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CHARACTERISTICS OF THE BREATHER AND ROGUE WAVES IN A (2+1)-DIMENSIONAL NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. Under investigation in this paper is a (2+1)-dimensional nonlinear Schrödinger equation (NLS), which is a generalisation of the NLS equation. By virtue of Wronskian determinants, an effective method is presented to succinctly construct the breather wave and rogue wave solutions of the equation. Furthermore, the main characteristics of the breather and rogue waves are graphically discussed. The results show that rogue waves can come from the extreme behavior of the breather waves. It is hoped that our results could be useful for enriching and explaining some related nonlinear phenomena.

1. Introduction

As we well know rogue waves (RWs), killer waves, extreme waves and giant waves are notorious for causing disastrous consequences in the ocean [1, 2], and appear both in the shallow waters and in the deep ocean [3]. An obvious feature of RWs is that they "suddenly come from nowhere and disappear with no trace", and it only takes seconds before they attack a ship. Nowadays, RWs have drawn more and more experimental and theoretical attention in many other fields such as optical fibres, Bose-Einstein (BE) condensates, capillary wave, plasma physics, finance and other related fields [4]-[12]. The most commonly used mathematical model for describing the rogue wave is the focusing nonlinear Schrödinger (NLS) equation [13]-[16]

$$iq_t + q_{xx} + \frac{1}{2}q|q|^2 = 0, \tag{1.1}$$

where subscripts denote partial derivatives, q denotes the wave envelope, x denotes the spatial variable and t denotes the temporal variable. In [17], Ma and his collaborator provide a direct and effective method for finding exact solutions to the NLS equation. Recently, by means of the Darboux transformation (DT), Hirota's bilinear (HB) method, Wronskian technique and orthogonal polynomials, etc., there have been a number of works to investigate exact solutions of other systems [18]-[31].

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Besides the NLS equation, it is known that most NLS-type equations also admit the RW solutions such as the Hirota equation [32], the Kundu-Eckhaus (KE) equation [33], the Davey-Stewarson (DS) equation [34], the derivative nonlinear Schrödinger (DNLS) equation [35] and other systems [36]-[43]. In recent years, many efforts were devoted to studying RWs for a (1+1)-dimensional model. However, there are few research studies on RWs for (2+1)-dimensional nonlinear equations.

In this paper, we will focus on a (2+1)-dimensional nonlinear Schrödinger equation [44, 45]

$$iq_t + q_{xy} + \frac{1}{2}q\partial_x^{-1}\partial_y|q|^2 = 0, \tag{1.2}$$

which can be rewritten in the following form:

$$\begin{cases} iq_t + q_{xy} + \dfrac{1}{2}Vq = 0, \\ \partial_x V = \partial_y |q|^2, \end{cases} \tag{1.3}$$

by introducing a new potential function $V = V(|q|)$, where $q = q(x, y, t)$ is a differentiable complex-value function. If $\partial_x = \partial_y$, equation (1.2) can be reduced to equation (1.1). To our knowledge, although there are many studies concerning the (1+1)-dimensional nonlinear Schrödinger equation (1.1), there are very few studies on equation (1.2).

The primary purpose of the present work is to employ a direct method to construct the breather and rogue wave solutions of equation (1.2) by using the Wronskian determinants technique [18, 19]. Additionally, the dynamic behaviors of the breather and rogue waves are also considered by choosing different parameters.

The structure of this paper is as follows. Some important formulas are briefly introduced in Section 2. In Section 3, we consider the rogue wave solutions of equation (1.2) by using Theorem 2.1. In Section 4, the dynamical behaviors of the breather and rogue waves are graphically discussed. Finally, some conclusions and discussions are presented in the last section.

2. The main formulas

In [46, 47], Matveev and co-authors present an important method to investigate the multi-rogue wave solutions of the NLS equation. Here we briefly recall some important formulas.

