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HAMEL BASES AND WELL–ORDERING THE CONTINUUM

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Abstract. In ZF, the existence of a Hamel basis does not yield a well–ordering of $\mathbb{R}$.

Throughout this paper, by a *Hamel basis* we always mean a basis for $\mathbb{R}$, construed as a vector space over $\mathbb{Q}$. We denote by E the *Vitali equivalence relation*, xEy iff $x - y \in \mathbb{Q}$ for x, $y \in \mathbb{R}$. We also write $[x]_E = \{y \colon yEx\}$ for the E–equivalence class of x. A transversal for the set of all E–equivalence classes picks exactly one member from each $[x]_E$. The range of any such transversal is also called a *Vitali set*.

A set $\Lambda \subset \mathbb{R}$ is a *Luzin set* iff Λ is uncountable but $\Lambda \cap M$ is at most countable for every meager set $M \subset \mathbb{R}$. A set $S \subset \mathbb{R}$ is a *Sierpiński set* iff S is uncountable but $S \cap N$ is at most countable for every null set $N \subset \mathbb{R}$ ("null" in the sense of Lebesgue measure). A set $B \subset \mathbb{R}$ is a *Bernstein set* iff $B \cap P \neq \emptyset \neq P \setminus B$ for every perfect set $P \subset \mathbb{R}$.

It has been well known for more than a century that the existence of a well–ordering of the reals implies the existence of all these "pathological" sets of reals: Hamel bases, Vitali sets, Luzin sets, Sierpiński sets,[1] and Bernstein sets; see, e.g., the thorough discussion in [4].[2]

D. Pincus and K. Prikry study the Cohen-Halpern-Lévy model H in [8] and show that there is a Luzin set in H, thereby establishing that in ZF, the existence of a Luzin set does not imply the existence of a well–ordering of the reals. We will recall their proof below; cf. Theorem 1.5.

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[1]To get Luzin and Sierpiński sets, one needs to make the additional hypothesis that CH holds true, unless, e.g., one works with the concept of *generalized* Luzin and Sierpiński sets which arises from the concept of Luzin and Sierpiński sets by replacing "at most countable" with "smaller than the continuum" and works under Martin's Axiom.

[2]A discussion of "paradoxical" decompositions of the unit ball à la Hausdorff and Banach–Tarski is beyond the scope of this paper; cf. also [4].

In ZF, the existence of a Hamel basis implies the existence of a Vitali set of reals; cf. Lemma 1.1 below. Feferman had observed that H has a Vitali set; cf. [8, p. 433]. Pincus and Prikry ask:

"We would be interested in knowing whether a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$ (the rationals) exists in H or in any other model in which $\mathbb{R}$ cannot be well ordered" ([8, p. 433]).

In [1], A. Blass shows that in ZF, if every vector space has a basis, then the axiom of choice holds true.

In the current paper we answer the question by Pincus and Prikry and show that H does have a Hamel basis. This will also give Feferman's result as a corollary; cf. Corollary 2.4 below.

We shall also show that H has a Bernstein set; cf. Theorem 1.7. There is no Sierpiński set in H, though; cf. Lemma 1.6. Therefore, in ZF not even the conjunction of the following statements (1), (3), (4), and (5) implies the existence of a well–ordering of the reals:

(1) There is a Luzin set.
(2) There is a Sierpiński set.
(3) There is a Bernstein set.
(4) There is a Vitali set.
(5) There is a Hamel basis.

In a sequel to the current paper, in [10], it is shown that in ZF plus DC, (5) does not yield a well–ordering of the reals.

1. Warm ups

In what follows, we shall sometimes think of reals as elements of the Baire space ${}^\omega\omega$, sometimes as elements of the Cantor space ${}^\omega 2$, and at other times think of them as actual reals. The attentive reader will have no problem sorting this out.

Let us first show that (5) implies (4). If X is a set of reals, then we write $\mathrm{span}(X)$ for the set of all $\sum_{n=1}^m q_n \cdot x_n$, where $m \in \mathbb{N}$, $m \geq 1$, $q_n \in \mathbb{Q}$, and $x_n \in X$ for all n, $1 \leq n \leq m$. By convention, we also declare $\mathrm{span}(\emptyset) = \{0\}$.

