PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 7, July 2018, Pages 3039–3047
http://dx.doi.org/10.1090/proc/14003
Article electronically published on February 28, 2018

ON ALGEBRAIC MULTIPLICITY OF (ANTI)PERIODIC EIGENVALUES OF HILL'S EQUATIONS

ZHIJIE CHEN AND CHANG-SHOU LIN

(Communicated by Lei Ni)

ABSTRACT. We construct two explicit examples of Hill's equations with complex-valued potentials such that the algebraic multiplicity of some (anti)periodic eigenvalue E equals $1 + 2p_i$ with $p_i \geq 1$, where p_i denotes the immovable part of E as a Dirichlet eigenvalue. These examples confirm a phenomena about Hill's equations in (Gesztesy and Weikard, Acta Math. **176** (1996), 73–107).

1. INTRODUCTION

Let $q(x)$ be a complex-valued continuous nonconstant periodic function of period Ω on $\mathbb{R}$. Consider the Hill's equation

$$y''(x) + q(x)y(x) = Ey(x), \quad x \in \mathbb{R}. \tag{1.1}$$

This equation has received an enormous amount of consideration due to its ubiquity in applications as well as its structural richness. A typical example is its connection with the KdV hierarchy and hence with integrable systems. We refer the readers to Gesztesy and Weikard's remarkable work [6] for an overview on this subject, where the intimate connection between Picard potentials and elliptic finite-gap solutions of the stationary KdV hierarchy was established for the first time.

Let $y_1(x)$ and $y_2(x)$ be any two linearly independent solutions of equation (1.1). Then so are $y_1(x + \Omega)$ and $y_2(x + \Omega)$, and hence there exists a monodromy matrix $M(E) \in \mathrm{SL}(2, \mathbb{C})$ such that

$$(y_1(x + \Omega), y_2(x + \Omega)) = (y_1(x), y_2(x))M(E).$$

Let

$$\Delta(E) := \mathrm{tr}M(E)$$

be the trace of the monodromy matrix, which is indeed an invariant of equation (1.1), i.e., it does not depend on the choice of linearly independent solutions $y_1(x), y_2(x)$. This $\Delta(E)$ plays a fundamental role in Floquet theory since it encodes all the spectrum information of the associated operator; see, e.g., [6] and the references therein. In particular, we define

$$d(E) := \mathrm{ord}_E(\Delta(\cdot)^2 - 4).$$

Then $d(E)$ is known to coincide with *the algebraic multiplicity of (anti)periodic eigenvalues*. A basic problem is to determine $d(E)$ for all (anti)periodic eigenvalues.

Received by the editors July 9, 2017, and, in revised form, October 18, 2017.
2010 *Mathematics Subject Classification.* Primary 34B30.
The research of the first author was supported by NSFC (No. 11701312).

If $q(x)$ is real-valued, then a classical result (cf. [8]) shows that the algebraic and geometric multiplicities of (anti)periodic eigenvalues *coincide* and hence $d(E) \leq 2$ for all E's. However, when $q(x)$ is either complex-valued or has singularities, this classical result might not hold. If the algebraic multiplicities are larger than geometric ones, the analysis of Hill's equation becomes much more difficult. In this note, we will give explicit examples that $d(E) \geq 3$ for some (anti)periodic eigenvalues.

Let $c(E, x, x_0)$ and $s(E, x, x_0)$ be the special fundamental system of solutions of (1.1) defined by

$$c(E, x_0, x_0) = s'(E, x_0, x_0) = 1, \; c'(E, x_0, x_0) = s(E, x_0, x_0) = 0.$$

