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A POINTWISE APPROACH TO RIGIDITY OF ALMOST GRAPHICAL SELF-SHRINKING SOLUTIONS OF MEAN CURVATURE FLOWS

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ABSTRACT. We prove rigidity of any properly immersed noncompact Lagrangian shrinker with single valued Lagrangian angle for Lagrangian mean curvature flows. Our pointwise approach also provides an elementary proof to the known rigidity results for graphical and almost graphical shrinkers of mean curvature flows.

1. INTRODUCTION

In this note, we prove the following:

Theorem 1.1. *If $u(x)$ is a smooth solution to the potential equation for Lagrangian shrinker $(x, Du(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$,*

$$\Theta = \sum_i^n \arctan \lambda_i = \frac{1}{2} x \cdot Du(x) - u(x), \tag{1.1}$$

on (bounded or unbounded) domain $\Omega \subset \mathbb{R}^n$ such that $|Du(x)| = \infty$ on the boundary $\partial\Omega$, where λ_is are the eigenvalues of D^2u, then the Lagrangian shrinker is a plane over $\Omega = \mathbb{R}^n$.

More generally, if L^n is a smooth, properly immersed (extrinsically complete), and noncompact Lagrangian shrinker in $\mathbb{R}^n \times \mathbb{R}^n$, where the Lagrangian angle Θ is a single valued function (zero Maslov class), then L^n is a Lagrangian plane.

Our pointwise approach to the shrinkers of Lagrangian mean curvature flows also provides a short proof for the rigidity of codimension one graphical and almost graphical shrinkers of mean curvature flows, which have been done via integral ways by Wang [W] and Ding-Xin-Yang respectively [DXY].

Theorem 1.2 ([W])**.** *Every smooth entire graphical self-shrinking hypersurface of the mean curvature flow must be a plane.*

Theorem 1.3 ([DXY])**.** *Every smooth, almost graphical, properly immersed (extrinsically complete), and noncompact self-shrinking (oriented) hypersurface of the mean curvature flow must be a plane or a cylinder with cross section being a self-shrinker in one lower dimensional space.*

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Here "almost graphical" means (one choice) of all unit normals of the hypersurface are on the closed upper hemisphere of the whole ambient Euclid space; "properness" or "extrinsic completeness" means the distance from every point of the shrinker boundary to the origin is infinite.

Self-shrinking solutions arise naturally at a minimum blowing-up-rate or type I singularity from Huisken's monotonicity formula [H] for mean curvature flows. These are immersions $F(p,t) : \Sigma \times (-\infty, 0) \to \mathbb{R}^{n+k}$ which deform homothetically $F(\Sigma,t) = \sqrt{-t}F(\Sigma,-1)$ under the mean curvature flow equation

$$(F_t)^{\perp} = \triangle_g F.$$

Here $(\ )^{\perp}$ is the normal component of the vector $(\ )$ and $\triangle_g F$ equals the mean curvature of $F(\Sigma,t)$ with g being the induced metric. Equivalently, the self-shrinker or shrinker $\Sigma = F(p,-1)$ satisfies

$$\triangle_g F = -\frac{1}{2}F^{\perp}.$$

When shrinker Σ is a codimension k graph $(x, f(x)) \subset \mathbb{R}^n \times \mathbb{R}^k$, for the profile $f(x)$ of the shrinking solution $\left(x, \sqrt{-t}f\left(x/\sqrt{-t}\right)\right)$, the above self-shrinking equation also takes the nondivergence as well as divergence form

$$g^{ij}D_{ij}f = \frac{1}{2}\left[x \cdot Df(x) - f(x)\right],$$
$$\triangle_g f^{\alpha} = \frac{1}{2}\left(\langle F, \nabla_g f^{\alpha}\rangle - f^{\alpha}\right)$$

for $\alpha = 1, \dots, k$. The equivalence of the two forms comes from a simple identity on the shrinker $\Sigma = F(p,-1)$:

$$g^{ij}D_{ij} - \frac{1}{2}x \cdot D = \triangle_g - \frac{1}{2}\langle F, \nabla_g\rangle. \tag{1.2}$$