Let n be any positive integer. We define two polynomials $u_{2n}(k)$ and $\Phi(k)$ by the following formulas:

$$u_{2n} := \prod_{j=1}^{n}\left(k^2 + i\cot\frac{a_j}{2}\right),\ a_j := \frac{(2j-1)\pi}{2n+1},$$

$$\Phi := i\sum_{l=1}^{2n}\varphi_l(ik)^l,\ \varphi_l \in \mathbb{R}. \tag{2.1}$$

In what follows, we consider the function F given by the following formulas:

$$F(k, x, y, t) := \frac{\exp\left(kx + ky + ik^2t + \Phi(k)\right)}{u_{2n}(k)}, \tag{2.2}$$

that for any k is a solution of the nonstationary linear Schrödinger equation with zero potential

$$-iF_t = F_{xx} = F_{yy}. \tag{2.3}$$

The same is true for $F_1, F_2, \dots, F_{2n}$ defined by the following formula:

$$\begin{aligned} &F_j(x,y,t) := D_k^{2j-1} F(k,x,y,t)|_{k=1},\ D_k := \frac{k^2}{k^2+1}\frac{\partial}{\partial k}, \\ &F_{n+j}(x,y,t) := D_k^{2j-1} F(k,x,y,t)|_{k=-1},\ j = 1,2,\dots,n. \end{aligned} \tag{2.4}$$

We suppose that W_1, W_2 have the following Wronskian determinants:

$$\begin{aligned} &W_1 := W(F_1, F_2, \dots, F_{2n}) \equiv \det A,\ A_{lj} = \partial_x^{l-1} F_j, \\ &W_2 :== W(F_1, F_2, \dots, F_{2n}, F). \end{aligned} \tag{2.5}$$

Then based on the above analysis and [47, 48], the following theorem can be constructed.

Theorem 2.1. *The function*

$$q_n(x,y,t) := (-1)^n u_{2n}(0) \exp\left(\frac{it}{2}\right) \frac{W_2|_{k=0}}{W_1}, \tag{2.6}$$

represents a family of solutions of equation (1.2) *that relies on* $2n$ *real parameters* $\varphi_1, \varphi_2, \dots, \varphi_{2n}$.

In the next section, Theorem 2.1 will be applied to investigate the multi-rogue wave solutions of the (2+1)-dimensional nonlinear Schrödinger equation (1.2).

3. Rogue wave solutions

In this section, we will investigate the rogue wave solutions of equation (1.2) by using Theorem 2.1. If taking $n = 1, \varphi_1 = 0, \varphi_2 = \sqrt{3}/4, \varphi_i = 0 (i \geq 3)$, the solution (2.6) yields the following form:

$$\begin{aligned} &q_1(x,y,t) = \left[1 - 4\frac{1+it}{1+(x+y)^2+t^2}\right] \exp\left(\frac{it}{2}\right), \\ &V = \frac{\left[(x+y)^2+t^2-3\right]^2 + 16t^2}{\left[1+(x+y)^2+t^2\right]^2}. \end{aligned} \tag{3.1}$$

Substituting $x = 0$ or $y = 0$ into (3.1), we can obtain the well-known Peregrine soliton [49]. As shown in Figures 1 and 2, for fixed t, solution (3.1) can be transformed into a W-shaped soliton. For fixed x or y, solution (3.1) is the well-known eye shaped rogue wave which has one local hump and two valleys. In addition, the profile of the rogue wave is symmetric about the t or x-axis.

In a similar way, if taking $n = 2$, $\varphi_1 = 3\varphi_3, \varphi_2 = 2\varphi_4 + \frac{(3+\sqrt{5})}{4}\sin\left(\frac{\pi}{5}\right), a = 48\varphi_3$, $b = 4(5+\sqrt{5})\sin(\pi/5) - 96\varphi_4$, the solution (2.6) yields the following form:

$$\begin{aligned} &q_2(x,y,t,a,b) = \left[1 - 12\frac{A_2 + iB_2}{C_2}\right] \exp\left(\frac{it}{2}\right), \\ &V = \frac{(C_2 - 12A_2)^2 + 144B_2^2}{C_2^2}, \end{aligned} \tag{3.2}$$

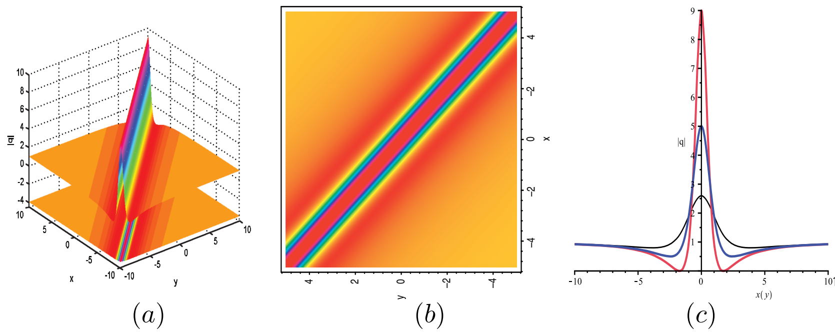


FIGURE 1. (Color online) W-shaped soliton wave via solution (3.1) with $t = 0$. **(a)** Perspective view of the real part of the wave. **(b)** The overhead view of the wave. **(c)** The wave propagation pattern of the wave along the $x(y)$ axis.