Then we have

$$\Delta(E) = c(E, x_0 + \Omega, x_0) + s'(E, x_0 + \Omega, x_0).$$

Define

$$p(E, x_0) := \mathrm{ord}_E s(\cdot, x_0 + \Omega, x_0),$$

$$p_i(E) := \min\{p(E, x_0) : x_0 \in \mathbb{R}\}.$$

It is known (cf. [6]) that $p(E, x_0)$ and $p_i(E)$ are the algebraic multiplicity of a Dirichlet eigenvalue and its immovable part, respectively. It was proved in [6, Theorem 3.2] that $d(E) - 2p_i(E) \geq 0$. Define

$$B_1 := \{E \,|\, 0 = p_i(E);\; 0 < d(E)\},$$

$$B_2 := \{E \,|\, 0 < p_i(E);\; 2p_i(E) < d(E)\}.$$

As mentioned before, $B_2 = \emptyset$ if $q(x)$ is real-valued. In the literature, there are references studying Hill's equation with complex-valued $q(x)$ under the restriction that the algebraic multiplicities of all (anti)periodic eigenvalues still coincide with the geometric multiplicities and hence are at most 2, i.e., $B_2 = \emptyset$; see, e.g., [1, 2]. On the other hand, Gesztesy and Weikard pointed out that (see [6, p. 94]): "*For B_2 to be nonempty, it is necessary that $d(\lambda) \geq 3$ for some (anti)periodic eigenvalue λ. While it seems difficult to construct an explicit example where $B_2 \neq \emptyset$, the very existence of this phenomenon has not been considered in previous work on the subject.*"

As far as we know, there still seem to be no explicit examples with $B_2 \neq \emptyset$. The purpose of this note is to construct two explicit examples of Hill's equations such that $B_2 \neq \emptyset$, i.e. $d(E) \geq 3$ for some (anti)periodic eigenvalue E, and hence confirms the aforementioned phenomena pointed out by Gesztesy and Weikard for complex-valued potentials. In particular, the algebraic and geometric multiplicities of (anti)periodic eigenvalues do not necessarily coincide for complex-valued potentials or real-valued potentials with singularities (see, e.g., Remark 2.3).

2. Two explicit examples

For $\tau \in \mathbb{C}$ with $\mathrm{Im}\,\tau > 0$, we let $\wp(z) = \wp(z; \tau)$ be the Weierstrass $\wp$-function with periods $\omega_1 = 1$ and $\omega_2 = \tau$, defined by

$$\wp(z; \tau) := \frac{1}{z^2} + \sum_{\omega \in (\mathbb{Z} + \mathbb{Z}\tau) \setminus \{0\}} \left(\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right),$$

and let $e_k = e_k(\tau) := \wp(\frac{\omega_k}{2};\tau)$ for $k \in \{1,2,3\}$, where $\omega_3 = 1+\tau$. It is well known that

$$\wp'(z)^2 = 4\prod_{k=1}^{3}(\wp(z)-e_k) = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

Let $\zeta(z) = \zeta(z;\tau) := -\int^z \wp(\xi;\tau)d\xi$ be the Weierstrass zeta function and let $\eta_k = \eta_k(\tau) := 2\zeta(\frac{\omega_k}{2};\tau)$. Clearly $\eta_3 = \eta_1 + \eta_2$ and

$$\zeta(z+\omega_k) = \zeta(z) + \eta_k, \quad k = 1,2,3.$$

To give the first example, we recall [9, Theorem 1.7 and Lemma 6.1] that $e_1(\cdot)+\eta_1(\cdot)$ is increasing along the line $\frac{1}{2}+i\mathbb{R}_{>0}$ and hence has a unique zero $\tau = \frac{1}{2}+ib_0$ with $b_0 \in (0,\frac{1}{2})$. Fix this $\tau = \frac{1}{2}+ib_0$ such that

$$e_1 + \eta_1 = e_1(\tau) + \eta_1(\tau) = 0. \tag{2.1}$$

Consider the following Hill's equation with Lamé potential $q(x) = -2\wp(x+\tau/2) = -2\wp(x+\tau/2;\tau)$ and period $\Omega = 1$:

$$y''(x) - 2\wp(x+\tau/2)y(x) = Ey(x), \quad x \in \mathbb{R}. \tag{2.2}$$

It is known (see, e.g., [7, Example 2.1] or [3, Example 7.4.1]) that the associated elliptic curve of (2.2) is

