

LOCAL CHARACTERIZATIONS FOR THE MATRIX MONOTONICITY AND CONVEXITY OF FIXED ORDER

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ABSTRACT. We establish local characterizations of matrix monotonicity and convexity of fixed order by giving integral representations connecting the Loewner and Kraus matrices, previously known to characterize these properties, to respective Hankel matrices. Our results are new already in the general case of matrix convexity, and our approach significantly simplifies the corresponding work on matrix monotonicity. We also obtain an extension of the original characterization for matrix convexity by Kraus and tighten the relationship between monotonicity and convexity.

1. INTRODUCTION

For an open interval (a, b) , we say that $f : (a, b) \rightarrow \mathbb{R}$ is matrix monotone (increasing) of order n (or n -monotone) if for any $n \times n$ Hermitian matrices A, B with spectra in (a, b) and $A \leq B$ we have $f(A) \leq f(B)$.¹ Analogously, $f : (a, b) \rightarrow \mathbb{R}$ is matrix convex of order n (or n -convex) if for any $n \times n$ Hermitian matrices A, B with spectra in (a, b) and $\lambda \in [0, 1]$ we have $f(\lambda A + (1 - \lambda)B) \leq \lambda f(A) + (1 - \lambda)f(B)$.

Ever since Charles Loewner (then known as Karl Löwner) introduced matrix monotone functions in 1934 [12], this class has been characterized in various ways. See for example [2, 8] for a survey and recent progress. The famous theorem established in Loewner's paper states that a function that is matrix monotone of all orders on an interval extends to an upper half-plane as a Pick-Nevanlinna function: an analytic function with non-negative imaginary part. Loewner's proof of this jewel is based on an important characterization in terms of divided differences, here denoted by $[\cdot, \cdot, \dots, \cdot]_f$. Recall that divided differences are defined recursively by $[\lambda]_f = f(\lambda)$ and for distinct $\lambda_1, \lambda_2, \dots, \lambda_n \in (a, b)$,

$$[\lambda_1, \lambda_2, \dots, \lambda_n]_f = \frac{[\lambda_1, \lambda_2, \dots, \lambda_{n-1}]_f - [\lambda_2, \lambda_3, \dots, \lambda_n]_f}{\lambda_1 - \lambda_n}.$$

If $f \in C^{n-1}(a, b)$, the divided difference has continuous extension to all tuples of not necessarily distinct n numbers on the interval [4].

Theorem 1 (Loewner). *A function $f : (a, b) \rightarrow \mathbb{R}$ is n -monotone (for $n \geq 2$) if and only if $f \in C^1(a, b)$ and the Loewner matrix*

$$(2) \quad L = ([\lambda_i, \lambda_j]_f)_{1 \leq i, j \leq n}$$

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¹As usual, the space of Hermitian matrices is equipped with the Loewner order, i.e., the partial order induced by the convex cone of positive semi-definite matrices.

is positive² for any tuple of numbers $(\lambda_i)_{i=1}^n$ on the same interval.

Similarly Kraus, a student of Loewner, introduced the matrix convexity in [11] and established a similar characterization:

Theorem 3 (Kraus). *A function $f : (a, b) \rightarrow \mathbb{R}$ is n -convex (for $n \geq 2$) if and only if $f \in C^2(a, b)$ and the Kraus matrix*

$$(4) \quad Kr = ([\lambda_i, \lambda_j, \lambda_0]_f)_{1 \leq i, j \leq n}$$

is positive for any tuple of numbers $(\lambda_i)_{i=1}^n \in (a, b)^n$ and $\lambda_0 \in (\lambda_i)_{i=1}^n$.

A different, local characterization for monotonicity was given by another student of Loewner, Dobsch, in [5]:

Theorem 5 (Dobsch, Donoghue). *A C^{2n-1} function $f : (a, b) \rightarrow \mathbb{R}$ is n -monotone if and only if the Hankel matrix*

$$(6) \quad M(t) = \left(\frac{f^{(i+j-1)}(t)}{(i+j-1)!} \right)_{1 \leq i, j \leq n}$$

is positive for any $t \in (a, b)$.