Lemma 1.1 (Folklore)**.** *In* ZF*, if there is a Hamel basis, then there is a Vitali set.*

Proof. Let B be a Hamel basis. Let $1 = \sum_{k=1}^n q_k \cdot z_k$, where $q_k \in \mathbb{Q} \setminus \{0\}$ and $z_k \in B$ for $1 \leq k \leq n$. It is straightforward to verify that $\mathrm{span}(B \setminus \{z_1\})$ is a Vitali set. □ (Lemma 1.1)

Let us now recall the Cohen-Halpern-Lévy model. We let $\mathbb{C}$ denote Cohen forcing, i.e., the collection of all finite sequences of natural numbers, ordered by end–extension. If I is any index set, then $\mathbb{C}(I)$ denotes the finite support product of I many copies of $\mathbb{C}$, i.e., $p \in \mathbb{C}(I)$ iff $p(\ell) \in \mathbb{C}$ for $\ell \in I$ and

$$\mathrm{supp}(p) = \{\ell \in I \colon p(\ell) \neq \emptyset\}$$

is finite. In what follows, $I \subseteq \omega$. If $I \cap J = \emptyset$, then $\mathbb{C}(I \cup J) \cong \mathbb{C}(I) \times \mathbb{C}(J)$.

Let us force with $\mathbb{C}(\omega)$ over L,[3] and let g be a generic filter. Let c_n, $n < \omega$, denote the Cohen reals which g adds. Let us write $A = \{c_n \colon n < \omega\}$ for the set of

[3] We might as well force over V rather than L, but forcing over L will simplify the notation a bit.

those Cohen reals. The model

$$H = H(L) = \mathsf{HOD}^{L[g]}_{A\cup\{A\}}$$

of all sets which inside $L[g]$ are hereditarily definable from parameters in $\mathrm{OR} \cup A \cup \{A\}$ is the Cohen–Halpern–Lévy model (over L); cf. [2, pp. 136–141], [3], and [8, p. 429]. As $L \subset H \subset L[g]$ and $\mathbb{C}(\omega)$ is countable, and hence trivially has the c.c.c., L, H, and $L[g]$ all have the same cardinals, and in particular $\omega_1^H = \omega_1^L$. It is well known that in H, the reals cannot be well–ordered and in fact A has no countable subset; cf., e.g., [2, pp. 136–141] and Lemma 1.2 below. Here and in what follows, a set X is called *countable* iff there is some bijection $f\colon \omega \to X$, and X is called *at most countable* iff X is finite or countable.

In particular, the Continuum Hypothesis fails in H: the set $A \subset \mathbb{R} \cap H$ is not countable, but H can see no surjection from A onto $\mathbb{R} \cap H$.

For any finite $a \subset A$, we write $L[a]$ for the model constructed from the finitely many reals in a. Fixing some Gödelization of formulae (or some enumeration of all the rud functions, resp.) at the outset, each $L[a]$ comes with a unique canonical global well–ordering $<_a$ of $L[a]$ by which we mean the one which is induced by the *natural* order of the elements of a and the fixed Gödelization device in the usual fashion. The assignment $a \mapsto <_a$, $a \in [A]^{<\omega}$, is hence in H.[4] This is a crucial fact.

Let us fix a bijection

$$e\colon \omega \to \omega \times \omega, \tag{1}$$

and let us write $((n)_0, (n)_1) = e(n)$.

We shall also make use of the following.

Lemma 1.2. (1) *Let $a \in [A]^{<\omega}$ and $X \subset L[a]$, $X \in H$, say $X \in \mathsf{HOD}^{L[g]}_{b\cup\{A\}}$, where $b \supseteq a$, $b \in [A]^{<\omega}$. Then $X \in L[b]$.*

(2) *There is no well–ordering of the reals in H.*

(3) *A has no countable subset in H.*

(4) *$[A]^{<\omega}$ has no countable subset in H.*

Proof sketch. (1) Every permutation $\pi\colon \omega \to \omega$ induces an automorphism e_π of $\mathbb{C}(\omega)$ by sending p to q, where $q(\pi(n)) = p(n)$ for all $n < \omega$. It is clear that no e_π moves the canonical name for A, call it $\dot{A}$. Let us also write $\dot{c}_n$ for the canonical name for c_n, $n < \omega$. Now if a, and b are as in the statement of (1), say $b = \{c_{n_1}, \ldots, c_{n_k}\}$, if p, $q \in \mathbb{C}(\omega)$, if $\pi \upharpoonright \{n_1, \ldots, n_k\} = \mathrm{id}$, $p \upharpoonright \{n_1, \ldots, n_k\}$ is compatible with $q \upharpoonright \{n_1, \ldots, n_k\}$, and $\mathrm{supp}(\pi(p)) \cap \mathrm{supp}(q) \subseteq \{n_1, \ldots, n_k\}$, if $x \in L$, if $\alpha_1, \ldots, \alpha_m$ are ordinals, and if φ is a formula, then

