PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 8, August 2018, Pages 3345–3351
http://dx.doi.org/10.1090/proc/14056
Article electronically published on April 18, 2018

STABILITY OF RIESZ BASES

VITALII MARCHENKO

(Communicated by Michael Hitrik)

Dedicated to the memory of Professor T. Kato on the occasion of the 100th anniversary of his birthday and of the 50th anniversary of his theorem on similarity for sequences of projections

ABSTRACT. The Kato Theorem on similarity for sequences of projections in a Hilbert space is extended to the case when both sequences consist of nonselfadjoint projections. Passing to subspaces, this leads to stability theorems for Riesz bases of subspaces, at least one of which is finite dimensional, and for arbitrary vector Riesz bases. The following is proved as an application. If $\{\phi_n\}_{n=1}^{\infty}$ is a Riesz basis and $|\theta_n| \leq C$ for large n, where the constant C depends only on $\{\phi_n\}_{n=1}^{\infty}$, then $\{\phi_n + \theta_n \phi_{n+1}\}_{n=1}^{\infty}$ also forms a Riesz basis.

1. INTRODUCTION

Riesz bases play an important role in an infinite-dimensional linear systems theory [7, 16, 19–22] and signal processing [5, 13]. Riesz bases also frequently appear as eigenvectors or root vectors (root subspaces) of various nonselfadjoint operators from mathematical physics, e.g. Hill and Dirac operators [8, 9].

The sequence of nontrivial closed subspaces $\{\mathfrak{N}_n\}_{n\in\mathbb{Z}_+}$ of a Hilbert space H is called a Riesz basis of subspaces provided that there exists an isomorphism S (bounded linear operator with bounded inverse) and an orthogonal basis $\{\mathfrak{M}_n\}_{n\in\mathbb{Z}_+}$ of H such that

$$\mathfrak{N}_n = S\mathfrak{M}_n$$

for $n \in \mathbb{Z}_+$. Hereinafter $\mathbb{Z}_+$ denotes the set of nonnegative integers. Recall also that the sequence $\{\phi_n\}_{n\in\mathbb{Z}_+} \subset H$ is a (vector) Riesz basis provided that $\phi_n = Se_n$ for $n \in \mathbb{Z}_+$, where S is an isomorphism and $\{e_n\}_{n\in\mathbb{Z}_+}$ is an orthonormal basis of H. Note that there are other equivalent definitions of Riesz basis, see, e.g., [5, 10, 14].

The study of stability of bases was initiated by R. E. A. C. Paley and N. Wiener in 1934. The celebrated Paley–Wiener Theorem states that if the sequence $\{\phi_n\}_{n\in\mathbb{Z}_+}$ is close to some orthonormal basis, then $\{\phi_n\}_{n\in\mathbb{Z}_+}$ forms a Riesz basis; see [18].

Fifty years ago T. Kato obtained the following theorem.

Theorem 1.1 ([12]). *Let $\{P_n\}_{n\in\mathbb{Z}_+}$ be a sequence of nonzero selfadjoint projections on H such that $\sum_{n=0}^{\infty} P_n = I$, $P_nP_m = \delta_n^m P_n$, $n, m \in \mathbb{Z}_+$, and $\{J_n\}_{n\in\mathbb{Z}_+}$ be a sequence*

Received by the editors March 27, 2017.
2010 *Mathematics Subject Classification.* Primary 46B15, 47A46.
This research was partially supported by the N. I. Akhiezer Foundation.

of nonzero projections on H satisfying $J_n J_m = \delta_n^m J_n$, $n, m \in \mathbb{Z}_+$. Suppose that

$$\dim P_0 = \dim J_0 < \infty, \tag{1.1}$$

$$\sum_{n=1}^{\infty} \|P_n(J_n - P_n)x\|^2 \le c^2\|x\|^2 \quad \textit{for all } x \in H, \tag{1.2}$$

where c is a constant such that $0 \le c < 1$. Then $\{J_n\}_{n\in\mathbb{Z}_+}$ is similar to $\{P_n\}_{n\in\mathbb{Z}_+}$, i.e., there exists an isomorphism S on H such that $J_n = SP_nS^{-1}$, $n \in \mathbb{Z}_+$.

