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REMARKS ON FACTORIALITY AND q-DEFORMATIONS

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Abstract. We prove that the mixed q-Gaussian algebra $\Gamma_Q(H_\mathbb{R})$ associated to a real Hilbert space $H_\mathbb{R}$ and a real symmetric matrix $Q = (q_{ij})$ with $\sup |q_{ij}| < 1$, is a factor as soon as $\dim H_\mathbb{R} \geq 2$. We also discuss the factoriality of q-deformed Araki-Woods algebras, in particular showing that the q-deformed Araki-Woods algebra $\Gamma_q(H_\mathbb{R}, U_t)$ given by a real Hilbert space $H_\mathbb{R}$ and a strongly continuous group U_t is a factor when $\dim H_\mathbb{R} \geq 2$ and U_t admits an invariant eigenvector.

1. Introduction

This paper studies the factoriality of some q-deformed von Neumann algebras. In the early 1990s, motivated by mathematical physics, Bożejko and Speicher introduced the von Neumann algebra $\Gamma_q(H_\mathbb{R})$ generated by q-Gaussian variables [BS91]. Since then, the von Neumann algebra $\Gamma_q(H_\mathbb{R})$ has been widely studied, and also its several generalizations have been introduced and fruitfully investigated. In particular, there are two interesting types of q-deformed algebras which generalize that of Bożejko and Speicher: the first one is the mixed q-Gaussian algebra introduced in [BS94], and the second one is the family of q-deformed Araki-Woods algebras constructed in [Hia03].

The question of factoriality of these q-deformed Neumann algebras remained a well-known problem in the field for many years. In 2005, Ricard [Ric05] proved that the von Neumann algebra $\Gamma_q(H_\mathbb{R})$ is a factor as soon as $\dim H_\mathbb{R} \geq 2$, which solved the problem for $\Gamma_q(H_\mathbb{R})$ in full generality (for earlier partial results see also [Śni04], [Kró06], [BKS97]). However, the analogous problem for mixed q-Gaussian algebras and q-deformed Araki-Woods algebras has remained open. Among the known results, the factoriality of mixed q-Gaussian algebras was proved by Królak [Kró00] when the underlying Hilbert space is infinite-dimensional, and very recently by Nelson and Zeng [NZ16] when the size of the deformation parameters is sufficiently small; similarly, the factoriality of q-deformed Araki-Woods algebras was only established by Hiai in [Hia03] when the 'almost periodic part' (see Section 4 for an explanation of this term) of the underlying Hilbert space is infinite-dimensional, and by Nelson in [Nel15] when q is small.

In this note we solve the problem of factoriality for mixed q-Gaussian algebras in full generality, following the ideas of [Ric05]. Our methods apply also to the q-deformed Araki-Woods algebras, and we show that the q-deformed Araki-Woods

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algebra $\Gamma_q(H_{\mathbb{R}}, U_t)$ is a factor as soon as $\dim H_{\mathbb{R}} \geq 2$ and the semigroup U_t admits an invariant eigenvector. We remark that after the completion of this work, we learned that the last result mentioned above was also obtained independently by Bikram and Mukherjee in [BM17], as a part of a detailed study of maximal abelian subalgebras in q-deformed Araki-Woods algebras.

The scalar products below are always linear on the left. The plan of the paper is as follows: in Section 2 we present a Hilbert space lemma providing estimates for certain commutators to be used later, in Section 3 we establish the factoriality of mixed q-Gaussian algebras in full generality, and in Section 4 we discuss several results concerning factoriality in the context of q-Araki-Woods von Neumann algebras.

2. A convergence lemma for q-commutation relations

The following purely Hilbert-space-theoretic lemma will play a key role in our discussions of factoriality in the following sections.

