r_{n-2,n} + z y_{n-2,n-1} b_{n-1}.$$

Then $c_{n-1,n} \in \mathcal{I}^{1/2^{n-3}}$ and

$$\begin{aligned} b_{n-2} c_{n-1,n} &= b_{n-2}^2 r_{n-2,n} + b_{n-2} z y_{n-2,n-1} b_{n-1} \\ &= a_{n-2,n} + b_{n-3} y_{n-2,n-1} b_{n-1} \\ &= a_{n-2,n} + c_{n-2,n-1} b_{n-1}; \end{aligned}$$

namely, the equality in (8) holds for $i = n-2$ and for $j = n$.

Step 8 (Assign a value to $c_{n,n}$ and verify the equality in (8) for $i = n-1$ and $j = n$). Let

$$c_{n,n} = b_{n-1}r_{n-1,n} + sb_{n-1}.$$

Then $c_{n,n} \in \mathcal{I}^{1/2^{n-3}}$ and

$$b_{n-1}c_{n,n} = b_{n-1}^2 r_{n-1,n} + b_{n-1}sb_{n-1} = a_{n-1,n} + c_{n-1,n-1}b_{n-1},$$

as required. □

Corollary 3.3. *Let $\mathcal{I}$ by any operator ideal such that $\mathcal{I}^t \subseteq \mathcal{I}$ for every $t > 0$. Then for every nilpotent element A of $\mathcal{I}$, there exist $B, C \in \mathcal{I}$ such that $A = BC - CB$.*

Examples of operator ideals $\mathcal{I}$ satisfying the conditions of Corollary 3.3 include:

(a) the ideal $\mathcal{K}$ of all compact operators;
(b) the ideal of all operators A whose singular numbers have polynomial decay: $s_n(A) = O(n^{-t})$ for some $t > 0$; note that this ideal is equal to the union of all Schatten p-class ideals, $p \geq 1$;
(c) the ideal of all operators A whose singular numbers have exponential decay: $s_n(A) = O(r^n)$ for some $0 < r < 1$;
(d) the ideal of all finite rank operators.

Question 3.4. Is $1/2^{n-3}$ the optimal (i.e., largest possible) exponent of $\mathcal{I}$ in Theorem 3.2? Clearly, the answer is yes when $n = 4$. But as far as we know, it is possible that the best exponent is $1/2$ for arbitrary n.

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