$$\begin{aligned} &p \Vdash^{\mathbb{C}(\omega)}_L \varphi(\check{x}, \check{\alpha}_1, \ldots, \check{\alpha}_m, \dot{c}_{n_1}, \ldots, \dot{c}_{n_k}, \dot{A}) \\ \Longleftrightarrow\ &\pi(p) \Vdash^{\mathbb{C}(\omega)}_L \varphi(\check{x}, \check{\alpha}_1, \ldots, \check{\alpha}_m, \dot{c}_{n_1}, \ldots, \dot{c}_{n_k}, \dot{A}) \end{aligned}$$

and $\pi(p)$ is compatible with q, so that the statement $\varphi(\check{x}, \check{\alpha}_1, \ldots, \check{\alpha}_m, \dot{c}_{n_1}, \ldots, \dot{c}_{n_k}, \dot{A})$ will be decided by conditions $p \in \mathbb{C}(\omega)$ with $\mathrm{supp}(p) \subseteq \{n_1, \ldots, n_k\}$. But every set in $L[b]$ is coded by a set of ordinals, so if X is as in (1), this shows that $X \in L[b]$.

(2) Every real is a subset of L. Hence by (1), if $L[g]$ had a well–ordering of the reals in $\mathsf{HOD}^{L[g]}_{a\cup\{A\}}$, some $a \in [A]^{<\omega}$, then every real of H would be in $L[a]$, which is nonsense.

[4] More precisely, the ternary relation consisting of all (a, x, y) such that $x <_a y$ is definable over H.

(3) Assume that $f\colon \omega \to A$ is injective, $f \in H$. Let $x \in {}^\omega\omega$ be defined by $x(n) = f((n)_0)((n)_1)$, so that $x \in H$. By (1), $x \in L[a]$ for some $a \in [A]^{<\omega}$. But then $\mathrm{ran}(f) \subset L[a]$, which is nonsense, as there is some $n < \omega$ such that $c_n \in \mathrm{ran}(f) \setminus a$.

(4) This readily follows from (3). □ (Lemma 1.2)

Let us recall another standard fact.

$$\text{If } a,\, b \in [A]^{<\omega}, \text{ then } L[a] \cap L[b] = L[a \cap b]. \tag{2}$$

To see this, let us assume without loss of generality that $a \setminus b \neq \emptyset \neq b \setminus a$, and say $a \setminus b = \{c_n \colon n \in I\}$ and $b \setminus a = \{c_n \colon n \in J\}$, where I and J are non–empty disjoint finite subsets of ω. Then $\mathbb{C}(I) \cong \mathbb{C} \cong \mathbb{C}(J)$, and $a \setminus b$ and $b \setminus a$ are mutually $\mathbb{C}$–generic over $L[a \cap b]$. But then $L[a] \cap L[b] = L[a \cap b][a \setminus b] \cap L[a \cap b][b \setminus a] = L[a \cap b]$; cf. [9, Problem 6.12].

For any $a \in [A]^{<\omega}$, we write $\mathbb{R}_a = \mathbb{R} \cap L[a]$ and $\mathbb{R}_a^+ = \mathbb{R}_a \setminus \bigcup\{\mathbb{R}_b \colon b \subsetneq a\}$. By [2, pp. 136–141], $(\mathbb{R}_a^+ \colon a \in [A]^{<\omega})$ is a partition of $\mathbb{R}$: By Lemma 1.2 (1),

$$\mathbb{R} \cap H = \bigcup\{\mathbb{R}_a^+ \colon a \in [A]^{<\omega}\}, \tag{3}$$

and $\mathbb{R}_a \cap \mathbb{R}_b = \mathbb{R}_{a \cap b}$ by (2), so that

$$\mathbb{R}_a^+ \cap \mathbb{R}_b^+ = \emptyset \text{ for } a, b \in [A]^{<\omega}, a \neq b. \tag{4}$$

For $x \in \mathbb{R}$, we shall also write $a(x)$ for the unique $a \in [A]^{<\omega}$ such that $x \in \mathbb{R}_a^+$, and we shall write $\#(x) = \mathrm{Card}(a(x))$.