Lemma 1. *Let $(H_n)_{n\geq 1}$ be a sequence of Hilbert spaces and write $H = \bigoplus_{n\geq 1} H_n$. Let $r, s \in \mathbb{N}$ and let $(a_i)_{1\leq i\leq r}$, $(b_j)_{1\leq j\leq s}$ be two families of operators on H which send each H_n into H_{n+1} or H_{n-1}, such that there exists $0 < q < 1$ with*

$$\|(a_i b_j - b_j a_i)|_{H_n}\| \leq q^n, \quad n \in \mathbb{N}.$$

Assume that $K_n \subset H_n$ is a finite-dimensional Hilbert subspace for each $n \geq 1$ such that for $K = \bigoplus_n K_n$ we have

$$a_i(K) \subset K, \quad 1 \leq i \leq r-1, \quad \text{and } a_r|_K = 0.$$

Then for any bounded nets $(\xi_\alpha), (\eta_\alpha) \subset K$ such that $\eta_\alpha \to 0$ weakly, we have

$$\langle a_1^* \cdots a_r^* \xi_\alpha, b_1 \cdots b_s \eta_\alpha \rangle \to 0.$$

Proof. Put

$$T_{ij}^{(n)} = (a_i b_j - b_j a_i)|_{H_n}, \quad 1 \leq i \leq r, 1 \leq j \leq s, n \geq 1.$$

Then for each i we may write

$$a_i b_1 \cdots b_s \xi - b_1 \cdots b_s a_i \xi = \sum_{j=1}^{s} b_1 \cdots b_{j-1} T_{ij}^{(m(j,n))} b_{j+1} \cdots b_s \xi, \quad \xi \in H_n,$$

where $m(j,n)$ is an integer greater than $n - s$. Iterating this formula we obtain

$$\begin{aligned}
&a_r \cdots a_1 b_1 \cdots b_s \xi \\
&= b_1 \cdots b_s a_r \cdots a_1 \xi \\
&\quad + \sum_{i=1}^{r} (a_r \cdots a_i b_1 \cdots b_s a_{i-1} \cdots a_1 \xi - a_r \cdots a_{i+1} b_1 \cdots b_s a_i \cdots a_1 \xi) \\
&= b_1 \cdots b_s a_r \cdots a_1 \xi \\
&\quad + \sum_{i=1}^{r} a_r \cdots a_{i+1} \left(\sum_{j=1}^{s} b_1 \cdots b_{j-1} T_{ij}^{(m'(i,j,n))} b_{j+1} \cdots b_s \right) a_{i-1} \cdots a_1 \xi,
\end{aligned}$$

where $\xi \in H_n$ and for each i, j, n the integer $m'(i,j,n)$ is greater than $n - s - r$. Now we consider two bounded nets $(\xi_\alpha), (\eta_\alpha) \subset K$ such that $\eta_\alpha \to 0$ weakly. Write

$$\eta_\alpha = (\eta_\alpha^{(n)})_{n\geq 1}, \quad \eta_\alpha^{(n)} \in K_n.$$

We have

$$\langle a_1^* \cdots a_r^* \xi_\alpha, b_1 \cdots b_s \eta_\alpha \rangle = \langle \xi_\alpha, a_r \cdots a_1 b_1 \cdots b_s \eta_\alpha \rangle,$$

and by the assumptions $a_r \cdots a_1 \eta_\alpha = 0$, so together with the previous computations for $a_r \cdots a_1 b_1 \cdots b_s \xi$, we obtain

$$\langle a_1^* \cdots a_r^* \xi_\alpha, b_1 \cdots b_s \eta_\alpha \rangle = \sum_{n \geq 1} \langle \xi_\alpha, T_n \eta_\alpha^{(n)} \rangle, \tag{2.1}$$

where

$$T_n = \sum_{i=1}^{r} a_r \cdots a_{i+1} \left(\sum_{j=1}^{s} b_1 \cdots b_{j-1} T_{ij}^{(m'(i,j,n))} b_j \cdots b_s \right) a_{i-1} \cdots a_1.$$