By employing standard regularization techniques, one could further extend this to merely C^{2n-3} functions with convex derivative of order $(2n - 3)$, a class of functions for which the property makes sense for almost every t , to obtain the complete local characterization of the matrix monotonicity of fixed order. The result has a striking consequence: n -monotonicity is a local property, meaning that if function has it in two overlapping intervals, then it has it for their union. This property is actually used in the proof, and although it was noted by Loewner to be easy ([12, p. 212, Theorem 5.6]), no rigorous proof was given until 40 years later in the monograph of Donoghue [6], and the proof is rather long when $n > 2$.

The main results of this paper establish novel integral representations connecting Hankel matrices to the Loewner and Kraus matrices. These identities give rise to a new simple proof for Theorem 5 and, more importantly, settle the conjecture in [9] (see also [10]) by establishing similar local characterization for the matrix convex functions.

Theorem 7. *A C^{2n} function $f : (a, b) \rightarrow \mathbb{R}$ is n -convex if and only if the Hankel matrix*

$$(8) \quad K(t) = \left(\frac{f^{(i+j)}(t)}{(i+j)!} \right)_{1 \leq i, j \leq n}$$

is positive for any $t \in (a, b)$.

Again, with regularizations we may extend this to give a complete local description of matrix convexity of fixed order, which as an immediate corollary gives the expected local property theorem for convexity.

Corollary 9. *For any positive integer n , n -convexity is a local property.*

As another byproduct, we obtain a slight improvement to Theorem 3, where λ_0 may now vary freely. This also implies through divided differences a rather direct connection between matrix monotonicity and convexity.

²Here and in the following, positivity of matrix means that it is positive semi-definite.

2. MATRIX MONOTONE FUNCTIONS

2.1. Integral representation. In this section we construct the integral representations for the Loewner matrices alluded to in the introduction.

Let $n \geq 2$, let (a, b) be an interval, and let $\Lambda = (\lambda_i)_{i=1}^n \in (a, b)^n$ be an arbitrary sequence of distinct points in (a, b) .

In the following the Loewner and respective Hankel matrices, introduced in the introduction in (2) and (6), for sufficiently smooth $f : (a, b) \rightarrow \mathbb{R}$ and $\lambda_0 \in (a, b)$ are denoted by $L(\Lambda, f)$ and $M_n(t, f)$ respectively.

Recall that as one easily verifies with Cauchy’s integral formula and induction, the divided differences can be written as

$$(10) \quad [\lambda_1, \lambda_2, \dots, \lambda_n]_f = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - \lambda_1) \cdots (z - \lambda_n)} dz,$$

for analytic f and suitable closed curve γ .³

Divided differences also admit a natural generalization for the mean value theorem [4]. Namely, for an open interval (a, b) , $f \in C^{n-1}(a, b)$, and any tuple of (not necessarily distinct) real numbers $\Lambda = (\lambda_i)_{i=1}^n \in (a, b)^n$ we have

$$(11) \quad [\lambda_1, \lambda_2, \dots, \lambda_n]_f = \frac{f^{(n-1)}(\xi)}{(n - 1)!}$$

for some $\xi \in [\min(\Lambda), \max(\Lambda)]$.

We shall also need the very basic properties of regularizations. Namely, for even, non-negative, and smooth function ϕ supported on $[-1, 1]$ and with integral 1 and integrable $f : (a, b) \rightarrow \mathbb{R}$, regularization (or ε -regularization, to be precise) of f , denoted by $f_\varepsilon : (a + \varepsilon, b - \varepsilon) \rightarrow \mathbb{R}$, is the convolution

$$f_\varepsilon(x) = \int_{-\infty}^{\infty} f(x - \varepsilon y)\phi(y)dy.$$

This is a smooth function, and for any continuity point $x \in (a, b)$ of f we clearly have $\lim_{\varepsilon \rightarrow 0} f_\varepsilon(x) = f(x)$. Note that regularizations of matrix monotone (convex) functions are obviously matrix monotone (convex) functions on a slightly smaller interval.