Adrian Mathias showed that there is an H–definable function which assigns to each $x \in H$ an ordering $<_x$ such that $<_x$ is a well–ordering iff x can be well–ordered in H; cf. [6, p. 182]. This gives the following as a special simple case.

Lemma 1.3 (A. Mathias)**.** *In H, the union of countably many countable sets of reals is countable.*

Proof. Let us work inside H. Let $(A_n \colon n < \omega)$ be such that for each $n < \omega$, $A_n \subset \mathbb{R}$ and there exists some surjection $f\colon \omega \to A_n$. For each such pair n, f let $y_{n,f} \in {}^\omega\omega$ be such that $y_{n,f}(m) = f((m)_0)((m)_1)$. If $a \in [A]^{<\omega}$ and $y_{n,f} \in \mathbb{R}_a$, then $A_n \in L[a]$. By (2), for each n there is a unique $a_n \in [A]^{<\omega}$ such that $A_n \in L[a_n]$ and $b \supset a_n$ for each $b \in [A]^{<\omega}$ such that $A_n \in L[b]$. Notice that A_n is also countable in $L[a_n]$.

Using the function $n \mapsto a_n$, an easy recursion yields a surjection $g\colon \omega \to \bigcup\{a_n \colon n < \omega\}$: first enumerate the finitely many elements of a_0 according to their natural order, then enumerate the finitely many elements of a_1 according to their natural order, etc. As A has no countable subset, $\bigcup\{a_n \colon n < \omega\}$ must be finite, say $a = \bigcup\{a_n \colon n < \omega\} \in [A]^{<\omega}$. But then $\{A_n \colon n < \omega\} \subset L[a]$. (We don't claim $(A_n \colon n < \omega) \in L[a]$.)

For each $n < \omega$, we may now let f_n be the $<_a$–least surjection $f\colon \omega \to A_n$. Then $f(n) = f_{(n)_0}((n)_1)$ for $n < \omega$ defines a surjection from ω onto $\bigcup\{A_n \colon n < \omega\}$, as desired. □ (Lemma 1.3)

Lemma 1.4 ([5, Theorem 3.20])**.** (1) *Let $a \in [A]^{<\omega}$. Then $\mathbb{R}_a$ is a null set in H.*

(2) *If $B \subset \mathbb{R} \cap H$, $B \in H$, and B is countable in $L[g]$, then B is a null set in H.*

Proof sketch. (1) Let $\mathbb{R} = {}^\omega 2$ in this argument, with the addition $+$ being the componentwise addition in $\mathbb{Z}/2\mathbb{Z}$. Let $n < \omega$ be such that $c_n \notin a$. It suffices to prove that $\mathbb{R}_a$ is null in $L[a \cup \{c_n\}]$.

In $L[a]$, let $\mathbb{R}_a = N \cup M$, where N is G_δ and null set, and M is F_σ and meager; cf., e.g., [7]. Inside $L[a \cup \{c_n\}]$, let us consider $N^* + c_n = \{x + c_n : x \in N^*\}$, where N^* is $L[a \cup \{c_n\}]$'s version of N.

Let $x \in \mathbb{R}_a$. As N is comeager in $L[a]$, $N + x$ is also comeager in $L[a]$, so that $c_n \in (N + x)^* = N^* + x$, see [9, Lemma 8.9 (2)], and hence $x \in N^* + c_n$. So $\mathbb{R}_a \subseteq N^* + c_n$. But N is null in $L[a]$, and hence N^* and $N^* + c_n$ are null in $L[a \cup \{c_n\}]$. $\mathbb{R}_a$ is therefore contained in a null set of $L[a \cup \{c_n\}]$ and is hence itself null.

(2) Say $f : \omega \to B$, $f \in L[g]$, is an enumeration of B, and let $\tau \in L^{\mathbb{C}(\omega)}$ be such that $\tau^g = f$. Let us write $\tau(n)$ for the canonical name for $f(n)$ induced by τ. We aim to find $N \in H$, a G_δ null set in H with a code in L such that $B \subset N$. Let $h : \mathbb{C}(\omega) \times \omega \to \omega$ be bijective.

Let $m < \omega$. Set $\epsilon^m = \frac{1}{m+1}$ and $\epsilon^m_n = \frac{1}{2^{n+1}} \cdot \epsilon^m$ for $n < \omega$, so that $\sum_{n=0}^{\infty} \epsilon^m_n = \epsilon^m$.