Recall that $\|T_{ij}^{(k)}\| \leq q^k$ for all i, j, k by assumption. So for each α and n

$$\|T_n \eta_\alpha^{(n)}\| \leq C(q, r, s) q^n \|\eta_\alpha^{(n)}\|,$$

where $C(q, r, s)$ is a constant independent of n. Together with (2.1) we have

$$|\langle a_1^* \cdots a_r^* \xi_\alpha, b_1 \cdots b_s \eta_\alpha \rangle| \leq C(q, r, s) \sup_\alpha \|\xi_\alpha\| \sum_{n \geq 1} q^n \|\eta_\alpha^{(n)}\|. \tag{2.2}$$

Since $\eta_\alpha \to 0$ weakly, we have for each $N \geq 1$,

$$\sum_{n=1}^{N} q^n \|\eta_\alpha^{(n)}\| \underset{\alpha}{\to} 0,$$

and on the other hand,

$$\sum_{n \geq N} q^n \|\eta_\alpha^{(n)}\| \leq \sup_n \|\eta_\alpha^{(n)}\| q^N / (1 - q).$$

Therefore by (2.2) we get

$$\forall N \geq 1, \quad \limsup_\alpha |\langle a_1^* \cdots a_r^* \xi_\alpha, b_1 \cdots b_s \eta_\alpha \rangle| \leq C'(r, s, q) q^N,$$

with a constant $C'(r, s, q)$ independent of N, which means that

$$\langle a_1^* \cdots a_r^* \xi_\alpha, b_1 \cdots b_s \eta_\alpha \rangle \to 0,$$

as desired. □

3. Factoriality of mixed q-Gaussian algebras

Let $N \in \mathbb{N}$, let $Q = (q_{ij})_{i,j=1}^N$ be a symmetric matrix with $q_{ij} \in (-1, 1)$, and let $H_\mathbb{R}$ be a finite-dimensional real Hilbert space with orthonormal basis $e_1, \ldots, e_N$. We recall briefly the construction of mixed Gaussian algebras, as introduced in [BS94]. Write $H = H_\mathbb{R} + \mathrm{i} H_\mathbb{R}$ to be the complexification of $H_\mathbb{R}$. Let $\mathcal{F}_Q(H)$ be the Fock space associated to the Yang-Baxter operator

$$T : H \otimes H \to H \otimes H, \quad e_i \otimes e_j \mapsto q_{ij} e_j \otimes e_i$$

constructed in [BS94]. Denote by $\langle \cdot, \cdot \rangle$ the inner product on $\mathcal{F}_Q(H)$ and let Ω be the vacuum vector. Denote by $\varphi(\cdot) = \langle \cdot \Omega, \Omega \rangle$ the vacuum state. The left creation operators l_i are defined by the formulas

$$l_i \xi = e_i \otimes \xi, \quad \xi \in \mathcal{F}_Q(H),$$

and their adjoints, the left annihilation operators, can be characterized by equalities

$$l_i^*\Omega = 0,$$

$$l_i^*(e_{j_1}\otimes\cdots\otimes e_{j_n}) = \sum_{k=1}^n \delta_{i,j_k} q_{ij_1}\cdots q_{ij_{k-1}} e_{j_1}\otimes\cdots\otimes e_{j_{k-1}}\otimes e_{j_{k+1}}\otimes\cdots\otimes e_{j_n}.$$

Similarly, we have the right creation/annihilation operators

$$r_i\xi = \xi\otimes e_i, \quad \xi\in\mathcal{F}_Q(H),$$

$$r_i^*\Omega = 0,$$

$$r_i^*(e_{j_1}\otimes\cdots\otimes e_{j_n}) = \sum_{k=1}^n \delta_{i,j_k} q_{ij_{k+1}}\cdots q_{ij_n} e_{j_1}\otimes\cdots\otimes e_{j_{k-1}}\otimes e_{j_{k+1}}\otimes\cdots\otimes e_{j_n}.$$