where

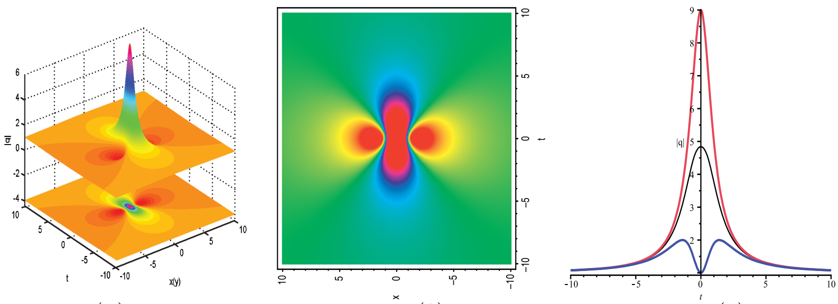
$$\begin{aligned}
&A_2(x,y,t,a,b) = (x+y)^4 + 6t^2(x+y)^2 + 6(x+y)^2\\
&+ 4a(x+y) + 5t^4 + 18t^2 - 4bt - 3,\\
&B_2(x,y,t,a,b) = t(x+y)^4 + \left(t^3 - 3t + b\right)(x+y)^2\\
&+ 4a(x+y)t + t^5 + 2t^3 - 2bt^2 - 15t + 2b,\\
&C_2(x,y,t,a,b) = \left[1 + (x+y)^2 + t^2\right]^3 - 4a(x+y)^3\\
&- 12\left(2t^2 - bt - 2\right)(x+y)^2 + 4\left[3a\left(t^2+1\right)(x+y)\right]\\
&+ 24t^4 - 4bt^3 + 96t^2 - 36bt + 4a^2 + 4b^2 + 8,
\end{aligned} \tag{3.3}$$

and a and b are arbitrary constants. When the parameters a, b are small enough, the solution (3.2) is very close to the P_2-breather. When the parameters a, b are large enough, we can obtain the form of a two-order rogue wave. As shown in Figures 3 and 4, for fixed x or y, the solution (3.2) has a single peak when $a = b = 0$, which can split into three peaks as $a^2 + b^2$ increase. The solution is called the "three sisters" in [50] and a "rogue wave triplet" in [51]. As depicted in Figure 5, for fixed t, the solution (3.2) can be transformed into a soliton solution. Furthermore, for fixed a and b, the wave will propagate along the $x(y)$-axis as the time changes.

Remark 3.1. On the (x,t)−plane, taking the transformation $X = x+y, T = t$, then q_1, q_2 in (3.2) can be reduced to rogue wave solutions of equation (1.1).

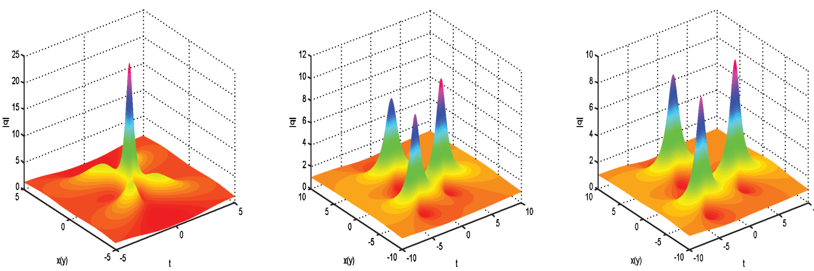
4. BREATHER WAVE SOLUTIONS

In this section, we will consider the dynamical behaviors of breather wave solutions. Based on the results in [20, 22, 23, 32], the first-order breather solution reads



