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# $L^p$ ALMOST CONFORMAL ISOMETRIES OF SUB-SEMI-RIEMANNIAN METRICS AND SOLVABILITY OF A RICCI EQUATION

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Abstract. Let $M$ be a smooth compact manifold without boundary. We consider two smooth Sub-Semi-Riemannian metrics on $M$. Under suitable conditions, we show that they are almost conformally isometric in an $L^p$ sense. Assume also that $M$ carries a Riemannian metric with parallel Ricci curvature. Then an equation of Ricci type is solvable in a specific sense, without assuming any proximity to a special metric.

## 1. Introduction

The goal of this note is to prove that the two principal results of D. DeTurck [11], established for positive definite symmetric bilinear forms and for some Einstein metrics can be extended significantly in various ways.

First, we can weaken the positive definiteness to tensor fields being only Sub-Semi-Riemmannian and having equal signature.

Next, we are able to replace Einstein metrics by parallel Ricci metrics (i.e. metrics with covariantly constant Ricci tensor).

Let $M$ be a smooth compact manifold without boundary. A Sub-Semi-Riemannian metric $\mathfrak{G}$ (SSR-metric for short) is a symmetric covariant 2-tensor field with constant signature.

Let us now state a lemma, interesting by itself, about almost conformal isometries, and whose proof deviates from DeTurck's original argument only slightly

**Lemma 1.1.** *Assume that $\mathfrak{G}$ and $\mathcal{G}$ are two smooth SSR-metrics on $M$ with equal signature. Let $g$ be a smooth Riemannian metric on $M$, let $p \in [1,\infty)$ and let $\varepsilon > 0$. Then there exist a smooth diffemorphism $\Phi$ and a smooth positive function $f$ such that $\Phi^*(f\mathcal{G}) - \mathfrak{G}$ is $\varepsilon$-close to zero in the $L^p$ norm relative to $g$.*

Before moving on to an application to the Ricci equation, let us introduce some notation. Given a smooth Riemannian manifold $(M, \mathfrak{g})$, we let $\mathrm{Ric}(\mathfrak{g})$ denote its Ricci curvature. Let $\Lambda$ be a real constant. We consider the operator

$$\mathrm{Ein}(\mathfrak{g}) := \mathrm{Ric}(\mathfrak{g}) + \Lambda\mathfrak{g}.$$

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This operator is geometric in the sense that for any smooth diffeomorphism $\varphi$ it holds that

$$\varphi^* \operatorname{Ein}(\mathfrak{g}) = \operatorname{Ein}(\varphi^* \mathfrak{g}).$$

We would like to invert Ein. For this, we choose on $M$ a tensor field $\mathcal{E} \in C^\infty(M, \mathcal{S}_2)$, where $\mathcal{S}_2$ is the set of covariant symmetric 2-tensors, and we look for a Riemannian metric $\mathfrak{g}$ such that

$$\operatorname{Ein}(\mathfrak{g}) = \mathcal{E}. \tag{1.1}$$

This is a geometrically natural and difficult quasilinear system to solve, even by perturbation methods. The prescribed Ricci curvature problem has a long history starting with the work of D. DeTurck [9]. It was studied in many different situations as the following non-exhaustive list illustrates: [11], [10], [13], [12], [1], [14], [2], [3], [6],[8], [7], [5], [4].

Motivated by the explosion of studies around the Ricci flow, and recently, some discrete versions thereof (e.g. $\operatorname{Ein}(g_{i+1}) = g_i$), a renewed interest arises for this kind of natural geometric equations. We invite the reader to look at the nice recent works of A. Pulemotov and Y. Rubinstein [16], as well as [17] for related results. Our contribution here is the following.

**Theorem 1.2.** *Assume that $M$ carries a Riemannian metric $g$ with parallel Ricci tensor. Let $\Lambda \in \mathbb{R}$ be such that $\operatorname{Ein}(g)$ is non-degenerate, and that $-2\Lambda$ is not in the spectrum of the Lichnerowicz Laplacian of $g$.*[1] *Then for any $\mathcal{E} \in C^\infty(M, \mathcal{S}_2)$ with the same signature as $\operatorname{Ein}(g)$ at each point of $M$, there exist a smooth positive function $f$ and a Riemannian metric $\mathfrak{g}$ in $C^\infty(M, \mathcal{S}_2)$ such that*

$$\operatorname{Ein}(\mathfrak{g}) = f\,\mathcal{E}.$$

The proof goes by combining Lemma 1.1, the local inversion result of Proposition 3.1 for weakly regular metrics (where the conformal factor $f$ is not required) and a regularity argument. We have then solved the problem up to a positive function $f$. Here we do not expect that $f$ can be taken equal to one in general; this will be the subject of future investigations.