Define the functions g_j for $1 \leq j \leq n$ by

$$(12) \quad g_{j,\Lambda}(t, y) = \prod_{k \neq j} (1 + y(t - \lambda_k)).$$

Define also the matrix $C(t) := C(t, \Lambda)$ by setting $C_{i,j}$ to be the coefficient of y^{i-1} in the polynomial $g_j(t, y)$; i.e., we have

$$(13) \quad g_j(t, y) = C_{1,j}(t) + C_{2,j}(t)y + \cdots + C_{n,j}(t)y^{n-1}.$$

Define polynomial p_Λ with $p_\Lambda(t) := \prod_{i=1}^n (t - \lambda_i)$. Also for any $z \in \mathbb{C}$ define function h_z by setting $h_z(x) = (z - x)^{-1}$.

Lemma 14. For $\Lambda = (\lambda_i)_{i=1}^n$ as before, $t \in \mathbb{R}$, and $z \in \mathbb{C}$ distinct from t , we have

$$C^T(t, \Lambda) M_n(t, h_z) C(t, \Lambda) = L(\Lambda, h_z) \frac{p_\Lambda(z)^2}{(z - t)^{2n}}.$$

³For our purposes, it is enough to consider f analytic in an open half-plane and γ a circle in this half-plane enclosing the points $\lambda_1, \lambda_2, \dots, \lambda_n$.

Proof. Write $D = C^T(t, \Lambda)M_n(t, h_z)C(t, \Lambda)$. Note that as we have $h_z^{(k)}(t)/k! = (z - t)^{-k-1}$, we may write $M_n(t, h_z) = \frac{1}{(z-t)^2}vv^T$ with $v = (1, \frac{1}{z-t}, \frac{1}{(z-t)^2}, \dots, \frac{1}{(z-t)^{n-1}})^T$. Thus

$$D = \frac{1}{(z - t)^2}(C(t, \Lambda)^T v)(C(t, \Lambda)^T v)^T.$$

One also easily sees that $(C(t, \Lambda)^T v)_i = g_i(t, \frac{1}{z-t})$ so that finally

$$\begin{aligned} D_{i,j} &= \frac{g_i(t, \frac{1}{z-t})g_j(t, \frac{1}{z-t})}{(z - t)^2} = \frac{1}{(z - t)^2} \prod_{k \neq i} \left(1 + \frac{t - \lambda_k}{z - t}\right) \prod_{k \neq j} \left(1 + \frac{t - \lambda_k}{z - t}\right) \\ &= [\lambda_i, \lambda_j]_{h_z} \frac{p_\Lambda(z)^2}{(z - t)^{2n}}. \end{aligned} \quad \square$$

Consider now the function

$$S(z, t) := S_\Lambda(z, t) := -\frac{(z - t)^{2n-2}}{p_\Lambda(z)^2}.$$

As $S(z, t)$ decays as z^{-2} , with the residue theorem we see that for suitable closed curve γ we have

$$0 = \frac{1}{2\pi i} \int_\gamma S(z, t) dz = \sum_{i=1}^n \operatorname{Res}_{z=\lambda_i} S(z, t).$$

Defining now the weight functions $I_i := I_{i,\Lambda}$ for $1 \leq i \leq n$ by

$$I_i(t) = \operatorname{Res}_{z=\lambda_i} S(z, t)$$

and

$$I(t) := I_\Lambda(t) := \sum_{\substack{1 \leq i \leq n \\ \lambda_i < t}} I_i(t),$$

we see by simple computation that the I_i s are polynomials such that $I_i(\lambda_i) = 0$ and I is hence a piecewise polynomial, continuous function supported on $[\min(\Lambda), \max(\Lambda)]$.

Note that with Cauchy’s integral formula we can also write I in the form

$$I(t) = \frac{1}{2\pi i} \int_{t-i\infty}^{t+i\infty} S(z, t) dz$$

whenever $t \notin \Lambda$.