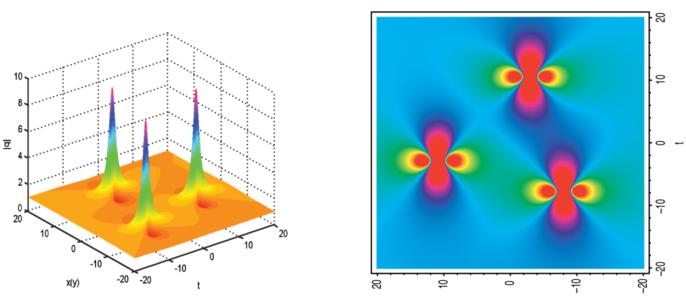
(a) (b) (c)

FIGURE 2. (Color online) Rogue wave solution (3.1) for equation (1.2) by taking $y = 0$ or $x = 0$. **(a)** Perspective view of the real part of the wave. **(b)** The overhead view of the wave. **(c)** The wave propagation pattern of the wave along the t axis.



(a) (b) (c)

FIGURE 3. (Color online) Rational solution (3.2) for equation (1.2) by taking suitable parameters: $a = b = 0$ for **(a)**, $a = b = 25$ for **(b)** and $a = b = 45$ for **(c)**, respectively.



(a) (b)

FIGURE 4. (Color online) Rational solution (3.2) for equation (1.2) by taking suitable parameters: $a = b = 500$. **(a)** Perspective view of the real part of the wave. **(b)** The overhead view of the wave.

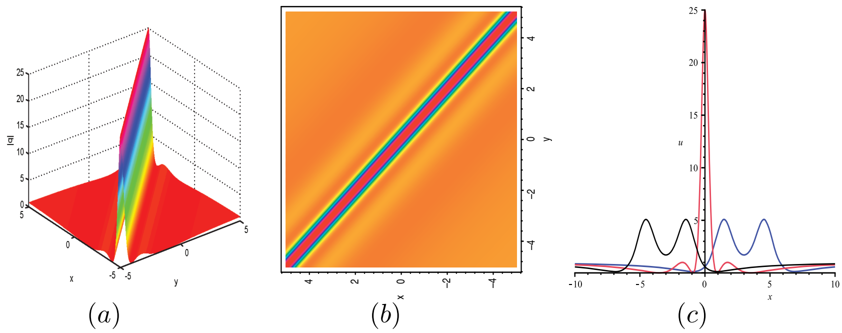


FIGURE 5. (Color online) Rational solution (3.2) for equation (1.2) by taking suitable parameters: $a = b = 0$. **(a)** Perspective view of the real part of the wave ($t = 0$). **(b)** The overhead view of the wave. **(c)** The wave propagation pattern of the wave along the x axis($t = -2, 0, 2$).

as

$$q_{\text{bw}}^{[1]} = \exp(i\theta)\left[\frac{(a^2 - 2\eta^2)\cosh(\omega_1) + 2i\eta\mu\sinh(\omega_1) + a\eta\cos(\omega_2)}{a\cosh(\omega_1) - \eta\cos(\omega_2)}\right], \tag{4.1}$$

where

$$\lambda = \xi + i\eta,\ k = -2\xi,\ \theta = kx + \delta y - k\delta t,\ \omega_1 = \eta\mu t,$$
$$\omega_2 = \eta\mu[x + y + (\xi - \delta/2)t],\ \mu = \sqrt{a^2 - \eta^2}. \tag{4.2}$$

and λ is a complex spectral parameter.

As shown in Figures 6 and 7, for fixed x or y, if $\mu^2 > 0$, the solution (4.1) is a periodic breather wave (i.e., Akhmediev breather wave), which can evolve periodically along a straight line with a certain angle with the x axis and y-axis. Its velocity, amplitude and width remain unchanged during the propagation (The number of the rogue wave in a single period.) Additionally, we can find that the wave is not the time-periodic breather but the space-periodic breather. For fixed t, the solution (4.1) is a periodic wave, Figure 7 is plotted for the periodic wave propagating in $x(y)$ direction with a single period. For simplicity, we set $\xi = 1, \eta = 1, \delta = 1$ in equation (4.1). When $|a|^2 > |\eta|^2$ and μ is imaginary, the hyperbolic and trigonometric functions in equation (4.1) convert to their analogues via the following relation:

$$\sinh(\Theta) = -i\sin(i\Theta),\ \cosh(\Theta) = \cos(i\Theta). \tag{4.3}$$

Substituting these transformations into equation (4.1), we can obtain a soliton solution in the following form:

$$q_{\text{km}}^{[1]} = \left[\frac{(a^2 - 2)\cos(mt) - 2im\sin(mt) + a\cosh(\xi_{\text{km}})}{a\cos(mt) - \cosh(\xi_{\text{km}})}\right]$$
$$\times \exp(i\theta), \tag{4.4}$$

where $\mu = im$, $\mu = \sqrt{a^2 - 1}$ and $\xi_{\text{km}} = m(x + y - 0.5t)$. As shown in Figures 8(a)-8(c), for fixed x or y, on the $(x(y), t)$ plane, the wave solution can represent the Kuznetzon-Ma (KM) [52, 53] breather that is spatially localized and temporally

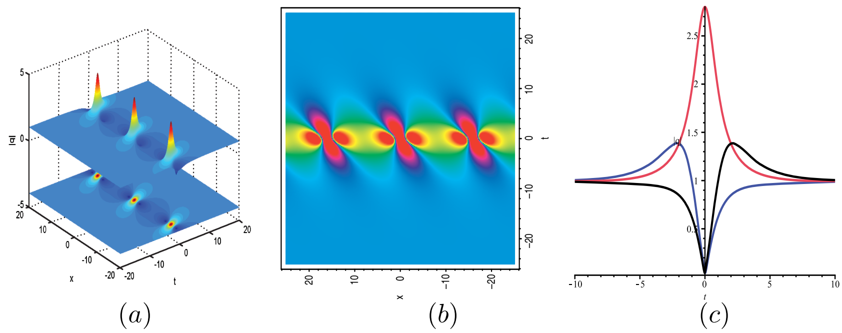


FIGURE 6. (Color online) Akhmediev breather wave via solution (4.1) with parameters: $y = 0, a = 1, \eta = 0.9, \xi = 0.5, \delta = 2$. **(a)** Perspective view of the real part of the wave. **(b)** The overhead view of the wave. **(c)** The wave propagation pattern of the wave along the t axis.

breathing. For fixed t, on the (x, y) plane, the wave can be transformed into a W-shaped soliton wave (see Figures 8(e) and (f)).

In what follows, based on the Taylor expansion of the function $q(x, y, t)$ at $\mu = 0$

$$\exp(\mu t) = 1 + \mu t + \frac{(\mu t)^2}{2} + O(\mu^3),$$

$$\cos[\mu(x + y + \gamma t)] = 1 - \frac{\mu^2}{2}(x + y + \gamma t)^2 + O(\mu^3), \tag{4.5}$$

we can obtain the following rogue wave solutions:

$$q_{\mathrm{rw}}^{[1]} = \lim_{a \to \eta} \left(q_{\mathrm{bw}}^{[1]} \right) = \left[-1 + \frac{q_{1N}}{q_{1D}} \right] a \exp(i\theta), \tag{4.6}$$

where

$$q_{1N} = 4 + 4i\eta^2 t, \; q_{1D} = 1 + \eta^4 t^2 + \eta^2 \left[x + y + (\xi - \delta/2)\, t \right]^2 . \tag{4.7}$$

Figures 9 and 10 are plotted for the rogue waves $|q|$ for equation (1.2) with suitable parameters, which are localized both in time and space, thus revealing the usual rogue wave features. Moreover, by comparing Figures 9 and 10, we find that the parameters can determine the central symmetry pattern of the rogue wave. If taking $\xi = \delta = 0$, the profile of the wave is symmetric about the t-axis (see Figure 10). Particularly, it is worth mentioning that the rogue wave can come from the extreme behavior of the breather wave for equation (1.2) (i.e., from Figure 6 to Figure 9).

Remark 4.1. The dynamical behaviors of breather waves are graphically discussed. To the best of our knowledge, the dynamical behaviors of breather waves for equation (1.2) have not been presented in previous literatures.