Using the correspondence between bases and sequences of projections (see Proposition 2.1 in [14]), Theorem 1.1 can be reformulated as a stability theorem for Riesz bases of subspaces in H when the given basis $\{\mathfrak{M}_n = P_nH\}_{n\in\mathbb{Z}_+}$ is orthogonal. The proof of the Kato Theorem is very elegant, constructive, and relies on stability theorems for Fredholm operators; see [11, 12]. The desired isomorphism S, which maps J_nH to P_nH for $n \in \mathbb{Z}_+$, is given in the following explicit form:

$$S = \sum_{n=0}^{\infty} P_n J_n.$$

The theorem of Kato was applied to the analysis of spectral expansions of perturbations of various unbounded selfadjoint operators on Hilbert spaces; see the papers of Clark [6] on perturbations of ordinary differential operators, Adduci and Mityagin on perturbations of selfadjoint operators with discrete spectrum [1, 2], Mityagin and Siegl on the study of singular perturbations of the harmonic oscillator type operators [17], and Chapter 5, §4, of the monograph of Kato devoted to the study of perturbations of selfadjoint operators [11]. Actually, all these studies are aimed at proving the fulfillment of conditions of Theorem 1.1 for some classes of perturbations of selfadjoint operators.

The purpose of this paper is the extension of Theorem 1.1 to the situation when the sequence $\{P_n\}_{n\in\mathbb{Z}_+}$ is nonselfadjoint and $\{P_nH\}_{n\in\mathbb{Z}_+}$ forms a Riesz basis.

2. The extension of the Kato Theorem

Let $\{\mathfrak{M}_n\}_{n\in\mathbb{Z}_+}$ be a complete sequence of closed subspaces in a Hilbert space H. Throughout the paper we will say that $\{\mathfrak{M}_n\}_{n\in\mathbb{Z}_+}$ is a Riesz basis with constant M provided that there is a constant $M \ge 1$ such that

$$\left\|\sum_{i=0}^{n} \delta_i y_i\right\| \le M \left\|\sum_{i=0}^{n} y_i\right\|$$

for any $n \in \mathbb{Z}_+$, $y_i \in \mathfrak{M}_i$, and $\delta_i \in \{0, 1\}$. The latter is an equivalent definition of the Riesz basis; see, e.g., Theorems 3.1 and 3.3 in [14]. Hence such a constant M exists for every Riesz basis. Moreover, it depends on the mutual arrangement of subspaces $\mathfrak{M}_n$, e.g., for an arbitrary orthogonal basis one has $M = 1$.

The extension of Theorem 1.1 is formulated as follows.

Theorem 2.1. *Let $\{\mathfrak{M}_n\}_{n\in\mathbb{Z}_+}$ be a Riesz basis of H with constant M and projections $\{P_n\}_{n\in\mathbb{Z}_+}$, and let $\{J_n\}_{n\in\mathbb{Z}_+}$ be a sequence of nonzero projections on H such that $J_nJ_m = \delta_n^m J_n$ $n, m \in \mathbb{Z}_+$. Suppose that condition* (1.1) *holds and for all $x \in H$ we have*

$$\sum_{n=1}^{\infty} \|P_n(J_n - P_n)x\|^2 \le \varsigma^2\|x\|^2, \tag{2.1}$$

where $\varsigma \in [0, 1/M)$*. Then* $\{J_n H\}_{n\in\mathbb{Z}_+}$ *also forms a Riesz basis of* H*.*

Proof. First we note that all Riesz bases of subspaces (with preassigned dimensions) in H are mutually isomorphic and corresponding sequences of projections are mutually similar; see, e.g., Theorem 3.3 in [14]. Hence, if $\{Q_n H\}_{n\in\mathbb{Z}_+}$ is an arbitrary orthogonal basis of subspaces and T is an isomorphism, then $\{TQ_nT^{-1}H\}_{n\in\mathbb{Z}_+}$ forms a Riesz basis. Moreover, $\{TQ_nT^{-1}H\}_{n\in\mathbb{Z}_+}$ is a Riesz basis with constant $\|T\|\,\|T^{-1}\|$, since

$$\left\|\sum_{n=0}^{\infty} \delta_n T Q_n T^{-1} x\right\| \leq \|T\| \left(\sum_{n=0}^{\infty} \delta_n \left\|Q_n T^{-1} x\right\|^2\right)^{1/2} \leq \|T\|\,\|T^{-1}\|\,\|x\|\,,$$

where $x \in H$ and $\delta_n \in \{0, 1\}$, $n \in \mathbb{Z}_+$, are chosen arbitrarily.