Remark 15. The weight function I and the analogous weight J to be introduced in the convex setting are examples of weights called Peano kernels or B-splines. The properties of these kernels are discussed for example in [3]. To stay self-contained, we give proofs of the crucial properties used in our discussion.

Lemma 16. *For $\Lambda = (\lambda_i)_{i=1}^n$ as before and for $z \in \mathbb{C}$ outside the interval $[\min(\Lambda), \max(\Lambda)]$, we have*

$$(2n - 1) \int_{-\infty}^{\infty} \frac{I(t)}{(z - t)^{2n}} dt = \frac{1}{p_\Lambda(z)^2}.$$

Proof. We simply compute that

$$\begin{aligned}
 (2n - 1) \int_{-\infty}^{\infty} \frac{I(t)}{(z - t)^{2n}} dt &= (2n - 1) \sum_{i=1}^n \int_{\lambda_i}^{\infty} \frac{I_i(t)}{(z - t)^{2n}} dt \\
 &= -(2n - 1) \sum_{i=1}^n \operatorname{Res}_{w=\lambda_i} \int_{\lambda_i}^{\infty} \frac{(w - t)^{2n-2}}{p_{\Lambda}(w)^2(z - t)^{2n}} dt \\
 &= \sum_{i=1}^n \operatorname{Res}_{w=\lambda_i} \frac{(1 - \frac{z-w}{z-\lambda_i})^{2n-1} - 1}{(w - z)p_{\Lambda}(w)^2} \\
 &= - \sum_{i=1}^n \operatorname{Res}_{w=\lambda_i} \frac{1}{(w - z)p_{\Lambda}(w)^2} \\
 &= \operatorname{Res}_{w=z} \frac{1}{(w - z)p_{\Lambda}(w)^2} - \frac{1}{2\pi i} \int_{\gamma} \frac{dw}{(w - z)p_{\Lambda}(w)^2} \\
 &= \frac{1}{p_{\Lambda}(z)^2},
 \end{aligned}$$

where we used the residue theorem for the function $(w \mapsto (w - z)^{-1}p_{\Lambda}(w)^{-2})$. \square

We are then ready to formulate and prove the integral representation of the Loewner matrix.

Theorem 17. *For $f \in C^{2n-1}(a, b)$ and Λ as before, we have*

$$L(\Lambda, f) = (2n - 1) \int_{-\infty}^{\infty} C^T(t, \Lambda)M_n(t, f)C(t, \Lambda)I_{\Lambda}(t)dt.$$

Proof. For entire f , by Lemmas 14, 16, Fubini, and (10) we have

$$\begin{aligned}
 (2n - 1) \int_{-\infty}^{\infty} C^T(t)M_n(t, f)C(t)I(t)dt &= \frac{1}{2\pi i} \int_{\gamma} \left((2n - 1) \int_{-\infty}^{\infty} C^T(t)M_n(t, h_z)C(t)I(t)dt \right) f(z)dz \\
 &= \frac{1}{2\pi i} \int_{\gamma} L(\Lambda, h_z) \left((2n - 1) \int_{-\infty}^{\infty} \frac{p_{\Lambda}(z)^2}{(z - t)^{2n}} I(t)dt \right) f(z)dz \\
 &= \frac{1}{2\pi i} \int_{\gamma} L(\Lambda, h_z)f(z)dz \\
 &= L(\Lambda, f).
 \end{aligned}$$

The general case now follows by uniformly approximating f and its derivatives up to order $(2n - 1)$ by entire functions on $[\min(\Lambda), \max(\Lambda)]$, say, by polynomials with a suitable application of the Weierstrass approximation theorem. \square

2.2. Positivity of the weight. In this section we prove the non-negativity of the weight function I introduced in the previous section. We begin with a simple lemma.