$$F^2 = R_3(E) = \prod_{k=1}^{3}(E-e_k). \tag{2.3}$$

It follows from [6, Theorem 4.1] and (2.3) that

$$B_1 \cup B_2 = \{e_1, e_2, e_3\}.$$

Furthermore, by defining

$$D(E) := E^{p_i(0)} \prod_{\lambda\in\mathbb{C}\setminus\{0\}} \left(1-\frac{E}{\lambda}\right)^{p_i(\lambda)}, \tag{2.4}$$

we have

$$R_3(E) = \prod_{k=1}^{3}(E-e_k) = C\frac{\Delta(E)^2-4}{D(E)^2},$$

where C is some nonzero constant. See also [6, Theorem 4.1]. Therefore,

$$d(e_k) = 1 + 2p_i(e_k), \quad k = 1,2,3. \tag{2.5}$$

Note that $e_1 < 0$; see Remark 2.3 below. The following result gives our first explicit example that $B_2 \neq \emptyset$.

Theorem 2.1. *For equation* (2.2) *with* $\tau = \frac{1}{2}+ib_0$*, there holds* $B_2 = \{e_1\}$*. Furthermore,*

$$d(e_1) = ord_{e_1}(\Delta(\cdot)^2 - 4) = 3.$$

Proof. Let $k \in \{1,2,3\}$. Consider the Lamé equation

$$y''(z) - 2\wp(z)y(z) = e_k y(z), \quad z \in \mathbb{C}. \tag{2.6}$$

Let $\sigma(z) = \sigma(z;\tau)$ be the Weierstrass sigma function associated with the lattice $\mathbb{Z}+\mathbb{Z}\tau$. Recall $\sigma'(z)/\sigma(z) = \zeta(z)$. A direct computation shows that

$$y_1(z) := e^{\frac{1}{2}\eta_k z}\frac{\sigma(z-\frac{\omega_k}{2})}{\sigma(z)}$$

is a solution of (2.6) and

$$y_1(-z) = -y_1(z), \quad y_1(z+1) = \varepsilon_1 y_1(z), \quad y_1(z+\tau) = \varepsilon_2 y_1(z), \tag{2.7}$$

with

$$(\varepsilon_1, \varepsilon_2) = \begin{cases} (1,-1) & \text{if } k=1, \\ (-1,1) & \text{if } k=2, \\ (-1,-1) & \text{if } k=3. \end{cases} \tag{2.8}$$

Here we used the Legendre relation $\tau\eta_1 - \eta_2 = 2\pi i$ and the transformation law of $\sigma(z)$ to obtain (2.7)–(2.8):

$$\sigma(z+\omega_l) = -e^{\eta_l(z+\frac{\omega_l}{2})}\sigma(z), \quad l=1,2,3. \tag{2.9}$$

Thus, $y_1(z)^2$ is even elliptic and

$$\frac{1}{y_1(z)^2} = e^{-\eta_k z}\frac{\sigma(z)^2}{\sigma(z-\frac{\omega_k}{2})^2} = \frac{\wp(z-\omega_k/2)-e_k}{C_1},$$

where C_1 is a nonzero constant. Define

$$\chi(z) := -\zeta(z-\omega_k/2) - e_k z;$$

then

$$\chi(z+1) = \chi(z) - \eta_1 - e_k. \tag{2.10}$$

Clearly

$$\chi'(z) = \wp(z-\omega_k/2) - e_k = \frac{C_1}{y_1(z)^2},$$

so a direct computation gives that

$$y_2(z) := y_1(z)\chi(z)$$

is another solution of (2.6) and is linearly independent with $y_1(z)$. Let

$$\tilde{y}_j(x) := y_j(x+\tau/2), \quad x\in\mathbb{R}, \quad j=1,2.$$

Then $\tilde{y}_1(x)$ and $\tilde{y}_2(x)$ are linearly independent solutions of equation (2.2) with $E = e_k$, i.e.,

$$y''(x) - 2\wp(x+\tau/2)y(x) = e_k y(x), \quad x\in\mathbb{R}. \tag{2.11}$$

Furthermore, (2.7) and (2.10) give

$$\tilde{y}_1(x+1) = \varepsilon_1\tilde{y}_1(x), \quad \tilde{y}_2(x+1) = \varepsilon_1\tilde{y}_2(x) - \varepsilon_1(\eta_1+e_k)\tilde{y}_1(x). \tag{2.12}$$

Case 1. $k=1$.