Parallel Ricci metrics are (locally) products of Einstein metrics (see, e.g., [18]). They exist on the simplest examples of manifolds that do not admit Einstein metrics, like $\mathbb{S}^1 \times \mathbb{S}^2$, $\Sigma_\gamma \times \mathbb{S}^2$, $(\gamma \geq 1)$ or $\Sigma_\gamma \times \mathbb{T}^2$, $(\gamma \geq 2)$, where $\Sigma_\gamma$ is a surface of genus $\gamma$. Here, we should stress that the latter manifolds provide examples in which, for suitable values of the parameter $\Lambda$, the tensor $\operatorname{Ein}(g)$ does *not* fulfill DeTurck's positivity assumption (see [11], Theorem 1.2, p. 358). These examples illustrate the interest of our Theorem 1.2.

Parallel Ricci metrics are also static solutions of geometric fourth order flows (e.g., $\partial_t g = \Delta_g \operatorname{Ric}(g)$). Finally they are particular cases of Riemannian manifolds with harmonic curvature (equivalently, Codazzi Ricci tensor).

Our global result shows again that metrics with covariantly constant Ricci tensor deserve particular attention.

## 2. $L^p$ closeness of some Sub-Semi-Riemannian metrics

We follow Section 3, called an "approximation lemma" in [11], in order to verify that all the steps can be adapted to SSR-metrics as above. This will prove Lemma 1.1.

[1]Similarly to D. DeTurck [11], we allow an eigenspace spanned by $g$ when $\Lambda = 0$.

We will keep almost the same notation as in [11], replacing $S$ by $\mathfrak{G}$ and $R$ by $\mathcal{G}$.

Let $\mathfrak{G}$ and $\mathcal{G}$ be as in the Introduction; we thus assume they have equal signature. For the rest of the section we fix a Riemannian metric $g$, an $\varepsilon > 0$ and $p \in [1, +\infty)$. All the measures, volumes, and norms are understood with respect to $g$.

At each point $x \in M$, since the two SSR-metrics $\mathfrak{G}$ and $\mathcal{G}$ have equal signature, there exists an orientation preserving automorphism $u_x$ of $T_xM$, such that

$$\mathcal{G}_x(u_x(\cdot), u_x(\cdot)) = \mathfrak{G}_x(\cdot, \cdot)\,.$$

For $x \in M$, the following construction can be performed using the $g$-exponential map at $x$. There exists an open set $U_x$ such that:
(i) $U_x$ is contained in a coordinate neighbourhood of $x$ such that, in this coordinate system centred at 0, up to a positive automorphism $u_x$ of $T_xM$, $\mathcal{G}$ is equal to $\mathfrak{G}$ at $x$:

$${}^t u_x\, \mathcal{G}_x\, u_x = \mathfrak{G}_x.$$

(ii) For any positive real $\alpha_x$, the linear change of coordinates

$$\Phi_x := \sqrt{\alpha_x}\, u_x$$

satisfies on $U_x$ the estimate (the left-hand side of which does not depend upon $\alpha_x$, and vanishes at the origin),

$$\left| (\Phi_x^* \frac{1}{\alpha_x} \mathcal{G})_y - \mathfrak{G}_y \right|^p \leq \min\left( \frac{\varepsilon^p}{2\mathrm{Vol}(M)}, |\mathfrak{G}_y|^p \right). \tag{2.1}$$

We consider a triangulation of $M$ where each simplex $S$ lies in the interior of some $U_x$ with $x \in \mathring{S}$. Since the point $x$ belongs to the *interior* of the simplex $S$, shrinking $\alpha_x$ if necessary, we are sure that $\Phi_x$ send $S$ into $S$ (the norm of $\Phi_x$ approaches zero when $\alpha_x$ tends to zero).