Lemma 18. *Let n be a positive integer and let the numbers $Z = (\zeta_i)_{i=1}^n$ be non-negative. Now if $f(t) = \prod_{i=1}^n (\zeta_i - t)^{-1}$, then for any non-negative integer k and $t < 0$ we have*

$$f^{(k)}(t) \geq 0.$$

Proof. The case of $n = 1$ is trivial; the general case now follows immediately from the product rule. □

Lemma 19. *For Λ as before, I_Λ is non-negative.*

Proof. We may clearly assume that Λ is strictly increasing. When checking the non-negativity at a point t , we may without loss of generality assume that $t = 0 \in [\lambda_1, \lambda_n]$. Also by continuity we may further assume that all the λ_i s are non-zero. We are left to investigate

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} S(z, 0) dz = -\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{z^{2n-2} dz}{p_\Lambda(z)^2}.$$

Making the change of variable $w = \frac{1}{z}$, we check that

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{dw}{p_Z(w)^2} \geq 0,$$

where $Z = \frac{1}{\Lambda}$; that is, $\zeta_i = \frac{1}{\lambda_i}$.

Let k ($< n$) be the number of the negative ζ_i s and denote $Z_- = (\zeta_i)_{i=1}^k$. Note that if we further write $f(t) = (\prod_{i>k}(t - \zeta_i))^{-2}$, we have by suitable variant of (10)

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{dw}{p_Z(w)^2} = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{f(w)dw}{p_{Z_-}(w)^2} = [\zeta_1, \zeta_1, \zeta_2, \zeta_2, \dots, \zeta_k, \zeta_k]_f,$$

which is positive in view of (11) and Lemma 18. □

2.3. Characterizations for the matrix monotonicity.

Proof of Theorem 5. The necessity of the condition can be found in [5]. For sufficiency note that by Theorem 17 we can write

$$L(\Lambda) = (2n - 1) \int_{-\infty}^{\infty} C^T(t)M(t)C(t)I(t)dt.$$

Now if $M(t) \geq 0$ for any $t \in (a, b)$, also $C^T(t)M(t)C(t) \geq 0$ for any $t \in (a, b)$. It follows from Lemma 19 that the integrand is a positive matrix, so indeed, L is positive as an integral of positive matrices. But now f is n -monotone by Theorem 1. □

Putting everything together we obtain complete characterizations of the class of n -monotone functions.

Theorem 20 (Loewner, Dobsch, Donoghue). *Let $n \geq 2$, and let (a, b) be an open interval. Now for $f : (a, b) \rightarrow \mathbb{R}$ the following are equivalent:*

- (i) f is n -monotone.
- (ii) $f \in C^1(a, b)$ and the Loewner matrix $L(\Lambda, f)$ is positive for any tuple $\Lambda \in (a, b)^n$.
- (iii) $f \in C^{2n-3}(a, b)$, $f^{(2n-3)}$ is convex, and the Hankel matrix $M_n(t, f)$, which makes sense almost everywhere, is positive for almost every $t \in (a, b)$.

Proof. As noted before, (i) \Leftrightarrow (ii) was proven in the original paper of Loewner [12]. For C^{2n-1} functions, (i) \Leftrightarrow (iii) is Theorem 5, and for merely C^{2n-3} functions the claim follows from standard regularization procedure, details of which can be found in [6]. For an alternate approach to the latter equivalence, see again [6]. □

Corollary 21. *For any positive integer n , n -monotonicity is a local property.*

3. MATRIX CONVEX FUNCTIONS

3.1. Integral representation. In this section we construct the integral representations for the Kraus matrices alluded to in the introduction.

Again, let $n \geq 2$, let (a, b) be an interval, and let $\Lambda = (\lambda_i)_{i=1}^n \in (a, b)^n$ be an arbitrary sequence of distinct points in (a, b) .

In the following the Kraus and the respective Hankel matrices, introduced in the introduction, for sufficiently smooth $f : (a, b) \rightarrow \mathbb{R}$ and $\lambda_0 \in (a, b)$ are denoted by $Kr(\lambda_0, \Lambda, f)$ and $K_n(t, f)$, respectively.