Then (2.1), (2.8), and (2.12) give

$$\tilde{y}_1(x+1) = \tilde{y}_1(x), \quad \tilde{y}_2(x+1) = \tilde{y}_2(x),$$

which implies that all solutions of equation (2.11) with $k=1$ are periodic, and so does $s(e_1,x,x_0)$ for any $x_0\in\mathbb{R}$. In particular, $s(e_1,x_0+1,x_0)=0$ for all $x_0\in\mathbb{R}$, and so $p_i(e_1)\geq 1$. Then (2.5) gives $d(e_1)\geq 3$ and $e_1\in B_2$.

Case 2. $k\in\{2,3\}$.

Then $e_k+\eta_1\neq 0$, and so (2.12) shows that the dimension of antiperiodic solutions of equation (2.11) is only 1. Together with [6, Proposition 3.1], we immediately obtain $p_i(e_k)=0$, i.e., $d(e_2)=d(e_3)=1$. Therefore,

$$e_2, e_3\in B_1, \quad \text{and so} \quad B_2 = \{e_1\}.$$

Finally, we need to prove $d(e_1) = 3$. It is known (cf. [6, Theorem 4.1]) that the spectrum of the operator H associated with $\frac{d^2}{dx^2} - 2\wp(x+\tau/2)$ in $L^2(\mathbb{R}, \mathbb{C})$ is given by

$$\sigma(H) = \{E \in \mathbb{C} : -2 \le \Delta(E) \le 2\} = \sigma_1 \cup \sigma_\infty,$$

where σ_1 is a bounded spectral arc, the two endpoints of which are in $\{e_1, e_2, e_3\}$, and σ_∞ is a semi-infinite spectral arc which extends to ∞ and the finite endpoint is also in $\{e_1, e_2, e_3\}$. Since $d(e_k)$'s are all odd, it is easy to prove (see e.g. [6, Theorem 4.1]) that there are $d(e_k)$ semiarcs meeting at e_k. Since $d(e_2) = d(e_3) = 1$ and $d(e_1) \ge 3$, we conclude that $d(e_1) = 3$, the two endpoints of σ_1 are e_2, e_3, the finite endpoint of σ_∞ is e_1, and the two spectral arcs σ_∞ and σ_1 intersect at e_1. The proof is complete. □

Remark 2.2. We emphasize that the above proof gives a clear picture of the spectrum: $\sigma(H) = \sigma_2 \cup \sigma_3 \cup \sigma_\infty$, where these three spectral curves have the same endpoint e_1, with other different endpoints e_2, e_3, and ∞, respectively.

Remark 2.3. For Hill's equation (1.1), we mentioned in Section 1 that if the nonconstant periodic function $q(x)$ is real-valued and continuous (in particular, no poles on $\mathbb{R}$), then

$$d(E) = \operatorname{ord}_E(\Delta(\cdot)^2 - 4) \le 2 \quad \forall\, E. \tag{2.13}$$