Working in L, for each pair $(p, k) \in \mathbb{C}(\omega) \times \omega$, write $n = h((p, k))$, and let us pick some $q \in \mathbb{C}(\omega)$, $q \leq p$, and some $s \in {}^{<\omega}\omega$ such that $q \Vdash^{\mathbb{C}(\omega)}_L \check{s} \subset \tau(k)$, and $\mu(U_s) \leq \epsilon^m_n$, and write $\mathcal{O}^m_n = U_s$. (Here, U_s is the basis clopen set $\{x : x \supset s\}$.)

Set $\mathcal{O}^m = \bigcup\{\mathcal{O}^m_n : n < \omega\}$. For a given $k < \omega$, the set $\{q \in \mathbb{C}(\omega) : \exists n\, q \Vdash^{\mathbb{C}(\omega)}_L \tau(k) \in \mathcal{O}^m_n\}$ is dense, so that $f(k) = (\tau(k))^g \in \mathcal{O}^m_n$ for some n. In other words, $B \subset \mathcal{O}^m$.

Set $N = \bigcap_{m<\omega} \mathcal{O}^m$, to be interpreted in H. We have that N is a G_δ null set inside H with a code in L, and $B \subset N$. □ (Lemma 1.4)

Theorem 1.5 (D. Pincus, K. Prikry)**.** *In H, there is a Luzin set.*

Proof. Let $\Lambda \in L$ be such that $L \models$ "Λ is a Luzin set". We aim to verify that Λ is Luzin in H. Λ is uncountable in L, so that also H can see a bijection of Λ with its own ω_1, as $\omega_1^H = \omega_1^L$. In particular, Λ is uncountable in H.

By Lemma 1.3, it suffices to verify that inside H,

$$\Lambda \setminus \mathcal{O} \text{ is at most countable,} \tag{5}$$

whenever $\mathcal{O}$ is a dense union of countably many open intervals with rational end–points.

Let $((p_n, q_n) : n < \omega)$ be an enumeration of all open intervals with rational end–points, and let $X \subset \omega$, $X \in H$, be such that

$$H \models \text{ "}\mathcal{O} = \bigcup\{(p_n, q_n) : n \in X\} \text{ is dense".}$$

Let us suppose that (5) were not true in H for this fixed $\mathcal{O}$. As $\Lambda \in L$, inside H there must then be a bijection from ω_1 onto $\Lambda \setminus \mathcal{O}$, so that by $\omega_1^{L[g]} = \omega_1^H$ also

$$\Lambda \setminus \mathcal{O} \text{ is uncountable in } L[g]. \tag{6}$$

Let $\tau \in L^{\mathbb{C}(\omega)}$ be a name for X, and let $p \in g$ be such that

$$p \Vdash^{\mathbb{C}(\omega)}_L \text{ "}\Lambda \setminus \bigcup\{(p_n, q_n) : n \in \tau\} \text{ is uncountable".}$$

As $\mathbb{C}(\omega)$ is countable, we may work in $L[g]$ and find some $q \in g$, $q \leq p$, such that for uncountably many $x \in \mathbb{R} \cap L$,

$$q \Vdash^{\mathbb{C}(\omega)}_L \text{ "}\check{x} \in \Lambda \setminus \bigcup\{(p_n, q_n) : n \in \tau\}\text{".} \tag{7}$$

Let us write U for the set of all $x \in \mathbb{R} \cap L$ with (7), so that U is an uncountable set of reals in L, and let

$$\mathcal{O}^* = \bigcup \{(p_n, q_n) \colon \exists r \leq q \, r \Vdash_L^{\mathbb{C}(\omega)} n \in \tau\},$$

as being defined in L.

Of course, $\mathcal{O}^* \supseteq \mathcal{O} \cap L$, so that $\mathcal{O}^*$ is open and dense in L. As Λ is a Luzin set in L, $\Lambda \setminus \mathcal{O}^*$ must be countable in L.

We have a contradiction with (6). □ (Theorem 1.5)

Lemma 1.6. *In H, there is no Sierpiński set.*

Proof. We shall prove that there is no set $S \in H$ of reals such that S is not at most countable in H and for each null set N of H, $S \cap N$ is at most countable.