The integral representation for the Kraus matrix is similar to that of the Loewner matrix. Fix again $n \geq 2$, open interval (a, b) , and $\Lambda = (\lambda_i)_{i=1}^n \in (a, b)^n$ an arbitrary sequence of distinct points on (a, b) . For fixed $\lambda_0 \in (a, b)$ the weights $J_{i, \lambda_0} := J_{i, \lambda_0, \Lambda}$, now for $0 \leq i \leq n$, are defined analogously as the residues at λ_i s of

$$T_{\lambda_0}(z, t) := T_{\lambda_0, \Lambda}(z, t) := -\frac{(z-t)^{2n-1}}{(z-\lambda_0)p_{\Lambda}(z)^2}$$

and

$$J_{\lambda_0}(t) := J_{\lambda_0, \Lambda}(t) := \sum_{\substack{0 \leq i \leq n \\ \lambda_i < t}} J_{i, \lambda_0}(t).$$

Lemma 22. *For $\Lambda = (\lambda_i)_{i=1}^n$, as before, $\lambda_0 \in (a, b)$ and $z \in \mathbb{C}$ outside the interval $[\min(\Lambda), \max(\Lambda)]$, we have*

$$2n \int_{-\infty}^{\infty} \frac{J_{\lambda_0}(t)}{(z-t)^{2n+1}} dt = \frac{1}{(z-\lambda_0)p_{\Lambda}(z)^2}.$$

Proof. The proof is almost identical to that of Lemma 16; we just perform the residue trick with the map $(w \mapsto (w-z)^{-1}(w-\lambda_0)^{-1}p_{\Lambda}(w)^{-2})$ instead. \square

Theorem 23. *For $f \in C^{2n}(a, b)$, Λ as before, and $\lambda_0 \in (a, b)$, we have*

$$Kr(\lambda_0, \Lambda, f) = 2n \int_{-\infty}^{\infty} C^T(t, \Lambda)K_n(t, f)C(t, \Lambda)J_{\lambda_0, \Lambda}(t)dt.$$

Proof. After noting that $K_n(t, h_z) = \frac{1}{z-t}M_n(t, h_z)$, the calculation is carried out as in the proof of Theorem 17, using Lemma 22 instead of Lemma 16. \square

3.2. Positivity of the weight.

Lemma 24. *For $\Lambda = (\lambda_i)_{i=1}^n$ as before and $\lambda_0 \in (a, b)$, $J_{\lambda_0, \Lambda}$ is non-negative.*

Proof. As in the proof of Lemma 19, we can assume that $t = 0$ is our point of inspection and that Λ is strictly increasing. We also make the same change of variables $Z = \frac{1}{\Lambda}$. Note that we may well assume that $\zeta_0 > 0$, since the other case would follow by reflecting the variables, that is, considering the sequence $-Z$ and $-\lambda_0$, instead. Now the inequality is reduced to an equivalent form

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{dw}{(\zeta_0 - w)p_Z(w)^2} \geq 0.$$

But as in the proof of Lemma 19, the left hand side can again be written as

$$[\zeta_1, \zeta_1, \zeta_2, \zeta_2, \dots, \zeta_k, \zeta_k]f,$$

where $f(t) = (\zeta_0 - t)^{-1} (\prod_{i>k} (t - \zeta_i))^{-2}$ and k is the number of negative ζ_i s. \square

3.3. Characterizations for the matrix convexity.

Proof of Theorem 7. The necessity of the condition was proven in [9]. For the other direction, by Theorem 23 we can write

$$Kr(\lambda, \Lambda) = 2n \int_{-\infty}^{\infty} C^T(t)K(t)C(t)J_{\lambda_0}(t)dt.$$

But as in the proof of Theorem 5, we now see that the Kraus matrix is an integral of positive matrices, hence positive, and Theorem 3 finishes the claim. \square

The next theorem finally completes the characterization of n -convex functions. The original characterization of Kraus is also improved.