Theorem 2.1 indicates that (2.13) does not necessarily hold for complex-valued potentials. On the other hand, since $\tau = \frac{1}{2} + ib_0 = 1 - \bar{\tau}$ gives

$$\overline{\wp(z;\tau)} = \wp(\bar{z}; 1-\bar{\tau}) = \wp(\bar{z};\tau),$$

we see that $\wp(x) = \wp(x;\tau)$ is real-valued for $x \in \mathbb{R}$, and hence $e_1 \in \mathbb{R}$. In fact, $e_1 < 0$ because of $\tau = \frac{1}{2} + ib_0$ with $b_0 \in (0, \frac{1}{2})$ and $e_1(\frac{1+i}{2}) = 0$. Consider Hill's equation (1.1) with potential $q(x) = -2\wp(x)$ which is real-valued but has singularities at $x \in \mathbb{Z}$:

$$y''(x) - 2\wp(x)y(x) = Ey(x), \quad x \in \mathbb{R}. \tag{2.14}$$

Then the proof of Theorem 2.1 also implies that $d(e_1) = \operatorname{ord}_{e_1}(\Delta(\cdot)^2 - 4) = 3$ for equation (2.14). Therefore, (2.13) does not necessarily hold for real-valued potentials with singularities. We refer the reader to [10] for Hill's equation with singular potentials.

Remark 2.4. Fix $k \in \{2, 3\}$. As in [5, Theorem 6.6], we can also prove the existence of τ (but not on the line $\frac{1}{2} + i\mathbb{R}_{>0}$) such that

$$e_k + \eta_1 = e_k(\tau) + \eta_1(\tau) = 0.$$

Fix any such τ. Then the same proof as Theorem 2.1 shows that, for equation (2.2) with this new τ, there holds $B_2 = \{e_k\}$ and $d(e_k) = 3$.

Now we introduce our second example. It was proved in [4] that

$$e_1'(\tau) = \frac{i}{\pi}\left[\tfrac{1}{6}g_2(\tau) + \eta_1(\tau)e_1(\tau) - e_1(\tau)^2\right]$$

has infinitey many zeros on $\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\}$. Fix any such τ such that

$$\tfrac{1}{6}g_2(\tau) + \eta_1(\tau)e_1(\tau) - e_1(\tau)^2 = 0. \tag{2.15}$$

It is easy to derive from (2.15) and

$$e_1 + e_2 + e_3 = 0, \quad g_2 = 2(e_1^2 + e_2^2 + e_3^2) \tag{2.16}$$

that

$$\tfrac{1}{6}g_2(\tau) + \eta_1(\tau)e_j(\tau) - e_j(\tau)^2 \neq 0, \quad j \in \{2,3\}, \tag{2.17}$$
$$12\eta_1(\tau)^2 - g_2(\tau) \neq 0.$$

Consider the following Hill's equation with Lamé potential $q(x) = -6\wp(x+\tau/2) = -6\wp(x+\tau/2;\tau)$ and period $\Omega = 1$:

$$y''(x) - 6\wp(x+\tau/2)y(x) = Ey(x), \quad x \in \mathbb{R}. \tag{2.18}$$

It is known (see, e.g., [7, Example 2.1] or [3, Example 7.4.1]) that the associated hyperelliptic curve of (2.18) is

$$F^2 = R_5(E) = (E^2 - 3g_2)\prod_{j=1}^{3}(E + 3e_j).$$

Again it follows from [6, Theorem 4.1] that

$$B_1 \cup B_2 = \{-3e_1, -3e_2, -3e_3, (3g_2)^{1/2}, -(3g_2)^{1/2}\}.$$

It is easy to see from (2.16) that

$$\{-3e_1, -3e_2, -3e_3\} \cap \{(3g_2)^{1/2}, -(3g_2)^{1/2}\} = \emptyset.$$

Define $D(E)$ as in (2.4); then

$$R_5(E) = (E^2 - 3g_2)\prod_{j=1}^{3}(E + 3e_j) = C\frac{\Delta(E)^2 - 4}{D(E)^2},$$

where C is some nonzero constant. Therefore,

$$d(-3e_j) = 1 + 2p_i(-3e_j), \quad j = 1,2,3. \tag{2.19}$$

Remark 2.5. It is well known that $g_2(\tilde{\tau}) = 0$ if and only if $\tilde{\tau} = \frac{ae^{\pi i/3}+b}{ce^{\pi i/3}+d}$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$. From here, we can actually prove that if $e_1'(\tau) = 0$ (i.e., (2.15) holds), then $e_1(\tau) - \eta_1(\tau) \neq 0$, and so $g_2(\tau) \neq 0$, which implies $(3g_2)^{1/2} \neq -(3g_2)^{1/2}$, i.e., (2.19) also hold for $\pm(3g_2)^{1/2}$. Since we do not need this fact in this paper, we refer to [4] for the proof of $g_2(\tau) \neq 0$, which is not trivial at all.