Let us suppose that $S \in H$ is such a set. By Lemma 1.4, we cannot have that $S \subseteq \mathbb{R}_a$ for some $a \in [A]^{<\omega}$, because if this were true, then $S \cap \mathbb{R}_a = S$ and S itself would have to be at most countable.

Therefore, the set

$$F = \{a \in [A]^{<\omega} \colon S \cap \mathbb{R}_a^+ \neq \emptyset\}$$

is not finite. We may then inside H define the function $f \colon F \to \mathbb{R} \cap H$ by setting $f(a)$ to be the $<_a$–least element of $S \cap \mathbb{R}_a^+$.

Write $B = \operatorname{ran}(f)$. Then $B \in H$, and B is countable inside $L[g]$. By Lemma 1.4 (2), B is then a null set in H. Therefore, $B = S \cap B$ must be countable in H, i.e., there is some bijective $h \in H$, $h \colon \omega \to B$.

However, $((a, \mathbb{R}_a^+) \colon a \in [A]^{<\omega}) \in H$, so that $x \mapsto a(x)$ is in H, and hence $a \circ h \in H$, where $(a \circ h)(n) = a(h(n))$, $n < \omega$. Then $a \circ h \colon \omega \to [A]^{<\omega}$ is injective, which contradicts Lemma 1.2 (4). □ (Lemma 1.6)

Theorem 1.7. *In H, there is a Bernstein set.*

Proof. In this proof, let us think of reals as elements of the Cantor space ${}^\omega 2$. Let us work in H.

We let

$$B = \{x \in \mathbb{R} \colon \exists \text{ even } n \, (2^n < \#(x) \leq 2^{n+1})\} \quad \text{and}$$
$$B' = \{x \in \mathbb{R} \colon \exists \text{ odd } n \, (2^n < \#(x) \leq 2^{n+1})\}.$$

Obviously, $B \cap B' = \emptyset$.

Let $P \subset \mathbb{R}$ be perfect. We aim to see that $P \cap B \neq \emptyset \neq P \cap B'$.

Say $P = [T] = \{x \in {}^\omega 2 \colon \forall n \, x \restriction n \in T\}$, where $T \subseteq {}^{<\omega}2$ is a perfect tree. Modulo some fixed natural bijection ${}^{<\omega}2 \leftrightarrow \omega$, we may identify T with a real. By (3), we may pick some $a \in [A]^{<\omega}$ such that $T \in L[a]$. Say $\operatorname{Card}(a) < 2^n$, where n is even.

Let $b \in [A]^{2^{n+1}}$, $b \supset a$, and let $x \in \mathbb{R}_b^+$. In particular, $\#(x) = 2^{n+1}$. It is easy to work in $L[b]$ and construct some $z \in [T]$ such that $x \leq_T z \oplus T$,[5] e.g., arrange that if $z \restriction m$ is the k^{th} splitting node of T along z, where $k \leq m < \omega$, then $z(m) = 0$ if $x(k) = 0$ and $z(m) = 1$ if $x(k) = 1$.

If we had $\#(z) \leq 2^n$, then $\#(z \oplus T) \leq \#(z) + \#(T) < 2^n + 2^n = 2^{n+1}$, so that $\#(x) < 2^{n+1}$ by $x \leq_T z \oplus T$. Contradiction! Hence $\#(z) > 2^n$. By $z \in L[b]$, $\#(z) \leq 2^{n+1}$. Therefore, $z \in P \cap B$.

The same argument shows that $P \cap B' \neq \emptyset$. B (and also B') is thus a Bernstein set. □ (Theorem 1.7)

[5]Here, $(x \oplus y)(2n) = x(n)$ and $(x \oplus y)(2n+1) = y(n)$, $n < \omega$.

2. A Hamel basis

The following is the main theorem of the current paper. Recall that for any $a \in [A]^{<\omega}$, we write $\mathbb{R}_a = \mathbb{R} \cap L[a]$. Let us now also write

$$\mathbb{R}_{<a} = \mathrm{span}(\bigcup\{\mathbb{R}_b \colon b \subsetneq a\}),$$

and $\mathbb{R}^*_a = \mathbb{R}_a \setminus \mathbb{R}_{<a}$. In particular, $\mathbb{R}_{<\emptyset} = \{0\}$ by our above convention that $\mathrm{span}(\emptyset) = \{0\}$, and $\mathbb{R}^*_\emptyset = (\mathbb{R} \cap L) \setminus \{0\}$.