Theorem 25. *Let $n \geq 2$, and let (a, b) be an open interval. Now for $f : (a, b) \rightarrow \mathbb{R}$ the following are equivalent:*

- (i) f is n -convex.
- (ii) $f \in C^2(a, b)$ and the Kraus matrix $Kr(\lambda_0, \Lambda, f)$ is positive for any tuple $\Lambda \in (a, b)^n$ and $\lambda_0 \in \Lambda$.
- (iii) $f \in C^2(a, b)$ and the Kraus matrix $Kr(\lambda_0, \Lambda, f)$ is positive for any tuple $\Lambda \in (a, b)^n$ and $\lambda_0 \in (a, b)$.
- (iv) $f \in C^{2n-2}(a, b)$, $f^{(2n-2)}$ is convex, and the Hankel matrix $K_n(t, f)$, which makes sense almost everywhere, is positive for almost every $t \in (a, b)$.

Proof. (i) \Leftrightarrow (ii) was proven in [11]. For C^{2n} functions (i) \Leftrightarrow (iv) is Theorem 7; the proof of Theorem 7 also gives (iv) \Rightarrow (iii) in this case. For merely C^{2n-2} functions these claims follow from regularization techniques as in the monotone case. (iii) \Rightarrow (ii) is trivial. \square

We also get an interesting corollary connecting the monotonicity to convexity, extending a result in [1].

Corollary 26. *Let $n \geq 2$, and let (a, b) be an open interval. If $f : (a, b) \rightarrow \mathbb{R}$ is n -convex, then for any $\lambda_0 \in (a, b)$ the function $g = (x \mapsto [x, \lambda_0]_f)$ is n -monotone.*

Proof. Simply note that $L(\Lambda, g) = Kr(\lambda_0, \Lambda, f)$. \square

Remark 27. The ideas introduced in the paper can be generalized to characterize a more general class of functions called matrix k -tone functions, introduced in [7]. A paper discussing related questions in this more general setting is in preparation.

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REFERENCES

- [1] Julius Bendat and Seymour Sherman, *Monotone and convex operator functions*, Trans. Amer. Math. Soc. **79** (1955), 58–71. MR0082655
- [2] Patrawut Chansangiam, *A survey on operator monotonicity, operator convexity, and operator means*, Int. J. Anal. (2015), Art. ID 649839, 8 pp. MR3426646
- [3] Carl de Boor, *A practical guide to splines*, Applied Mathematical Sciences, vol. 27, Springer-Verlag, New York-Berlin, 1978. MR507062

- [4] Carl de Boor, *Divided differences*, *Surv. Approx. Theory* **1** (2005), 46–69. MR2221566
- [5] Otto Dobsch, *Matrixfunktionen beschränkter Schwankung* (German), *Math. Z.* **43** (1938), no. 1, 353–388. MR1545729
- [6] William F. Donoghue Jr., *Monotone matrix functions and analytic continuation*, *Die Grundlehren der mathematischen Wissenschaften*, Band 207, Springer-Verlag, New York-Heidelberg, 1974. MR0486556
- [7] Uwe Franz, Fumio Hiai, and Éric Ricard, *Higher order extension of Löwner’s theory: operator k -tone functions*, *Trans. Amer. Math. Soc.* **366** (2014), no. 6, 3043–3074. MR3180739
- [8] Frank Hansen, *The fast track to Löwner’s theorem*, *Linear Algebra Appl.* **438** (2013), no. 11, 4557–4571. MR3034551
- [9] Frank Hansen and Jun Tomiyama, *Differential analysis of matrix convex functions*, *Linear Algebra Appl.* **420** (2007), no. 1, 102–116. MR2277632
- [10] Frank Hansen and Jun Tomiyama, *Differential analysis of matrix convex functions. II*, *JIPAM. J. Inequal. Pure Appl. Math.* **10** (2009), no. 2, Article 32, 5 pp. MR2511925
- [11] Fritz Kraus, *Über konvexe Matrixfunktionen* (German), *Math. Z.* **41** (1936), no. 1, 18–42. MR1545602
- [12] Karl Löwner, *Über monotone Matrixfunktionen* (German), *Math. Z.* **38** (1934), no. 1, 177–216. MR1545446
- [13] The Sage Developers, *SageMath, the Sage Mathematics Software System (Version 7.1)*, 2016, <http://www.sagemath.org>.

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