It follows that for our Riesz basis $\{\mathfrak{M}_n = P_n H\}_{n\in\mathbb{Z}_+}$ with constant M there exist an orthogonal basis $\left\{\overline{P}_n H\right\}_{n\in\mathbb{Z}_+}$ and an isomorphism S such that

$$\left\{P_n H = S\overline{P}_n S^{-1} H\right\}_{n\in\mathbb{Z}_+}, \quad \|S\|\,\|S^{-1}\| \leq M. \tag{2.2}$$

In the following steps of the proof we use this orthogonal basis $\left\{\overline{P}_n H\right\}_{n\in\mathbb{Z}_+}$ and isomorphism S.

So $P_n = S\overline{P}_n S^{-1}$, $n \in \mathbb{Z}_+$, where S is an isomorphism such that (2.2) holds. Then condition (2.1) turns into

$$\sum_{n=1}^{\infty} \left\|S\overline{P}_n S^{-1}(J_n - S\overline{P}_n S^{-1})x\right\|^2 \leq \varsigma^2 \|x\|^2. \tag{2.3}$$

Since

$$\|Sz\| \geq \frac{\|z\|}{\|S^{-1}\|}$$

for any $z \in H$, condition (2.3) yields that

$$\sum_{n=1}^{\infty} \left\|\overline{P}_n S^{-1} J_n x - \overline{P}_n S^{-1} x\right\|^2 \leq \varsigma^2 \left\|S^{-1}\right\|^2 \|x\|^2. \tag{2.4}$$

Next we define a new sequence of projections as follows:

$$\overline{J}_n = S^{-1} J_n S, \quad n \in \mathbb{Z}_+.$$

Clearly, $\overline{J}_n \overline{J}_m = S^{-1} J_n S S^{-1} J_m S = \delta_n^m \overline{J}_n$, $n, m \in \mathbb{Z}_+$. Since S is an isomorphism, by (1.1) we have

$$\dim \overline{J}_0 = \dim J_0 = \dim P_0 = \dim \overline{P}_0.$$

Further, for any $x \in H$ we consider $y = S^{-1}x$. Then condition (2.4) implies the following: for any $y \in H$ we have

$$\sum_{n=1}^{\infty} \left\|\overline{P}_n \left(\overline{J}_n - \overline{P}_n\right) y\right\|^2 \leq \varsigma^2 \|S^{-1}\|^2 \|S\|^2 \|y\|^2 = c^2 \|y\|^2.$$

Taking into account (2.2) we have that $c = \varsigma \|S^{-1}\| \|S\| \leq \varsigma M \in [0, 1)$. The next step is to apply Theorem 1.1 and conclude that there exists an isomorphism $\overline{S}$ such that $\overline{J}_n = \overline{S}\overline{P}_n \overline{S}^{-1}$, $n \in \mathbb{Z}_+$. Consequently $\overline{S}\overline{P}_n \overline{S}^{-1} = S^{-1} J_n S$, $n \in \mathbb{Z}_+$, and, finally,

$$J_n = S\overline{S}\overline{P}_n \overline{S}^{-1} S^{-1} = S\overline{S} S^{-1} P_n S \overline{S}^{-1} S^{-1},\ n \in \mathbb{Z}_+.$$

The application of Theorem 3.3 from [14] completes the proof. $\square$

3. Stability of vector Riesz bases

Let $\{\phi_n\}_{n\in\mathbb{Z}_+}$ be a Riesz basis of H. Then $\{\phi_n\}_{n\in\mathbb{Z}_+}$ will be called a Riesz basis with constant M provided that the sequence $\{Lin\{\phi_n\}\}_{n\in\mathbb{Z}_+}$ of one-dimensional subspaces forms a Riesz basis of subspaces in H with constant M.

Using the general form of one-dimensional projection in a Hilbert space, we deduce from Theorem 2.1 the following result.