We consider the associated mixed q-Gaussian algebra $\Gamma_Q(H_\mathbb{R})$ generated by the self-adjoint variables $s_j = l_j^* + l_j$. Denote

$$q = \max_{i,j} |q_{ij}| < 1.$$

By a *word* in $\mathcal{F}_Q(H)$ we mean a vector in $\mathcal{F}_Q(H)$ of the form $\zeta_1\otimes\cdots\otimes\zeta_n$ with some $n\geq 1$ and $\zeta_1,\ldots,\zeta_n\in H$. Królak [Kró00] proved that any word $\xi\in\mathcal{F}_Q(H)$ corresponds to a *Wick product* $W(\xi)\in\Gamma_Q(H_\mathbb{R})$ with $W(\xi)\Omega=\xi$. Also, [BS94] remarked that $J\Gamma_Q(H_\mathbb{R})J$ is the commutant of $\Gamma_Q(H_\mathbb{R})$, where J is the conjugation operator given by

$$J(e_{i_1}\otimes\cdots\otimes e_{i_n}) = e_{i_n}\otimes\cdots\otimes e_{i_1}.$$

We write

$$W_r(\xi) = JW(J\xi)J, \quad \xi\in\oplus_n H^{\otimes n}.$$

Then $W_r(\xi)\in\Gamma_Q(H_\mathbb{R})'$.

Lemma 2. *For each $n\in\mathbb{N}$ and $i,j=1,\ldots,N$ the operators $T_i^{(n)}$ on $H^{\otimes n}$ characterized by the equalities*

$$l_i^* r_j - r_j l_i^* = \delta_{ij}\oplus_n T_i^{(n)}$$

satisfy the norm estimate $\|T_i^{(n)}\|\leq q^n$.

Proof. The case of $n=0$ is obvious and we take $n\geq 1$ in the following. Observe that

$$\begin{aligned} l_i^* r_j(e_{j_1}\otimes\cdots\otimes e_{j_n}) &= l_i^*(e_{j_1}\otimes\cdots\otimes e_{j_n}\otimes e_j) \\ &= \sum_{k=1}^n \delta_{i,j_k} q_{ij_1}\cdots q_{ij_{k-1}} e_{j_1}\otimes\cdots\otimes e_{j_{k-1}}\otimes e_{j_{k+1}}\otimes\cdots\otimes e_{j_n}\otimes e_j \\ &\quad + \delta_{ij} q_{ij_1}\cdots q_{ij_n} e_{j_1}\otimes\cdots\otimes e_{j_n}, \end{aligned}$$

and

$$r_j l_i^*(e_{j_1}\otimes\cdots\otimes e_{j_n}) = \sum_{k=1}^n \delta_{i,j_k} q_{ij_1}\cdots q_{ij_{k-1}} e_{j_1}\otimes\cdots\otimes e_{j_{k-1}}\otimes e_{j_{k+1}}\otimes\cdots\otimes e_{j_n}\otimes e_j.$$

Now take

$$T_i^{(n)}: H^{\otimes n}\to H^{\otimes n}, \quad e_{j_1}\otimes\cdots\otimes e_{j_n}\mapsto \delta_{ij} q_{ij_1}\cdots q_{ij_n} e_{j_1}\otimes\cdots\otimes e_{j_n}.$$

The eigenspace of $T_i^{(n)}$ corresponding to $\delta_{ij}q_{ij_1}\cdots q_{ij_n}$ is spanned by the vectors of the type $E_{\{j_1,\dots,j_n\}} = \{e_{j'_1}\otimes\cdots\otimes e_{j'_n} : q_{ij_1}\cdots q_{ij_n} = q_{ij'_1}\cdots q_{ij'_n}\}$, which are orthogonal for distinct $\underline{j} = \{j_1,\dots,j_n\}$. So

$$\|T_i^{(n)}\| \le \max\{q_{ij_1}\cdots q_{ij_n} : 1\le j_1,\dots,j_n\le N\} \le q^n$$

and $T_i^{(n)}$ is the desired operator. □

Now the following main result is in reach. The idea is partially inspired by the proof in [Ric05] in conjunction with Lemma 1.