Let $\Omega$, $\Omega_1$, $\Omega_2$, $\Omega_3$ be open neighbourhoods of the $(n-1)$-dimensional skeleton of the triangulation, with the properties:

$$\mathrm{Vol}(\Omega) < \frac{\varepsilon^p}{2(\max_M |\mathcal{G}| + 2\max_M |\mathfrak{G}|)^p},$$

and

$$\Omega_3 \subset \overline{\Omega}_3 \subset \Omega_2 \subset \overline{\Omega}_2 \subset \Omega_1 \subset \overline{\Omega}_1 \subset \Omega.$$

The rest of the proof in Section 3 of [11] is based on triangular inequalities between norms of tensors and can be implemented here without any change. For a better understanding, though, we provide further details of the figure on page 368 of [11], specifying the estimates that occur on the different parts of the simplex; see Figure 1. On the picture, we have denoted the error $|\Phi^*(f\mathcal{G}) - \mathfrak{G}|$ by $e$:

$$e = |\Phi^*(f\mathcal{G}) - \mathfrak{G}|.$$

On the inner part $T$ of the simplex, $e$ is estimated by (2.1). The transition of the diffeomorphism $\Phi$, on the middle annulus $R_2 = S \cap (\Omega_1 \backslash \Omega_2)$, from $\Phi_x$ to the identity, still exists because our $\Phi_x = \sqrt{\alpha_x}\, u_x$ is an orientation preserving map with norm less than 1 as in [11].

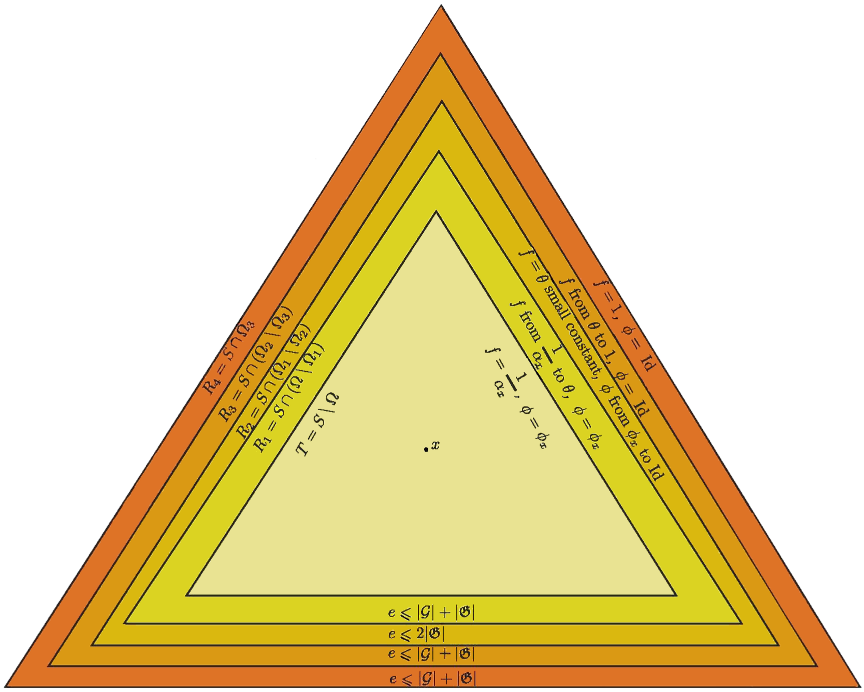


FIGURE 1. The simplex $S$ with the values of $f$ and $\Phi$, and the estimates on $e$.

*Remark* 2.1. The proof above only uses that at each point $x \in M$, the two symmetric 2-tensor fields $\mathfrak{G}$ and $\mathcal{G}$ have equal signature. Thus the conclusion of Lemma 1.1 stays true for symmetric 2-tensor fields having varying signatures and not only for SSR-metrics.

**Example 2.2.** The simplest non-trivial example consists of a product of manifolds $M = \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ with two SSR-metrics of the form

$$\mathfrak{G}(x, y, z) = -g_{\mathcal{X},y,z}(x) \oplus g_{\mathcal{Y},x,z}(y) \oplus 0_{\mathcal{Z}},$$

where $g_{\mathcal{X},y,z}$ is a family of Riemannian metrics on $\mathcal{X}$, depending on the parameters $y, z$ and smooth in all of its arguments. Here, any of the three manifolds but one may be reduced to a point.