Theorem 2.6. *For equation* (2.18) *with* τ *satisfying* (2.15)*, there holds* $B_2 = \{-3e_1\}$*, i.e.,*

$$d(-3e_1) = ord_{-3e_1}(\Delta(\cdot)^2 - 4) \geq 3.$$

Proof.

Step 1. We prove that $(3g_2)^{1/2}, -(3g_2)^{1/2} \in B_1$.

Denote $E_\pm = \pm(3g_2)^{1/2}$ and consider the the Lamé equation

$$y''(z) - 6\wp(z)y(z) = E_\pm y(z), \quad z \in \mathbb{C}. \tag{2.20}$$

Let $(a_\pm, -a_\pm)$ be a pair of complex numbers such that

$$\wp(a_\pm) = \wp(-a_\pm) = E_\pm/6 = \pm(g_2/12)^{1/2}. \tag{2.21}$$

Since $\wp'' = 6\wp^2 - g_2/2$, we obtain $\wp''(a_\pm) = \wp''(-a_\pm) = 0$. Furthermore, a direct computation shows that

$$y_1(z) := \wp(z) - \wp(a_\pm) = \wp(z) - \frac{E_\pm}{6}$$

is a solution of equation (2.20). Note that $y_1(z)$ is even elliptic, so

$$y_1(z+1) = y_1(z), \quad y_1(z+\tau) = y_1(z). \tag{2.22}$$

Since $\wp''(a_\pm) = 0$ implies that the residue of $\frac{1}{y_1(z)^2}$ at $a_\pm$ and $-a_\pm$ are both 0, we easily obtain

$$\frac{\wp'(a_\pm)^2}{y_1(z)^2} = \wp(z-a_\pm) + \wp(z+a_\pm) - 2\wp(a_\pm).$$

Define

$$\chi(z) := -(\zeta(z-a_\pm) + \zeta(z+a_\pm) + 2\wp(a_\pm)z);$$

then

$$\chi(z+1) = \chi(z) - 2(\eta_1 + \wp(a_\pm)). \tag{2.23}$$

Define $y_2(z) := y_1(z)\chi(z)$. Since $\chi'(z) = \frac{\wp'(a_\pm)^2}{y_1(z)^2}$, as before, $y_2(z)$ is another solution of (2.20) and is linearly independent with $y_1(z)$. Let

$$\tilde{y}_j(x) := y_j(x+\tau/2), \quad x \in \mathbb{R}, \quad j = 1, 2.$$

Then $\tilde{y}_1(x)$ and $\tilde{y}_2(x)$ are linearly independent solutions of equation (2.18) with $E = E_\pm$, i.e.,

$$y''(x) - 6\wp(x+\tau/2)y(x) = E_\pm y(x), \quad x \in \mathbb{R}.$$

Furthermore, (2.22) and (2.23) give

$$\tilde{y}_1(x+1) = \tilde{y}_1(x), \quad \tilde{y}_2(x+1) = \tilde{y}_2(x) - 2(\eta_1 + \wp(a_\pm))\tilde{y}_1(x).$$

Recall from (2.17) and (2.21) that $\eta_1 + \wp(a_\pm) \neq 0$. Then as in the proof of Theorem 2.1, we conclude that $p_i(E_\pm) = 0$, i.e., $E_\pm \in B_1$.

Step 2. We prove that $-3e_2, -3e_3 \in B_1$ and $-3e_1 \in B_2$.