When shrinker Σ is a Lagrangian or "gradient" graph $L = (x, Du(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$, the potential equation (1.1) is revealed by integrating the nondivergence equation. For each oriented tangent plane to any Lagrangian submanifold $L^n \subset \mathbb{R}^n \times \mathbb{R}^n$, there are n canonical angles up to a multiple of 2π formed with the x-plane $\mathbb{R}^n$; the sum of those angles is called the Lagrangian angle. For example, when a Lagrangian submanifold is a graph over x-space, it must be a "gradient" one $(x, Du(x))$. The Lagrangian angle is a single valued function $\Theta = \sum_i^n \arctan\lambda_i\left(D^2u\right)$; for circle $(\cos\theta, \sin\theta)$ on $\mathbb{R}^1 \times \mathbb{R}^1$, its Lagrangian angle is multiple valued $\theta + \pi/2$.

Our current work grew out of an attempt at the rigidity issue for extrinsically complete graphical shrinkers defined on bounded domains. We are grateful to Tom Ilmanen for this question. Our resolution requires one to exploit both vertical and horizontal parts of position vector of shrinkers for a "full" barrier instead of just the horizontal part as in the joint work [CCY] with Chau and Chen.

Heuristically, our argument goes as follows. A geometric quantity, which is the corresponding cosine to the slope of the codimension one almost graphical shrinker, or the Lagrangian angle of the Lagrangian shrinker, satisfies an elliptic equation with self-similar term (Step 1's). This amplifying force term forces the geometric quantity to go up near infinity by the "full" barrier (Step 2's). Hence the geometric quantity is constant by the strong minimum principle, and in turn the rigidity follows from the second fundamental form term in the cosine equation for the codimension one shrinkers (Step 3 of Section 2) and the quadratic excess terms in the potential equation for the Lagrangian shrinkers (Step 3 of Section 3), respectively.

2. Proof of Theorems 1.2 and 1.3

Step 1. Starting from the self-shrinking equation, a simple calculation [EH, p. 471] shows that the cosine of the angle between a unit normal N of the oriented immersed shrinker Σ and a fixed direction $e_{n+1} = (0, \ldots, 0, 1)$ in $\mathbb{R}^n \times \mathbb{R}^1$, the nonnegative $w = \langle N, e_{n+1} \rangle$, satisfies

$$\mathcal{L}w = \triangle_g w - \frac{1}{2} \langle X, \nabla_g w \rangle = - |A|^2 w \leq 0,$$

where $|A|$ denotes the norm of the second fundamental form of the hypersurface Σ; moreover, a straightforward calculation shows that the distance $|X|$ from any point on the shrinker Σ^n to the origin satisfies

$$\mathcal{L} |X|^2 = \triangle_g |X|^2 - \frac{1}{2} \left\langle X, \nabla_g |X|^2 \right\rangle = 2n - |X|^2 .$$

Step 2. Based on $|X|^2$ we construct a barrier to force w to attain its global minimum at a finite point of Σ. Set

$$b = -\varepsilon \left(|X|^2 - K^2 \right) + \min_{\bar{\mathfrak{B}}_K(0) \cap \Sigma} w,$$

where ε is any fixed small positive number and $\bar{\mathfrak{B}}_K(0)$ is the ball in $\mathbb{R}^{n+1}$ centered at the origin with radius K such that $K \geq \sqrt{2n}$ and $\mathfrak{B}_K(0) \cap \Sigma$ is not empty. Now $\mathcal{L}b = -\varepsilon \left(2n - |X|^2 \right) \geq 0$ on each unbounded component of $\Sigma \backslash \bar{\mathfrak{B}}_K(0)$ and on the infinite and finite boundary of each such component $w \geq b$. Here we used the properness of Σ, or $|X| = \infty$ at the (infinite) boundary of Σ for the boundary comparison. By the comparison principle we see $w \geq b$ on all those unbounded components of $\Sigma \backslash \bar{\mathfrak{B}}_K(0)$. By letting ε go to zero, we then conclude that w achieves its global minimum at a finite point on Σ (could be outside $\bar{\mathfrak{B}}_K(0)$). The strong minimum principle implies that w is a constant.