## 3. Solvability of a Ricci type equation

We revisit Section 2 of [11] called "perturbation lemma".

We first need to introduce some operators. The divergence of a symmetric 2-tensor field and its $L^2$-adjoint acting on one form is

$$(\delta h)_j := -\nabla^i h_{ij}\ , \quad (\delta^* v)_{ij} := \frac{1}{2}(\nabla_i v_j + \nabla_j v_i).$$

The gravitational operator acting on symmetric 2-tensors is

$$G(h) := h - \frac{1}{2}\,\mathrm{Tr}_g(h) g.$$

The Lichnerowicz Laplacian is[2]

$$\Delta_L = \nabla^*\nabla + 2\,\mathrm{Ric} - 2\,\mathrm{Riem}\,.$$

It is part of the linearisation of the Ricci operator:

$$D\,\mathrm{Ric}(g) = \frac{1}{2}\Delta_L + \delta^*\delta\, G.$$

The Hodge Laplacian acting on one form is

$$\Delta_H = \Delta + \mathrm{Ric} = \nabla^*\nabla + \mathrm{Ric} = d^*d + dd^*.$$

We also define the following Laplacian:

$$\Delta_V := 2\,\delta\, G\,\delta^* = \nabla^*\nabla - \mathrm{Ric} = \Delta_H - 2\,\mathrm{Ric}\,.$$

We let $V$ denote its finite dimensional kernel, composed of smooth one forms (by elliptic regularity).

We start with the equivalent of Proposition 2.1 in [11].

**Proposition 3.1.** *Let $(M,g)$ be a smooth Riemannian manifold with parallel Ricci curvature. Let $\Lambda \in \mathbb{R}$ be such that* $\mathrm{Ein}(g)$ *is non-degenerate and that $-2\Lambda$ is not in the spectrum of the Lichnerowicz Laplacian. Then for any $\mathcal{E}$ close to* $\mathrm{Ein}(g)$ *in $H^{k+1,p}(M,\mathcal{S}_2)$, there exists a Riemannian metric $\mathfrak{g}$ in $H^{k+1,p}(M,\mathcal{S}_2)$ such that*

$$\mathrm{Ein}(\mathfrak{g}) = \mathcal{E}.$$

In [11], the proof of the corresponding proposition is given by a succession of lemmas. We thus revisit them one after the other. Some care is needed because we have to replace Ric with Ein and $-\Delta_L$ with $\Delta_L + 2\Lambda$. Furthermore, in our context, the operator $\Delta_L + 2\Lambda$ has trivial kernel whereas the kernel of $\Delta_L$ is non-empty in [11], spanned by $g$. We clearly also have for any Riemannian metric $g$ the Bianchi identity

$$\delta G(\mathrm{Ein}(g)) = 0.$$

We start with a local study of the action of the diffeomorphim group on the covariant symmetric 2-tensors, near a non-degenerate parallel one. The result obtained remains in the spirit of the local study near a Riemannian metric by Berger, Ebin, or Palais (see e.g. Lemma 2.3 of [11]). Here the metric tensor is replaced by a non-degenerate parallel tensor field. In the following, we let $\mathcal{T}_1$ denote the set of one forms.

**Lemma 3.2.** *Let $E$ be a smooth, non-degenerate and parallel symmetric 2-tensor field. Let $\mathcal{X}$ be a smooth Banach submanifold of $H^{k,p}(M,\mathcal{S}_2)$, whose tangent space at $E$ complements $\delta^*(H^{k,p}(M,\mathcal{T}_1))$. Then for any $\mathcal{E}$ close enough to $E$ in $H^{k,p}(M,\mathcal{S}_2)$, there exists an $H^{k+1,p}$ diffeomorphism $\Phi$ close to the identity such that $\Phi^*\mathcal{E} \in \mathcal{X}$.*

*Proof.* The tensor field $E$ being parallel, its Lie derivative in the direction of a vector field $v$ is

$$\mathcal{L}_v E = 2\delta^*(Ev).$$

Locally, the submanifold $\mathcal{X}$ can be seen as the image of an immersion $\mathfrak{X} : U \longrightarrow H^{k,p}(M,\mathcal{S}_2)$, with $\mathfrak{X}(0) = E$. We define $\mathcal{T}^\perp$ to be the set of vector fields $v \in H^{k+1,p}(M,\mathcal{T}_1)$ such that $E\,v$ is $L^2$-orthogonal[3] to $\ker\delta^*$.