Theorem 3.1. *Let $\{\phi_n\}_{n\in\mathbb{Z}_+}$ be a Riesz basis of H with constant M and corresponding biorthogonal sequence $\{\phi_n^*\}_{n\in\mathbb{Z}_+}$. Assume that $\big(\{\psi_n\}_{n\in\mathbb{Z}_+}, \{\psi_n^*\}_{n\in\mathbb{Z}_+}\big)$ is a biorthogonal system in H satisfying*

$$0 < \inf_n \|\psi_n\| \le \sup_n \|\psi_n\| < \infty. \tag{3.1}$$

If for all $x \in H$ we have

$$\sum_{n=1}^{\infty} |\langle x, \psi_n^*\rangle\langle \psi_n, \phi_n^*\rangle - \langle x, \phi_n^*\rangle|^2 \|\phi_n\|^2 \le \varsigma^2 \|x\|^2,$$

where $\varsigma \in [0, 1/M)$, then $\{\psi_n\}_{n\in\mathbb{Z}_+}$ is a Riesz basis of H.

Further we apply Theorem 3.1 to the construction of Riesz bases in H. Let $\{e_n\}_{n\in\mathbb{Z}_+}$ be an orthonormal basis of H. Then $e_n + \frac{1}{n+1} e_{n+1}$, $n \in \mathbb{Z}_+$, forms a Riesz basis of H ([4], Example 1.2); see also [3]. This can also be deduced from Theorem 1.1. Our next aim is to show that similar facts take place if we consider Riesz bases instead of orthonormal ones.

Let $\{\phi_n\}_{n\in\mathbb{Z}_+}$ be a Riesz basis with corresponding biorthogonal sequence $\{\phi_n^*\}_{n\in\mathbb{Z}_+}$. Consider the following systems:

$$\psi_n^* = \phi_n^* + \overline{\theta}_n \phi_{n+1}^*,\ n \in \mathbb{Z}_+,$$

$$\psi_n = \phi_n - \theta_{n-1}\phi_{n-1} + \theta_{n-1}\theta_{n-2}\phi_{n-2} + \cdots + (-1)^n \prod_{k=0}^{n-1} \theta_k \phi_0,\ n \in \mathbb{Z}_+,$$

where $\{\theta_n\}_{n\in\mathbb{Z}_+}$ is a sequence of complex numbers and $\theta_{-j} = 0$ for $j \in \mathbb{N}$.

Proposition 3.2. *Let $\{\phi_n\}_{n\in\mathbb{Z}_+}$ be a Riesz basis of H with constant M and corresponding biorthogonal sequence $\{\phi_n^*\}_{n\in\mathbb{Z}_+}$. Suppose that there exists $N \in \mathbb{N}$ such that $|\theta_n| \le C$ for $n \ge N$, where*

$$C < \frac{\inf\limits_{n\ge N+1} \|\phi_n\|}{2M^2 \sup\limits_{n\ge N} \|\phi_n\|}. \tag{3.2}$$

Then $\{\psi_n\}_{n\in\mathbb{Z}_+}$ and $\{\psi_n^*\}_{n\in\mathbb{Z}_+}$ are Riesz bases of H.

Proof. First we prove the proposition for the case $N = 1$.

Direct computations show that $\big(\{\psi_n\}_{n\in\mathbb{Z}_+}, \{\psi_n^*\}_{n\in\mathbb{Z}_+}\big)$ is a biorthogonal system in H satisfying (3.1). Clearly, $\langle \psi_n, \phi_n^*\rangle = 1$ and $\langle x, \psi_n^*\rangle = \langle x, \phi_n^*\rangle + \theta_n \langle x, \phi_{n+1}^*\rangle$ for each $n \in \mathbb{Z}_+$. Consequently, for all $x \in H$,

$$\sum_{n=1}^{\infty} |\langle x, \psi_n^*\rangle\langle \psi_n, \phi_n^*\rangle - \langle x, \phi_n^*\rangle|^2 \|\phi_n\|^2 = \sum_{n=1}^{\infty} |\theta_n|^2 |\langle x, \phi_{n+1}^*\rangle|^2 \|\phi_n\|^2 \tag{3.3}$$

$$\le C^2 \sup_{n\in\mathbb{N}} \|\phi_n\|^2 \sum_{n=2}^{\infty} |\langle x, \phi_n^*\rangle|^2. \tag{3.4}$$