Let $\{i, j, k\} = \{1, 2, 3\}$. Consider the Lamé equation

$$y''(z) - 6\wp(z)y(z) = -3e_k y(z), \quad z \in \mathbb{C}. \tag{2.24}$$

Then a direct computation shows that

$$y_1(z) = e^{\frac{1}{2}(\eta_i+\eta_j)z} \frac{\sigma(z-\frac{\omega_i}{2})\sigma(z-\frac{\omega_j}{2})}{\sigma(z)^2} \tag{2.25}$$

is a solution of equation (2.24). By the Legendre relation $\tau\eta_1 - \eta_2 = 2\pi i$ and the transformation law (2.9) of $\sigma(z)$, we easily obtain $y_1(-z) = y_1(z)$ and

$$y_1(z+1) = \varepsilon_1 y_1(z), \quad y_1(z+\tau) = \varepsilon_2 y_1(z), \tag{2.26}$$

where $(\varepsilon_1, \varepsilon_2)$ is given by (2.8). Thus, $y_1(z)^{-2}$ is even elliptic and there exists a constant $C_3 \neq 0$ such that

$$\begin{aligned}\frac{C_3}{y_1(z)^2} &= \frac{1}{(\wp(z)-\wp(\frac{\omega_i}{2}))(\wp(z)-\wp(\frac{\omega_j}{2}))} \\ &= c_1\left(\wp(z-\tfrac{\omega_i}{2}) - e_i\right) + c_2\left(\wp(z-\tfrac{\omega_j}{2}) - e_j\right),\end{aligned}$$

where

$$c_1 = \frac{2}{(e_i-e_j)\wp''(\frac{\omega_i}{2})}, \quad c_2 = \frac{2}{(e_j-e_i)\wp''(\frac{\omega_j}{2})}.$$

By $\wp''(\frac{\omega_i}{2}) = 6e_i^2 - g_2/2$ and $g_2 = 4(e_k^2 - e_ie_j)$, a direct computation gives

$$c_1 + c_2 = \frac{-12(e_i+e_j)}{\wp''(\frac{\omega_i}{2})\wp''(\frac{\omega_j}{2})} = \frac{12e_k}{\wp''(\frac{\omega_i}{2})\wp''(\frac{\omega_j}{2})},$$

$$c_1e_i + c_2e_j = \frac{2g_2 - 12e_k^2}{\wp''(\frac{\omega_i}{2})\wp''(\frac{\omega_j}{2})}.$$

Define

$$\chi(z) := -c_1\left(\zeta(z - \tfrac{\omega_i}{2}) + e_iz\right) - c_2\left(\zeta(z - \tfrac{\omega_j}{2}) + e_jz\right);$$

then

$$\chi'(z) = \frac{C_3}{y_1(z)^2}$$

and

$$\begin{aligned}\chi(z+1) - \chi(z) &= -\eta_1(c_1 + c_2) - (c_1e_i + c_2e_j)\\ &= -12\frac{e_k\eta_1 + \frac{g_2}{6} - e_k^2}{\wp''(\frac{\omega_i}{2})\wp''(\frac{\omega_j}{2})} =: A_k.\end{aligned} \tag{2.27}$$

Since (2.15) and (2.17) give $A_1 = 0$ and $A_j \neq 0$ for $j = 2, 3$, the same proof as Theorem 2.1 shows that $p_i(-3e_2) = p_i(-3e_3) = 0$, i.e., $-3e_2, -3e_3 \in B_1$, and $p_i(-3e_1) \geq 1$, i.e., $d(-3e_1) \geq 3$ and $-3e_1 \in B_2$.

In conclusion, $B_2 = \{-3e_1\}$ and $B_1 = \{-3e_2, -3e_3, (3g_2)^{1/2}, -(3g_2)^{1/2}\}$. This completes the proof. □

Remark 2.7. If we apply the similar argument as in Theorem 2.1 by using the spectrum of the associated operator for equation (2.18), we can only obtain $d(-3e_1) \in \{3, 5\}$. It seems difficult to determine $d(-3e_1)$ explicitly. We tend to believe that $d(-3e_1) = 3$.