[2]Different sign convention from DeTurck

[3]A closed complementing space suffices.

Let the map

$$F : U \times \mathcal{T}^{\perp} \times H^{k,p}(M,\mathcal{S}_2) \longrightarrow H^{k,p}(M,\mathcal{S}_2)$$

be defined as

$$F(k,Y,\mathcal{E}) = \Phi^*_{Y,1}(\mathcal{E}) - \mathfrak{X}(k),$$

where $\Phi_{Y,1}$ is the flow of the vector field $Y$ at time 1. We have $F(0,0,E) = 0$ and the linearisation of $F$ in the first two variables is

$$D_{(k,Y)}F(0,0,E)(l,X) = 2\delta^*(E\,X) - D\mathfrak{X}(0)l.$$

Now, since

$$H^{k,p}(M,\mathcal{S}_2) = \delta^*(H^{k+1,p}(M,\mathcal{T}_1)) \oplus \operatorname{Im} D\mathfrak{X}(0),$$

and since $E$ is non-degenerate, the derivative $D_{(k,Y)}F(0,0,E)$ is an isomorphism. From the implicit function theorem, for $\mathcal{E}$ close to $E$, there exists $k$ and $Y$, small such that $F(k,Y,\mathcal{E}) = 0$. □

Let us recall Lemma 2.5 in [11]:[4]

**Lemma 3.3.** *For $k \geq 1$, we have*

$$H^{k,p}(M,\mathcal{S}_2) = \frac{(\ker \delta G \cap H^{k,p}(M,\mathcal{S}_2))}{\delta^*(V)} \oplus \delta^*(H^{k+1,p}(M,\mathcal{T}_1)) \oplus G\delta^*(V).$$

The equivalent of Lemma 2.6 in [11] reads (we do not have to quotient by $R\,g$ because there is no kernel in our case, so that no adjustment with a constant $c$ is needed).

**Lemma 3.4.** *Suppose that $k \geq 0$, $p > n$, and that $g$ satisfies the hypotheses of Theorem* 1.2. *Let*

$$K = \frac{(\ker \delta G \cap H^{k+2,p}(M,\mathcal{S}_2))}{\delta^*(V)} \oplus G\delta^*(V)$$

*and define $F : K \longrightarrow H^{k,p}(M,\mathcal{S}_2)$ by*[5]

$$F(b) := \operatorname{Ein}(g+b).$$

*Then there exists a neighbourhood $U$ of* 0 *such that $F(U)$ is a Banach submanifold of $H^{k,p}(M,\mathcal{S}_2)$ whose tangent space at $F(0)=\operatorname{Ein}(g)$ complements $\delta^*(H^{k+1,p}(M,\mathcal{T}_1))$.*

*Proof.* We have to show that the derivative of $F$ at 0 is injective and its image has the closed subspace $\delta^*(H^{k+1,p})$ as complementary. A metric with parallel Ricci tensor is a local Einstein product so it is smooth. We first show that the spaces $\operatorname{Im}\delta^*$ and $\ker \delta G$ are "stable" (modulo two points of regularity) by $\Delta_L + 2\Lambda$ but also by $D\operatorname{Ein}(g)$ (when the metric is Ricci parallel). Indeed, we recall that in that case [15]:

$$\delta\,\Delta_L = \Delta_H\,\delta,$$

and the adjoint version:

$$\Delta_L\,\delta^* = \delta^*\,\Delta_H.$$

We deduce that

$$(\Delta_L + 2\Lambda)\,\delta^* = \delta^*\,(\Delta_H + 2\Lambda) = \delta^*\,(\Delta_V + 2\operatorname{Ein}),$$

[4]It seems there is a misprint in the proof of this lemma: The Ricci term for $\delta G\delta^*$ at the top of page 362 in [11] has a wrong sign.