Theorem 3. *For each $1\le i\le n$, the von Neumann subalgebra generated by s_i is maximal abelian in $\Gamma_Q(H_{\mathbb{R}})$. In particular, $\Gamma_Q(H_{\mathbb{R}})$ is a factor if $n\ge 2$.*

Proof. By [BKS97], we know that the spectral measure of s_i is the q-semicircular law with $q = q_{ii}$. Therefore the von Neumann algebra M generated by s_i is diffuse and abelian, and hence isomorphic to the von Neumann algebra $L^\infty([0,1],dm)$ where dm denotes the Lebesgue measure on $[0,1]$. As a result, we may find a sequence of unitaries $(u_\alpha)_{\alpha\in\mathbb{N}}\subset M$ which correspond to Rademacher functions via this isomorphism. In particular, we have

$$u_\alpha = u_\alpha^*,\quad u_\alpha^2 = 1,\quad u_\alpha\Omega\to 0 \text{ weakly in } \mathcal{F}_Q(H).$$

Now assume $x\in\Gamma_Q(H_{\mathbb{R}})$ with $xs_i = s_ix$, and hence

$$xy = yx,\quad y\in M.$$

Let $\mathcal{F}_Q(\mathbb{C}e_i)\subset\mathcal{F}_Q(H)$ be the Fock space associated to e_i. Observe that for any vector $\xi\in\bigcup_{m\in\mathbb{N}}H^{\otimes m}$ and all $\alpha\ge 1$ we have

(3.1)
$$\langle\xi, x\Omega\rangle = \varphi(x^*W(\xi)) = \varphi(x^*u_\alpha^2W(\xi)) = \varphi(u_\alpha x^*u_\alpha W(\xi)) = \langle W_r(\xi)u_\alpha\Omega, xu_\alpha\Omega\rangle.$$

We remark that if further ξ is orthogonal to $\mathcal{F}_Q(\mathbb{C}e_i)$, then

$$\forall y\in\Gamma_Q(H_{\mathbb{R}}),\quad \langle W_r(\xi)u_\alpha\Omega, yu_\alpha\Omega\rangle\to 0. \tag{3.2}$$

To see this, it suffices to consider the case $y\Omega\in H^{\otimes n}$ for an arbitrary $n\ge 0$ since it is easy to see that the functionals $y^*\Omega\mapsto\langle W_r(\xi)u_\alpha\Omega, yu_\alpha\Omega\rangle$ extend to uniformly bounded functionals on $\mathcal{F}_Q(H)$ thanks to the traciality of φ ([BS94, Theorem 4.4]). Now by the Wick formula in [Kró00, Theorem 1], it is enough to prove the convergence

$$\langle r_{i_1}\cdots r_{i_s}r^*_{i_{s+1}}\cdots r^*_{i_p}u_\alpha\Omega, l_{j_1}\cdots l_{j_t}l^*_{j_{t+1}}\cdots l^*_{j_q}u_\alpha\Omega\rangle\to 0 \tag{3.3}$$

for any fixed indices $i_1,\dots,i_p,j_1,\dots,j_q$ with some $i_k\ne i$. Denote

$$s' = \min\{k : i_k\ne i\}.$$

If $s'>s$, we have $r^*_{i_{s'}}\cdots r^*_{i_p}u_\alpha\Omega = 0$ for all $\alpha\ge 1$ and the convergence (3.3) becomes trivial. So we assume in the following $s'\le s$. Note that by definition

$$r_il_j - l_jr_i = 0,\quad r_i^*l_j^* - l_j^*r_i^* = 0,$$

and by Lemma 2

$$\|(l_i^*r_j - r_jl_i^*)|_{H^{\otimes n}}\| \le q^n,\quad n\ge 1.$$

Also, observe that by the choice of s',

$$r^*_{i_{s'}}|_{\mathcal{F}_Q(\mathbb{R}e_i)} = 0,\quad r^*_{i_k}(\mathcal{F}_Q(\mathbb{R}e_i))\subset\mathcal{F}_Q(\mathbb{R}e_i),\quad 1\le k<s'.$$