The proof of Claim 2.2 below will show that

$$\mathbb{R} \cap H = \mathrm{span}(\bigcup\{\mathbb{R}^*_a \colon a \in [A]^{<\omega}\}). \tag{8}$$

Also, we have that $\mathbb{R}^*_a \subset \mathbb{R}^+_a$, so that by (4),

$$\mathbb{R}^*_a \cap \mathbb{R}^*_b = \emptyset \text{ for } a, b \in [A]^{<\omega}, a \neq b. \tag{9}$$

Theorem 2.1. *In H, there is a Hamel basis.*

Proof. We call $X \subset \mathbb{R}^*_a$ *linearly independent over* $\mathbb{R}_{<a}$ iff whenever

$$\sum_{n=1}^{m} q_n \cdot x_n \in \mathbb{R}_{<a},$$

where $m \in \mathbb{N}$, $m \geq 1$, and $q_n \in \mathbb{Q}$ and $x_n \in X$ for all n, $1 \leq n \leq m$, then $q_1 = \ldots = q_m = 0$. In other words, $X \subset \mathbb{R}^*_a$ is linearly independent over $\mathbb{R}_{<a}$ iff

$$\mathrm{span}(X) \cap \mathbb{R}_{<a} = \{0\}.$$

We call $X \subset \mathbb{R}^*_a$ *maximal linearly independent over* $\mathbb{R}_{<a}$ iff X is linearly independent over $\mathbb{R}_{<a}$ and no $Y \supsetneq X$, $Y \subset \mathbb{R}^*_a$ is still linearly independent over $\mathbb{R}_{<a}$. In particular, $X \subset \mathbb{R}^*_\emptyset = (\mathbb{R} \cap L) \setminus \{0\}$ is linearly independent over $\mathbb{R}_{<\emptyset} = \{0\}$ iff X is a Hamel basis for $\mathbb{R} \cap L$.

For any $a \in [A]^{<\omega}$, we let b_a denote the $<_a$–least set $X \subset \mathbb{R}^*_a$, $X \in L[a]$, which is maximal linearly independent over $\mathbb{R}_{<a}$. By the above crucial fact, the function $a \mapsto b_a$ is well defined and *exists inside* H. In particular,

$$B = \bigcup\{b_a \colon a \in [A]^{<\omega}\}$$

is an element of H.

We claim that B is a Hamel basis for the reals of H, which will be established by Claims 2.2 and 2.3.

Claim 2.2. $\mathbb{R} \cap H \subset \mathrm{span}(B)$.

Proof of Claim 2.2. Assume not, and let $n < \omega$ be the least size of some $a \in [A]^{<\omega}$ such that $\mathbb{R}^*_a \setminus \mathrm{span}(B) \neq \emptyset$. Pick $x \in \mathbb{R}^*_a \setminus \mathrm{span}(B) \neq \emptyset$, where $\mathrm{Card}(a) = n$.

We must have $n > 0$, as $b_\emptyset$ is a Hamel basis for the reals of L. Then, by the maximality of b_a, while b_a is linearly independent over $\mathbb{R}_{<a}$, $b_a \cup \{x\}$ cannot be linearly independent over $\mathbb{R}_{<a}$. This means that there are $q \in \mathbb{Q}$, $q \neq 0$, $m \in \mathbb{N}$, $m \geq 1$, and $q_n \in \mathbb{Q} \setminus \{0\}$ and $x_n \in b_a$ for all n, $1 \leq n \leq m$, such that

$$z = q \cdot x + \sum_{n=1}^{m} q_n \cdot x_n \in \mathbb{R}_{<a}.$$

By the definition of $\mathbb{R}_{<a}$ and the minimality of n, $z \in \mathrm{span}(\bigcup\{b_c \colon c \subsetneq a\})$, which then clearly implies that $x \in \mathrm{span}(\bigcup\{b_c \colon c \subseteq a\}) \subset \mathrm{span}(B)$.

This is a contradiction! □ (Claim 2.2)

Claim 2.3. B is linearly independent.