[5]To avoid ambiguities, we may take any fixed closed complementing space $W$ for $\delta^*(V)$ in $\ker \delta G \cap H^{k+2,p}(M,\mathcal{S}_2)$ instead of the first factor of $K$.

and

$$D\,\mathrm{Ein}(g)\,\delta^* = \frac{1}{2}\delta^*\,(\Delta_H + 2\Lambda) + \frac{1}{2}\delta^*\,\Delta_V = \delta^*\,(\Delta_V + \mathrm{Ein}) = \delta^*\,(\Delta + \Lambda)$$

thus the "stability" of $\mathrm{Im}\,\delta^*$ by both operators above.

When restricted to the kernel of $\delta\,G$, we trivially have

$$D\,\mathrm{Ein}(g) = \frac{1}{2}(\Delta_L + 2\Lambda),$$

but also, by linearising $\delta\,G\;\mathrm{Ein}(g) = 0$ for instance,

$$\delta\,G\,(\Delta_L + 2\Lambda) = 0,$$

whence the stability of $\ker\delta\,G$.

We also remark that with the formula above, if $v \in V$ we have

$$(\Delta_L + 2\Lambda)(\delta^* v) = 2\delta^*(\mathrm{Ein}\,v) = 2\,\mathrm{Ein}\,\delta^* v\,,$$

as well as

$$D\,\mathrm{Ein}(g)(\delta^* v) = \delta^*(\mathrm{Ein}\,v) = \mathrm{Ein}\,\delta^* v\,.$$

Given any function $u$, it is well known that $\Delta_L(ug) = (\Delta u)g$, so

$$(\Delta_L + 2\Lambda)(d^* w\,g) = [(\Delta + 2\Lambda)d^* w]g = [d^*(\Delta_H + 2\Lambda)w]g.$$

We obtain that

$$(\Delta_L + 2\Lambda)G\delta^* = G\delta^*(\Delta_H + 2\Lambda).$$

If $v \in V$, we deduce

$$(\Delta_L + 2\Lambda)G\delta^* v = G\delta^*(2\,\mathrm{Ein}\,v).$$

Assume that $-2\Lambda$ is not an eigenvalue of $\Delta_L$; then $\Delta_L + 2\Lambda$ is an isomorphism from $H^{k+2,p}(M,\mathcal{S}_2)$ to $H^{k,p}(M,\mathcal{S}_2)$. The image of the splitting in Lemma 3.3 by $\Delta_L + 2\Lambda$ leads to[6]

$$H^{k,p}(M,\mathcal{S}_2) = \frac{(\ker\delta G \cap H^{k,p}(M,\mathcal{S}_2))}{\delta^*(\mathrm{Ein}\,V)} \oplus \delta^*(H^{k+1,p}(M,\mathcal{T}_1)) \oplus G\delta^*(\mathrm{Ein}\,V).$$

The two first factors are the same as the image by $D\,\mathrm{Ein}(g)$ of the corresponding spaces in Lemma 3.3. Let us study the image of the third one. For $v \in V$, we compute

$$\begin{aligned}\delta^*\delta G G\delta^* v &= \delta^*\delta G(\delta^* v + \frac{1}{2}d^* v\,g) = \frac{1}{2}\delta^*\delta G(d^* v\,g) = \frac{2-n}{4}\delta^*\delta(d^* v\,g)\\ &= \frac{n-2}{4}\delta^* dd^* v = \frac{n-2}{2}\delta^*\delta\delta^* v = -G\delta^*\delta\delta^* v.\end{aligned} \tag{3.1}$$

We deduce that

$$D\,\mathrm{Ein}(g)G\delta^* V = \left[G\delta^*(\mathrm{Ein}\ .) + \frac{n-2}{2}\delta^*\delta\delta^*\right]V. \tag{3.2}$$

Let us define

$$\mathcal{F} := \delta^*(H^{k+1,p}(M,\mathcal{T}_1)) \oplus G\delta^*(\mathrm{Ein}\,V).$$

We now prove that

$$\mathcal{F} = \delta^*(H^{k+1,p}(M,\mathcal{T}_1)) \oplus D\,\mathrm{Ein}(g)G\delta^* V. \tag{3.3}$$

[6] Here we also have to replace the first factor by $(\Delta_L + 2\Lambda)W$ when a choice of $W$ was made in the first factor of $K$.