Further, since $\{\phi_n\}_{n\in\mathbb{Z}_+}$ is a Riesz basis with constant M, we have

$$\sum_{n=0}^{\infty} |\langle x, \phi_n^*\rangle|^2 \|\phi_n\|^2 \leq 4M^2 \|x\|^2$$

for all $x \in H$; see, e.g., [14], Remark 4.7. Hence

$$\sum_{n=2}^{\infty} |\langle x, \phi_n^*\rangle|^2 \leq \frac{4M^2}{\inf\limits_{n\geq 2} \|\phi_n\|^2} \|x\|^2 \tag{3.5}$$

for all $x \in H$. Using (3.5) we can continue estimate (3.3) and obtain

$$\sum_{n=1}^{\infty} |\langle x, \psi_n^*\rangle\langle \psi_n, \phi_n^*\rangle - \langle x, \phi_n^*\rangle|^2 \|\phi_n\|^2 \leq 4M^2C^2 \frac{\sup\limits_{n\in\mathbb{N}} \|\phi_n\|^2}{\inf\limits_{n\geq 2} \|\phi_n\|^2} \|x\|^2 = \varsigma^2 \|x\|^2,$$

where $\varsigma = 2MC \frac{\sup_{n\in\mathbb{N}} \|\phi_n\|}{\inf_{n\geq 2} \|\phi_n\|}$. By virtue of (3.2) we have that $\varsigma < 1/M$. Applying Theorem 3.1 yields that $\{\psi_n\}_{n\in\mathbb{Z}_+}$ is a Riesz basis of H. Hence a biorthogonal sequence $\{\psi_n^*\}_{n\in\mathbb{Z}_+}$ also forms a Riesz basis.

The proof of the proposition in the case when $N \geq 2$ relies on similar arguments. We consider projections P_0 and J_0 of dimension N and use Theorem 2.1 for this case. □

Note that every Riesz basis is bounded from below and from above, so $\{\phi_n\}_{n\in\mathbb{Z}_+}$ satisfy (3.1) and condition (3.2) always makes sense. For normalized bases we have the following direct corollary from Proposition 3.2.

Corollary 3.3. *Let $\{\phi_n\}_{n\in\mathbb{Z}_+}$ be a normalized Riesz basis of H with constant M and corresponding biorthogonal sequence $\{\phi_n^*\}_{n\in\mathbb{Z}_+}$. If there exists $N \in \mathbb{N}$ such that $|\theta_n| \leq C$ for $n \geq N$, where*

$$C < \frac{1}{2M^2}, \tag{3.6}$$

then $\{\psi_n\}_{n\in\mathbb{Z}_+}$ and $\{\psi_n^*\}_{n\in\mathbb{Z}_+}$ are Riesz bases of H.

For the case when $\{\phi_n\}_{n\in\mathbb{Z}_+}$ is orthonormal we have $M = 1$, and Corollary 3.3 can be improved, since the condition $C < 1/2$ is too strong. To improve the result for this case, using similar arguments we deduce from Theorem 1.1 the following result and weaken condition (3.6).

Proposition 3.4. *Let $\{\phi_n\}_{n\in\mathbb{Z}_+}$ be an orthonormal basis of H. If there exists $N \in \mathbb{N}$ and $C < 1$ such that $|\theta_n| \leq C$ for $n \geq N$, then $\{\psi_n\}_{n\in\mathbb{Z}_+}$ and $\{\psi_n^*\}_{n\in\mathbb{Z}_+}$ are Riesz bases of H.*

4. Remarks

1) Proposition 3.4 can be deduced from the Paley–Wiener Theorem and is sharp in the following sense. Let $\{e_n\}_{n\in\mathbb{Z}_+}$ be an orthonormal basis of H and assume that there exists $N \in \mathbb{N}$ such that $|\theta_n| \geq 1$, $n \geq N$. Then $\{\varphi_n = e_n + \theta_n e_{n+1}\}_{n\in\mathbb{Z}_+}$ is complete and a minimal sequence in H but not uniformly minimal. Hence $\{\varphi_n\}_{n\in\mathbb{Z}_+}$ does not form a Schauder basis.