So now applying Lemma 1 to the families of operators $r_{i_1}^*, \ldots, r_{i_{s'}}^*$ and $l_{j_1}, \ldots, l_{j_t}, l_{j_{t+1}}^*, \ldots, l_{j_q}^*$, we obtain the convergence (3.3). As a consequence, the convergence (3.2) holds as well, which, together with (3.1), yields that

$$\langle \xi, x\Omega \rangle = 0.$$

This means that $x\Omega \in \mathcal{F}_Q(\mathbb{C}e_i)$ since ξ is arbitrarily chosen in a dense subset of $\mathcal{F}_Q(\mathbb{C}e_i)^\perp$. We can then deduce that $x \in M$ using the second quantization of the projection $P : H_\mathbb{R} \to \mathbb{R}e_i$ (see [LP99, Lemma 3.1]). Thus we have shown that the von Neumann subalgebra M generated by s_i is maximal abelian in $\Gamma_Q(H_\mathbb{R})$.

Also, if $x \in \Gamma_Q(H_\mathbb{R}) \cap \Gamma_Q(H_\mathbb{R})'$, then the above argument shows that $x\Omega \in \bigcap_{i=1}^n \mathcal{F}_Q(\mathbb{C}e_i)$, so $x\Omega \in \mathbb{C}\Omega$. Therefore $\Gamma_Q(H_\mathbb{R})$ is a factor. □

4. Factoriality of q-Araki-Woods algebras

Now we discuss the factoriality of q-Araki-Woods algebras. We refer to [Hia03] for the detailed description of the construction of these algebras and only sketch the outline below. Following the notation of [Hia03], given a real Hilbert space $H_\mathbb{R}$ with a strongly continuous group U_t of orthogonal transformations on $H_\mathbb{R}$, we may introduce a deformed inner product $\langle \cdot, \cdot \rangle_U$ on $H_\mathbb{C} := H_\mathbb{R} + \mathrm{i}H_\mathbb{R}$. Denote by H the completion of $H_\mathbb{C}$ with respect to $\langle \cdot, \cdot \rangle_U$ and denote by $\mathcal{F}_q(H)$ the q-Fock space associated to H. We define the left and right creation operators

$$l(\xi)\eta = \xi \otimes \eta, \quad r(\xi)\eta = \eta \otimes \xi, \quad \xi \in H, \eta \in \mathcal{F}_q(H),$$

and the left and right annihilation operators

$$l^*(\xi) = l(\xi)^*, \quad r^*(\xi) = r(\xi)^*, \quad \xi \in H.$$

We denote by $\Gamma_q(H_\mathbb{R}, U_t)$ (resp., $C_q^*(H_\mathbb{R}, U_t)$) the von Neumann algebra (resp., C*-algebra) generated by $\{l(e) + l^*(e) : e \in H_\mathbb{R}\}$ in $B(\mathcal{F}_q(H))$, to be called the q-Araki-Woods von Neumann algebra. Properties of the vacuum state guarantee the existence of the *Wick product* map $W : \Gamma_q(H_\mathbb{R}, U_t)\Omega \to \Gamma_q(H_\mathbb{R}, U_t)$ such that $W(\xi)\Omega = \xi$. On the other hand, denote

$$H'_\mathbb{R} = \{\xi \in H : \ \forall \eta \in H_\mathbb{R}, \ \langle \xi, \eta \rangle \in \mathbb{R}\}.$$

Then the von Neumann algebra $\Gamma_{q,r}(H_\mathbb{R}, U_t)$ generated by $\{r(e) + r^*(e) : e \in H'_\mathbb{R}\}$ in $B(\mathcal{F}_q(H))$ is the commutant of $\Gamma_q(H_\mathbb{R}, U_t)$, and again there exists a right Wick product $W_r : \Gamma_{q,r}(H_\mathbb{R}, U_t)\Omega \to \Gamma_{q,r}(H_\mathbb{R}, U_t)$ such that $W_r(\xi)\Omega = \xi$. We denote by I the standard complex conjugation on $H_\mathbb{R} + \mathrm{i}H_\mathbb{R}$, and by I_r the complex conjugation on $H'_\mathbb{R} + \mathrm{i}H'_\mathbb{R}$. The following observations are well known and we state them here for later use.