The fact that $\mathcal{F}$ is the sum of the factors is clear by (3.2). Let $w$ be in the intersection of the factors, so

$$w = \delta^* u = G\delta^* \operatorname{Ein} v + \delta^*\delta\delta^* \frac{n-2}{2} v,$$

for some $u \in H^{k+1,p}(M, \mathcal{T}_1)$ and $v \in V$. Because of the decomposition of $\mathcal{F}$, we deduce that $G\delta^* \operatorname{Ein} v = 0$ thus $(\Delta_L + 2\Lambda)G\delta^* v = 0$; then $G\delta^* v = 0$ and finally, by (3.1), $\frac{n-2}{2}\delta^*\delta\delta^* v = 0$ so $w = 0$. We have obtained:

$$H^{k,p}(M, \mathcal{S}_2) = \operatorname{Im} DF(0) \oplus \delta^*(H^{k+1,p}(M, \mathcal{T}_1)).$$

We claim that $DF(0)$ is injective. Indeed, let $h$ be in the kernel of $DF(0)$; then $h = [u] + G\delta^* v$ with $[u]$ in the first summand of $K$. Thus

$$[\Delta_L + 2\Lambda][u] + D\operatorname{Ein}(g)G\delta^* v = 0\,,$$

and in view of the decomposition (3.3) we obtain

$$[\Delta_L + 2\Lambda][u] = D\operatorname{Ein}(g)G\delta^* v = 0\,.$$

This implies $[u] = 0$ and, from (3.2),

$$v \in G\delta^* \operatorname{Ein} V \cap \delta^*(H^{k+1,p}(M, \mathcal{T}_1)) = \{0\}\,,$$

so $h = 0$. ☐

From Lemma 3.2 with $E = \operatorname{Ein}(g)$ and Lemma 3.4 we directly deduce:

**Lemma 3.5.** *If $\mathcal{E} \in H^{k,p}$ and $|\mathcal{E} - \operatorname{Ein}(g)|_{k,p} < \varepsilon$, then there exist a metric $\mathfrak{g} \in H^{k+2,p}$ and a diffeomorphism $\varphi \in H^{k+1,p}$ for which $\operatorname{Ein}(\mathfrak{g}) = \varphi^*\mathcal{E}$.*

We will complete the proof of Proposition 3.1, where now $\mathcal{E} \in H^{k+1,p}$, but $\mathfrak{g}$ and $\varphi$ still come from Lemma 3.5, so $\varphi$ is a priori not regular enough. Inspection of pages 364-365 in [11] shows that it suffices to replace $\operatorname{Ric}(\mathfrak{g})$ by $\operatorname{Ein}(\mathfrak{g})$ to obtain that $\varphi$ is in fact in $H^{k+2,p}$. We conclude that $(\varphi^{-1})^*\mathfrak{g} \in H^{k+1,p}$, and that its image by Ein is $\mathcal{E}$. At this level we also use that $\operatorname{Ein}(\mathfrak{g})$ is non-degenerate (see equation (2.8) there).

Theorem 1.2 is now a direct consequence of Lemma 1.1, Proposition 3.1 with $k = 0$, and the regularity result of [12].

**Example 3.6.** Recalling that the Ricci curvature of a product of Riemannian manifolds is the direct sum of the Ricci curvatures of each factor, we see that a product of Einstein manifolds clearly satisfies the assumption of Theorem 1.2. The simplest example combining the three possibilities of Einstein constants is the following. Let us consider three compact Einstein manifolds $(\mathcal{X}, g_-)$, $(\mathcal{Y}, g_+)$, $(\mathcal{Z}, g_0)$ with Ricci curvatures given by $\operatorname{Ric}(g_-) = -g_-$, $\operatorname{Ric}(g_+) = g_+$, $\operatorname{Ric}(g_0) = 0$. Then $M = \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ endowed with

$$g = g_- \oplus g_+ \oplus g_0,$$

has parallel Ricci curvature equal to

$$\operatorname{Ric}(g) = -g_- \oplus g_+ \oplus 0.$$

In this example, the kernel of $\Delta_L$ contains the parallel tensors

$$h = c_- g_- \oplus c_+ g_+ \oplus c_0 g_0,$$

for any constants $c_-, c_+, c_0$. Here, we only have to choose $\Lambda$ in order to destroy this kernel and make $\operatorname{Ein}(g)$ non-degenerate.

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