Remark. In the case of the shrinker being an entire graph $\Sigma = (x, f(x))$, the argument in this Step 2 is “cleaner”.

Step 3. If constant $w > 0$, then by the equation for w, one sees that $|A| = 0$ and the almost graphical shrinker is a plane. If constant $w \equiv 0$, then vertical vector $(0, \cdots, 0, 1)$ is tangent to Σ everywhere, and in turn the almost graphical shrinker is a cylinder $\Sigma^n = \bar{\Sigma}^{n-1} \times \mathbb{R}^1$ with $\bar{\Sigma}^{n-1}$ being a shrinker in $\mathbb{R}^n$.

3. Proof of Theorem 1.1

Step 1. When the Lagrangian shrinker is locally a graph $L = (x, Du(x))$, as calculated in [CCY, p. 232], we have the equation for Θ:

$$g^{ij} D_{ij} \Theta - \frac{1}{2} x \cdot D\Theta = 0.$$

Because of (1.2), Θ also satisfies a divergence equation

$$\mathcal{L}\Theta = \triangle_g \Theta - \frac{1}{2} \langle X, \nabla_g \Theta \rangle = 0,$$

with X being the position vector of L in $\mathbb{R}^n \times \mathbb{R}^n$. Note that this divergence operator $\mathcal{L}$ is invariant under any parametrization of L. Again, a straightforward calculation shows that

$$\mathcal{L}\left|X\right|^2 = \triangle_g \left|X\right|^2 - \frac{1}{2}\left\langle X, \nabla_g \left|X\right|^2\right\rangle = 2n - \left|X\right|^2.$$

Step 2. We take the same barrier

$$b = -\varepsilon\left(\left|X\right|^2 - K^2\right) + \min_{\bar{\mathfrak{B}}_K(0)\cap L} \Theta,$$

where ε is any fixed small positive number and $\bar{\mathfrak{B}}_K(0)$ is the ball in $\mathbb{R}^n \times \mathbb{R}^n$ centered at the origin with radius K such that $K \geq \sqrt{2n}$ and $\mathfrak{B}_K(0) \cap L$ is not empty. Now $\mathcal{L}b = -\varepsilon\left(2n - \left|X\right|^2\right) \geq 0$ on each unbounded component of $L \backslash \bar{\mathfrak{B}}_K(0)$ and on the infinite and finite boundaries of each such component $\Theta \geq b$. Here we used the properness of L, or $|X| = \infty$ at the (infinite) boundary of L, for the boundary comparison. In the case $L = (x, Du(x)) \subset \Omega \times \mathbb{R}^n$, $|X| = \infty$ is because of either $|Du(x)| = \infty$ or $|x| = \infty$ on the (infinite) boundary of L. By the comparison principle we see $\Theta \geq b$ on all those unbounded components of $L\backslash\bar{\mathfrak{B}}_K(0)$. By letting ε go to zero, we then conclude that Θ achieves its global minimum at a finite point on L (could be outside $\bar{\mathfrak{B}}_K(0)$). The strong minimum principle implies that Θ is a constant.

Step 3. We go to the potential equation (1.1) to capture the flatness of the Lagrangian shrinker. Near a closest point P on L to the origin, one can represent L as a "gradient" graph $(x, Dv(x))$ over the Lagrangian plane through the origin and parallel to the tangent plane of L at P. Here we use the abused notation x for the coordinates on the ground, or the Lagrangian plane. Because of the constancy of the Lagrangian angle Θ, the potential equation for v becomes

$$c = \frac{1}{2}x \cdot Dv(x) - v(x).$$

This c may differ from the constant Θ by another one due to a possible $U(n)$ coordinate rotation in $\mathbb{C}^n = \mathbb{R}^n \times \mathbb{R}^n$. Euler's homogeneous function theorem implies that the smooth function $v(x) + c$ around $x = 0$ is a polynomial of degree two. We immediately see that near P, $L = (x, Dv(x))$ is a piece of the above "ground" plane. Because of the analyticity of L, as the potential equation is analytic, we conclude that Lagrangian shrinker L is a Lagrangian plane and over $\Omega = \mathbb{R}^n$ in the graphical case.

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