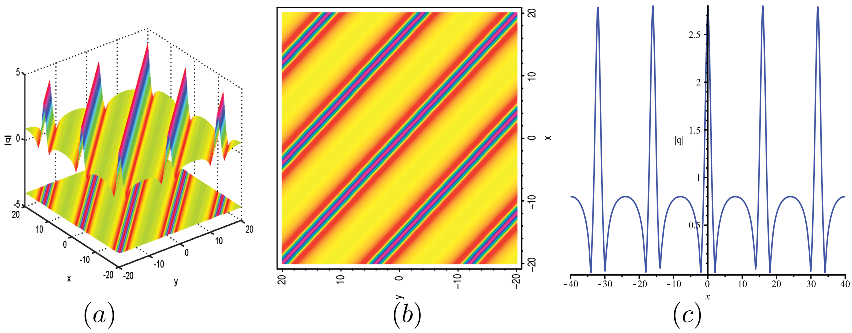


FIGURE 7. (Color online) A periodic wave for equation (1.2) with parameters: $t = 0, a = 1, \eta = 0.9, \xi = 0.5, \delta = 2$. **(a)** Perspective view of the real part of the wave. **(b)** The overhead view of the wave. **(c)** The wave propagation pattern of the wave along the x axis.

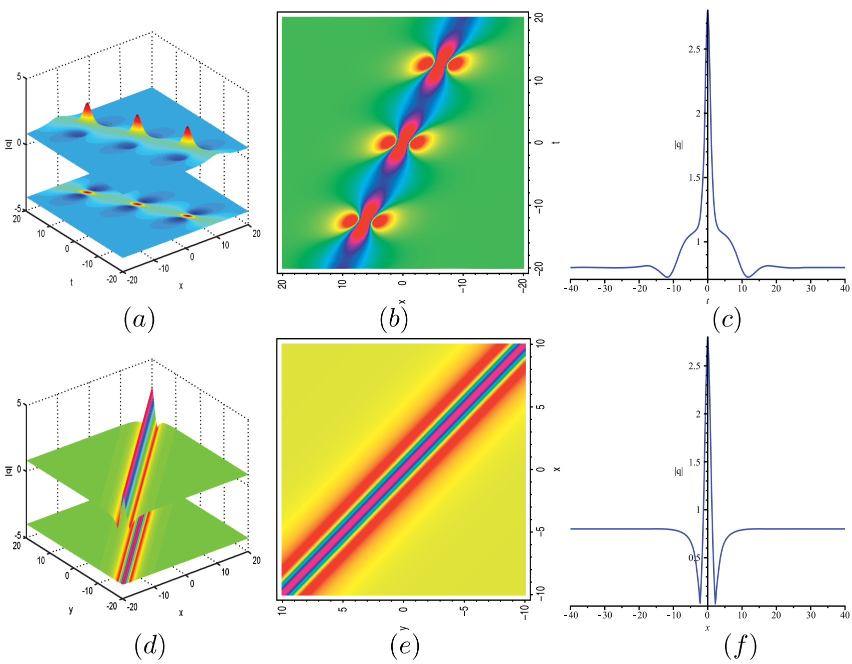


FIGURE 8. (Color online) Kuznetzon-Ma breather wave via solution (4.4) with parameters: $m = 0.5$. **(a)** Perspective view of the real part of the wave ($y = 0$). **(b)** The overhead view of the wave. **(c)** The wave propagation pattern of the wave along the t axis. **(d)** Perspective view of the real part of the solitary wave ($t = 0$). **(e)** The overhead view of the wave. **(f)** The wave propagation pattern of the wave along the x axis.

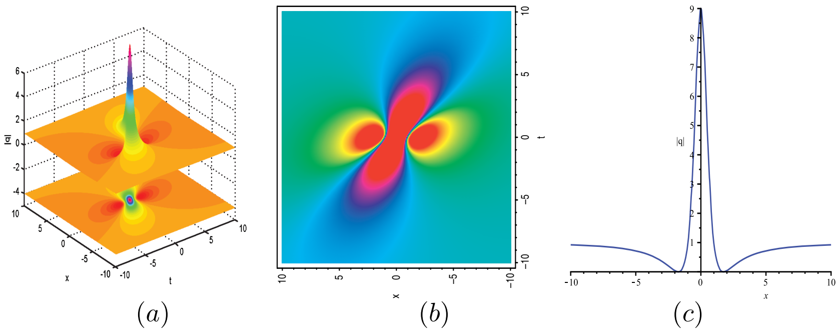


FIGURE 9. (Color online) Rogue wave (4.6) for equation (1.2) by choosing suitable parameters: $y = 0, \xi = 1, \delta = 1, \eta = 1$. **(a)** Perspective view of the real part of the wave. **(b)** The overhead view of the wave. **(c)** The wave propagation pattern of the wave along the x axis.