2) Theorem 2.1 is a much more subtle result than the Bari–Marcus Theorem (see Theorem 5.2 and Remark 5.3 in [10, Chapter VI]). Indeed, in Theorem 2.1 we

do not need to require that $\{J_nH\}_{n\in\mathbb{Z}_+}$ is complete and that for each $k \in \mathbb{Z}_+$ the minimal angle between J_kH and $\overline{Lin}\{J_nH\}_{n\neq k}$ is positive.

3) Point 2 is also confirmed by the following observation. Consider the case when $\dim J_n = \dim P_n = 1$ for all n. Then [10, Chapter VI, §5] the Bari–Marcus Theorem reduces to the Bari Theorem (Theorem 2.3 in [10, Chapter VI, §2]; see also [3]). Let $\{\phi_n\}_{n\in\mathbb{Z}_+}$ be a Riesz basis of H. Using the Bari Theorem we can obtain that $\{\phi_n + \theta_n\phi_{n+1}\}_{n\in\mathbb{Z}_+}$ is also a Riesz basis, provided that $\{\theta_n\}_{n\in\mathbb{Z}_+} \in \ell_2$. The last is a much weaker statement than Proposition 3.2. From this point of view, it can be said that the condition of quadratic closeness of subspaces turns out to be much more stronger than condition (2.1).

4) In applications all subspaces are usually finite dimensional. However, Theorem 2.1 does not work in the case when all subspaces J_nH have infinite dimension, in contrast to the Bari–Marcus Theorem.

5) For the case when $P_0 = P_0^*$ and under a stronger condition on ς in (2.1), namely $\varsigma \in [0, 1/(2M))$, Theorem 2.1 was previously obtained by the author in [14]; see Theorem 7.5. But for the case $M = 1$, i.e., when $\{\mathfrak{M}_n\}_{n\in\mathbb{Z}_+}$ is orthogonal, this result did not coincide with the Kato Theorem, which was unnatural. Moreover, it was deduced as a consequence of a more general stability result of Kato type for bases of subspaces in Banach spaces with the so-called Schauder–Orlicz decompositions; see Theorem 7.2 in [14]. In the present paper we essentially weaken the condition on ς, drop the condition $P_0 = P_0^*$, and present a simple direct proof of the main result, based on the Kato Theorem. Thereby we eliminate an unnatural gap between stability result for Riesz bases and the Kato Theorem in such a way that condition (2.1) approaches condition (1.2) when $M \to 1$. Thus, Theorem 2.1 approaches the Kato Theorem when $M \to 1$, i.e., when $\{\mathfrak{M}_n\}_{n\in\mathbb{Z}_+}$ becomes more and more close to the orthogonal basis, which corresponds to the nature of stability.

6) The sequence of nontrivial closed subspaces $\{\mathfrak{M}_n\}_{n\in\mathbb{Z}_+}$ of a Banach space X is called by an unconditional basis of subspaces provided that each $x \in X$ has a unique unconditionally convergent expansion

$$x = \sum_{n=0}^{\infty} x_n,$$

where $x_n \in \mathfrak{M}_n$. The results on the stability of unconditional bases of subspaces in ℓ_p spaces, similar to Theorem 2.1, were obtained in [15]. Moreover, it was shown that the properties of unconditional bases of subspaces in ℓ_p spaces depend on the best constants in the Khintchine inequality; see [14, 15].

7) It is known that every Riesz basis of subspaces is an unconditional basis of subspaces and vice versa; see Theorem 5.1 in [10, Chapter VI] or Theorem 3.3 in [14]. Consequently, an unconditional basis of subspaces (with preassigned dimensions) in a Hilbert space is unique, up to an isomorphism. However, are there other Banach spaces with a unique, up to an isomorphism, unconditional basis of subspaces?