Lemma 4. (1) *Suppose that $e_1, \ldots, e_n \in H_\mathbb{C}$. Then we have the following Wick formula:*
(4.1)

$$W(e_1 \otimes \cdots \otimes e_n) = \sum_{k=0}^{n} \sum_{i_1, \ldots, i_k, j_{k+1}, \ldots, j_n} l(e_{i_1}) \ldots l(e_{i_k}) l^*(Ie_{j_{k+1}}) \ldots l^*(Ie_{j_n}) q^{i(I_1, I_2)},$$

where $I_1 = \{i_1, \ldots, i_k\}$ and $I_2 = \{j_{k+1}, \ldots, j_n\}$ form a partition of the set $\{1, \ldots, n\}$ and $i(I_1, I_2)$ is the number of crossings. A similar formula holds for $W_r(e_1 \otimes \cdots \otimes e_n)$ as well.

(2) *Let $f \in H_\mathbb{R}$, $e \in H'_\mathbb{R} + \mathrm{i}H'_\mathbb{R}$. If $\langle e, f \rangle = 0$; then $\langle I_r e, f \rangle = 0$.*

Proof. (1) See [BKS97, Proposition 2.7], [Was17, Lemma 3.1].

(2) Write $e = e_1 + \mathrm{i}e_2$ with $e_1, e_2 \in H'_{\mathbb{R}}$. Since $\langle e_1, f\rangle \in \mathbb{R}, \langle e_2, f\rangle \in \mathbb{R}$, we see that the identity $\langle e, f\rangle = 0$ yields

$$\langle e_1, f\rangle = \langle e_2, f\rangle = 0.$$

Therefore

$$\langle I_r e, f\rangle = \langle e_1 - \mathrm{i}e_2, f\rangle = 0.$$

□

According to Shlyakhtenko [Shl97], we have the decomposition

$$(H_{\mathbb{R}}, U_t) = (K_{\mathbb{R}}, U'_t) \oplus (L_{\mathbb{R}}, U''_t),$$

where U'_t is almost periodic and U''_t is ergodic. Then $K_{\mathbb{R}} \subset H_{\mathbb{R}}$ is the real closed subspace spanned by eigenvectors of $U_t = A^{\mathrm{i}t}$. Let $K_{\mathbb{C}} = K_{\mathbb{R}} + \mathrm{i}K_{\mathbb{R}}$ be the complexification and K be the completion of $K_{\mathbb{C}}$ with respect to the deformed norm as above, and similarly for L. Note that the orthogonal projection $P : H_{\mathbb{R}} \to K_{\mathbb{R}}$ commutes with U_t. So by the second quantization, $\Gamma_q(K_{\mathbb{R}}, U_t|_K)$ embeds as a von Neumann subalgebra of $\Gamma_q(H_{\mathbb{R}}, U_t)$. For an operator T we denote by $\mathcal{F}_q(T)$ its second quantization.

The following observation shows that in looking at the center of the q-Araki-Woods algebra it suffices to consider the 'K-part' of the algebra (we do not really use this fact in what follows).