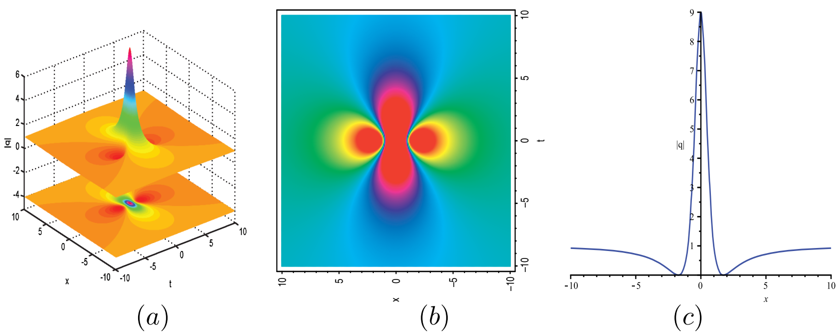


FIGURE 10. (Color online) Symmetry rogue wave for equation (1.2) by choosing suitable parameters: $y = 0, \xi = 0, \delta = 0, \eta = 1$. **(a)** Perspective view of the real part of the wave. **(b)** The overhead view of the wave. **(c)** The wave propagation pattern of the wave along the x axis.

5. CONCLUSIONS AND DISCUSSIONS

In this work, a (2+1)-dimensional nonlinear Schrödinger equation (1.2) has been systematically investigated, which can be reduced to the NLS equation (1.1). Based on Wronskian determinants, we construct its multi-rogue wave solutions. In order to make the readers understand the solutions better, we have made some graphical analysis of these solutions. Furthermore, the main characteristics of the breathers and rogue waves are graphically discussed. Meanwhile, it is very necessary to point out that the rouge wave comes from the extreme behavior of the breather wave (depicted in Figures 6 and 9).

In [17], three ansätze of transformations are analyzed and used to construct exact solutions to the nonlinear Schrödinger equation. If $\mu = 1/2$, the solutions obtained in [17] solve the standard NLS equation (1.1). In our paper, by virtue of Wronskian determinants and the Darboux transformation method, an effective method is presented to succinctly construct the breather and rogue wave solutions of the equation (1.2). Taking the transformations $X = x+y$, $T = t$, the solutions obtained in this paper can be reduced to the solutions of the standard NLS equation (1.1). The reduced solutions are similar to the solutions (3.26) and (3.27) presented in [17]. By comparing the reduced solutions with the solutions presented in [17], one can find that the solutions presented (3.26) and (3.27) in [17] should be breather wave and rogue wave solutions of the (1+1)-dimensional NLS equation. By using the method proposed in [17], one can also construct the $(2+1)$-dimensional solutions by using the similar ansätzes. In [18, 19], based on the bilinear formalism, a Wronskian technique leading to rational solutions is also presented by Ma and his collaborators for the KdV equation and the Boussinesq equation, respectively.

In [44], Hirota's method is used to construct the soliton solutions of the equation (1.2). Strachan has obtained its one-soliton and two-soliton solutions, respectively. In [45], the bilinear form is used to construct the soliton solutions of the equation (1.2). Radha and Lakshmanan have obtained the bilinear form directly from the P-analysis of the equation (1.2), which can then be used to generate its soliton solutions. They also indicate the absence of two genuine nonparallel ghost solitons which in isolation can produce a vanishing physical field in order to give rise to a 'dromion'. In their paper, the soliton solutions are also presented for the equation (1.2).

In our paper, by virtue of Wronskian determinants and the Darboux transformation method, an effective method is presented to succinctly construct the breather and rogue wave solutions of the equation (1.2). Furthermore, the main characteristics, such as dynamical behaviors, of these solutions are graphically discussed. As we stated before that the solution presented in this paper can be reduced to a W-shaped soliton, the eye shaped rogue wave, the Kuznetzon-Ma breather wave, etc. One can verify that the breather and rogue wave solutions provided here are different from the soliton solutions obtained by Strachan [44], and Radha, Lakshmanan in [45]. Our results show that rogue waves can come from the extreme behavior of the breather waves. We hope that our results could be useful for enriching and explaining some related nonlinear phenomena. There is still a lot of work to do for multi-breather waves in the near future.

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