Remark 2.8. Fix $k \in \{2, 3\}$. It was also proved in [4] the existence of infinitely many τ's such that

$$e_k'(\tau) = \frac{i}{\pi}\left[\tfrac{1}{6}g_2(\tau) + \eta_1(\tau)e_k(\tau) - e_k(\tau)^2\right] = 0.$$

Fix any such τ. Then the same proof as Theorem 2.6 shows that, for equation (2.18) with this new τ, there holds $B_2 = \{-3e_k\}$, i.e., $d(-3e_k) \geq 3$. Moreover, it was also proved in [4] the existence of infinitely many τ's such that

$$\eta_1'(\tau) = \frac{i}{2\pi}\left(\eta_1(\tau)^2 - \tfrac{1}{12}g_2(\tau)\right) = 0.$$

Fix any such τ. Then either $\eta_1 + \wp(a_+) = 0$ or $\eta_1 + \wp(a_-) = 0$, and hence for equation (2.18) with this new τ, either $(3g_2)^{1/2} \in B_2$ or $-(3g_2)^{1/2} \in B_2$.

Our examples suggest that for any $n \in \mathbb{N}_{\geq 3}$, there should be τ such that for the Lamé equation

$$y''(x) - n(n+1)\wp(z + \tau/2; \tau)y(x) = Ey(x), \quad x \in \mathbb{R},$$

there exists some (anti)periodic eigenvalue E with $d(E) \geq 3$. We should return to this problem in the future.

Acknowledgment

We would like to thank the referee for a careful reading and valuable comments.

References

[1] Björn Birnir, *Complex Hill's equation and the complex periodic Korteweg-de Vries equations*, Comm. Pure Appl. Math. **39** (1986), no. 1, 1–49. MR820337

[2] Björn Birnir, *Singularities of the complex Korteweg-de Vries flows*, Comm. Pure Appl. Math. **39** (1986), no. 3, 283–305. MR829842

[3] Ching-Li Chai, Chang-Shou Lin, and Chin-Lung Wang, *Mean field equations, hyperelliptic curves and modular forms: I*, Camb. J. Math. **3** (2015), no. 1-2, 127–274. MR3356357

[4] Z. Chen and C. S. Lin, *Critical points of classical modular forms from elliptic function theory.* preprint, 2017.

[5] Zhijie Chen, Ting-Jung Kuo, Chang-Shou Lin, and Chin-Lung Wang, *Green function, Painlevé VI equation, and Eisenstein series of weight one*, J. Differential Geom. **108** (2018), no. 2, 185–241. MR3763067

[6] Fritz Gesztesy and Rudi Weikard, *Picard potentials and Hill's equation on a torus*, Acta Math. **176** (1996), no. 1, 73–107. MR1395670

[7] F. Gesztesy and R. Weikard, *Lamé potentials and the stationary (m)KdV hierarchy*, Math. Nachr. **176** (1995), 73–91. MR1361127

[8] Otto Haupt, *Über lineare homogene Differentialgleichungen 2. Ordnung mit periodischen Koeffizienten. Bemerkung zur Arbeit gleichen Titels von Herrn Hamel* (German), Math. Ann. **79** (1918), no. 3, 278–285. MR1511929

[9] Chang-Shou Lin and Chin-Lung Wang, *Elliptic functions, Green functions and the mean field equations on tori*, Ann. of Math. (2) **172** (2010), no. 2, 911–954. MR2680484

[10] Rudi Weikard, *On Hill's equation with a singular complex-valued potential*, Proc. London Math. Soc. (3) **76** (1998), no. 3, 603–633. MR1620492

Department of Mathematical Sciences, Yau Mathematical Sciences Center, Tsinghua University, Beijing, 100084, People's Republic of China

Email address: `zjchen2016@tsinghua.edu.cn`

Taida Institute for Mathematical Sciences (TIMS), Center for Advanced Study in Theoretical Sciences (CASTS), National Taiwan University, Taipei 10617, Taiwan

Email address: `cslin@math.ntu.edu.tw`