Proof of Claim 2.3. Assume not. This means that there are $1 \leq k < \omega$, $a_i \in [A]^{<\omega}$ pairwise different, $m_i \in \mathbb{N}$, $m_i \geq 1$ for $1 \leq i \leq k$, and $q^i_n \in \mathbb{Q} \setminus \{0\}$ and $x^i_n \in b_{a_i}$ for all i and n with $1 \leq i \leq k$ and $1 \leq n \leq m_i$ such that

$$\sum_{n=1}^{m_1} q^1_n \cdot x^1_n + \ldots + \sum_{n=1}^{m_k} q^k_n \cdot x^k_n = 0. \tag{10}$$

By the properties of b_{a_i}, $\sum_{n=1}^{m_i} q^i_n \cdot x^i_n \in \mathbb{R}^*_{a_i}$, so that (10) buys us that there are $z_i \in \mathbb{R}^*_{a_i}$, $z_i \neq 0$, $1 \leq i \leq k$, such that

$$z_1 + \ldots + z_k = 0. \tag{11}$$

There must be some i such that there is no j with $a_j \supsetneq a_i$, which implies that $a_j \cap a_i \subsetneq a_i$ for all $j \neq i$. Let us assume without loss of generality that $a_j \cap a_1 \subsetneq a_1$ for all j, $1 < j \leq k$.

Let $a_1 = \{c_\ell \colon \ell \in I\}$, where $I \in [\omega]^{<\omega}$, and let $a_j \cap a_1 = \{c_\ell \colon \ell \in I_j\}$, where $I_j \subsetneq I$, for $1 < j \leq l$.

In what follows, a *nice name* τ for a real is a name of the form

$$\tau = \bigcup_{n,m<\omega} \{(n,m)^\vee\} \times A_{n,m}, \tag{12}$$

where each $A_{n,m}$ is a maximal antichain of conditions of the forcing in question deciding that $\tau(\check{n}) = \check{m}$.

We have that z_1 is $\mathbb{C}(I)$–generic over L, so that we may pick a nice name $\tau_1 \in L^{\mathbb{C}(I)}$ for z_1 with $(\tau_1)^{g \restriction I} = z_1$. Similarly, for $1 < j \leq k$, z_j is $\mathbb{C}(I_j)$–generic over $L[g \restriction (\omega \setminus I)]$, so that we may pick a nice name $\tau_j \in L[g \restriction (\omega \setminus I)]^{\mathbb{C}(I_j)}$ for z_j with $(\tau_j)^{g \restriction I_j} = z_j$. We may construe each τ_j, $1 < j \leq k$, as a name in $L[g \restriction (\omega \setminus I)]^{\mathbb{C}(I)}$ by replacing each $p \colon I_j \to \mathbb{C}$ in an antichain as in (12) by $p' \colon I \to \mathbb{C}$, where $p'(\ell) = p(\ell)$ for $\ell \in I_j$ and $p'(\ell) = \emptyset$ otherwise. Let $p \in g \restriction I$ be such that

$$p \Vdash^{\mathbb{C}(I)}_{L[g \restriction (\omega \setminus I)]} \tau_1 + \tau_2 + \ldots + \tau_k = 0.$$

We now have that inside $L[g \restriction (\omega \setminus I)]$, there are nice $\mathbb{C}(I)$–names τ'_j, $1 < j \leq k$ (namely, τ_j, $1 < j \leq k$), such that still inside $L[g \restriction (\omega \setminus I)]$

(1) $p \Vdash^{\mathbb{C}(I)} \tau_1 + \tau'_2 + \ldots + \tau'_k = 0$, and
(2) for all j, $1 < j \leq k$ and for all p in one of the antichains of the nice name τ'_j, $\operatorname{supp}(p) \subseteq I_j$.

Both (1) and (2) are arithmetic in real codes for $\tau_1, \tau'_2, \ldots, \tau'_k$, so that by $\tau_1 \in L^{\mathbb{C}(I)}$ and Σ^1_1–absoluteness between L and $L[g \restriction (\omega \setminus I)]$ there are inside L nice $\mathbb{C}(I)$–names τ'_j, $1 < j \leq k$, such that in L, (1) and (2) hold true. But then, writing $z'_j = (\tau'_j)^{g \restriction I}$, we have by (2) that $z'_j \in \mathbb{R}_{I_j}$ for $1 < j \leq k$, and $z_1 + z'_2 + \ldots + z'_k = 0$ by (1). But then $z_1 \in \mathbb{R}^*_I \cap \mathbb{R}_{<I}$, which is absurd. □ (Claim 2.3)

This finishes the proof of Theorem 2.1. □ (Theorem 2.1)

In the light of Lemma 1.1, Theorem 2.1 reproves Feferman's result.

Corollary 2.4 (S. Feferman)**.** *In H, there is a Vitali set.*

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