References

[1] James Adduci and Boris Mityagin, *Eigensystem of an L^2-perturbed harmonic oscillator is an unconditional basis*, Cent. Eur. J. Math. **10** (2012), no. 2, 569–589. MR2886559

[2] James Adduci and Boris Mityagin, *Root system of a perturbation of a selfadjoint operator with discrete spectrum*, Integral Equations Operator Theory **73** (2012), no. 2, 153–175. MR2921063

[3] N. K. Bari, *Biorthogonal systems and bases in Hilbert space* (Russian), Moskov. Gos. Univ. Učenye Zapiski Matematika **148(4)** (1951), 69–107. MR0050171
[4] Peter G. Cazassa and Ole Christensen, *Perturbation of operators and applications to frame theory*, dedicated to the memory of Richard J. Duffin. J. Fourier Anal. Appl. **3** (1997), no. 5, 543–557. MR1491933
[5] Ole Christensen, *An introduction to frames and Riesz bases*, 2nd ed., Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, [Cham], 2016. MR3495345
[6] Colin Clark, *On relatively bounded perturbations of ordinary differential operators*, Pacific J. Math. **25** (1968), 59–70. MR0226447
[7] Ruth F. Curtain and Hans Zwart, *An introduction to infinite-dimensional linear systems theory*, Texts in Applied Mathematics, vol. 21, Springer-Verlag, New York, 1995. MR1351248
[8] Plamen Djakov and Boris Mityagin, *Criteria for existence of Riesz bases consisting of root functions of Hill and 1D Dirac operators*, J. Funct. Anal. **263** (2012), no. 8, 2300–2332. MR2964684
[9] Plamen Djakov and Boris Mityagin, *Riesz bases consisting of root functions of 1D Dirac operators*, Proc. Amer. Math. Soc. **141** (2013), no. 4, 1361–1375. MR3008883
[10] I. C. Gohberg and M. G. Kreĭn, *Introduction to the theory of linear nonselfadjoint operators*, Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. 18, American Mathematical Society, Providence, R.I., 1969. MR0246142
[11] Tosio Kato, *Perturbation theory for linear operators*, reprint of the 1980 edition. Classics in Mathematics, Springer-Verlag, Berlin, 1995. MR1335452
[12] Tosio Kato, *Similarity for sequences of projections*, Bull. Amer. Math. Soc. **73** (1967), 904–905. MR0216324
[13] Nir Lev, *Riesz bases of exponentials on multiband spectra*, Proc. Amer. Math. Soc. **140** (2012), no. 9, 3127–3132. MR2917085
[14] Vitalii Marchenko, *Isomorphic Schauder decompositions in certain Banach spaces*, Cent. Eur. J. Math. **12** (2014), no. 11, 1714–1732. MR3225827
[15] Vitalii Marchenko, *Stability of unconditional Schauder decompositions in ℓ_p spaces*, Bull. Aust. Math. Soc. **92** (2015), no. 3, 444–456. MR3415621
[16] A. I. Miloslavskiĭ, *Stability of certain classes of evolution equations* (Russian), Sibirsk. Mat. Zh. **26** (1985), no. 5, 118–132, 206. MR808708
[17] Boris Mityagin and Petr Siegl, *Root system of singular perturbations of the harmonic oscillator type operators*, Lett. Math. Phys. **106** (2016), no. 2, 147–167. MR3451535
[18] Raymond E. A. C. Paley and Norbert Wiener, *Fourier transforms in the complex domain*, American Mathematical Society Colloquium Publications, vol. 19, reprint of the 1934 original. American Mathematical Society, Providence, RI, 1987. MR1451142
[19] Rabah Rabah, Grigory M. Sklyar, and Alexander V. Rezounenko, *Generalized Riesz basis property in the analysis of neutral type systems* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **337** (2003), no. 1, 19–24. MR1993989
[20] R. Rabah, G. M. Sklyar, and A. V. Rezounenko, *Stability analysis of neutral type systems in Hilbert space*, J. Differential Equations **214** (2005), no. 2, 391–428. MR2145255
[21] Rabah Rabah and Grigory M. Sklyar, *The analysis of exact controllability of neutral-type systems by the moment problem approach*, SIAM J. Control Optim. **46** (2007), no. 6, 2148–2181. MR2369313
[22] G. M. Sklyar and P. Polak, *On asymptotic estimation of a discrete type C_0-semigroups on dense sets: application to neutral type systems*, Appl. Math. Optim. **75** (2017), no. 2, 175–192. MR3621839

Mathematical Division of B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine
Email address: `v.marchenko@ilt.kharkov.ua`