Lemma 5. (1) *The semigroup $\mathcal{F}_q(U_t)$ admits no eigenvectors in $\mathcal{F}_q(K)^\perp \subset \mathcal{F}_q(H)$;*

(2) *Assume $x \in \Gamma_q(H_{\mathbb{R}}, U_t) \cap \Gamma_q(H_{\mathbb{R}}, U_t)'$. Then*

$$x\Omega \in \mathcal{F}_q(K) \quad \textit{and} \quad x \in \Gamma_q(K_{\mathbb{R}}, U_t|_{K_{\mathbb{R}}}).$$

Proof. (1) Let (e_i) be an orthonormal basis in $H_{\mathbb{R}}$. Since $\mathcal{F}_q(P)$ is the orthogonal projection onto $\mathcal{F}_q(K)$, we have

$$\mathcal{F}_q(P)(\mathcal{F}_q(K)^\perp) = 0.$$

Hence

$$\mathcal{F}_q(K)^\perp = \overline{\mathrm{span}}\{e_{i_1} \otimes \cdots \otimes e_{i_n} : n \geq 1\ \exists 1 \leq m \leq n, e_{i_m} \in L_{\mathbb{R}}\}.$$

Denote

$$K_n = \overline{\mathrm{span}}\{e_{i_1} \otimes \cdots \otimes e_{i_n} \in \mathcal{F}_q(K)^\perp\} = \mathrm{span}\{H_{i_1} \otimes \cdots \otimes H_{i_n}, H_i = K \text{ or } L, \exists H_i = L\}.$$

Note that U_t is unitarily equivalent to a multiplier map on some $L^2(\mu)$. So by the definition of K and L and the fact that at least one of H_{i_k} is equal to L, it is easy to see that $\mathcal{F}_q(U_t)|_{H_{i_1} \otimes \cdots \otimes H_{i_n}} = U_t|_{H_{i_1}} \otimes \cdots \otimes U_t|_{H_{i_n}}$ admits no eigenvectors. Since each H_{i_n} is invariant under U_t, $\mathcal{F}_q(U_t)$ admits no eigenvectors in K_n either. Then the lemma follows immediately. Indeed, let

$$\xi = \sum_n \xi_n \in \mathcal{F}_q(K)^\perp, \quad \xi_n \in K_n,$$

be an eigenvector. Then we get

$$\sum_n (U_t \xi_n - \lambda \xi_n) = 0$$

for some λ and hence $U_t \xi_n - \lambda \xi_n = 0$ for all n, which yields a contradiction.

(2) Assume $x \in \Gamma_q(H_\mathbb{R}, U_t) \cap \Gamma_q(H_\mathbb{R}, U_t)'$. Note that x is in the centralizer of the vacuum state φ. So we have for all $t \in \mathbb{R}$,

$$\sigma_t(x)\Omega = \Delta^{\mathrm{it}} x \Delta^{-\mathrm{it}}\Omega = x\Omega.$$

Recall the Tomita-Takesaki theory for $\Gamma_q(H_\mathbb{R}, U_t)$ and the vacuum state. We see that $x\Omega$ is a fixed point of $\mathcal{F}_q(U_t)$, and hence $(\mathcal{F}_q(P)^\perp)(x\Omega)$ is an eigenvector by orthogonal decomposition. So by the above lemma $(\mathcal{F}_q(P)^\perp)(x\Omega) = 0$. That is, $x\Omega \in \mathcal{F}_q(K)$ and $x \in \Gamma_q(K_\mathbb{R}, U_t|_K)$. □

Proposition 6. *Let $D_\mathbb{R} \subset H_\mathbb{R}$ be a real finite-dimensional Hilbert subspace and let M be a diffuse abelian von Neumann subalgebra of $\Gamma_q(H_\mathbb{R}, U_t)$ such that $M\Omega \subset \mathcal{F}_q(D)$, where $D = D_\mathbb{R} + iD_\mathbb{R}$. Assume $x \in \Gamma_q(H_\mathbb{R}, U_t) \cap M'$.*

(1) *If $x \in C_q^*(H_\mathbb{R}, U_t)$, then $x\Omega \in \mathcal{F}_q(D)$.*

(2) *If M is contained in the centralizer of $\Gamma_q(H_\mathbb{R}, U_t)$, then $x\Omega \in \mathcal{F}_q(D)$.*

Proof. The proof is similar to that of Theorem 3, so we only present a sketch. Since M is diffuse and $M\Omega \subset \mathcal{F}_q(D)$, we may find a sequence of unitaries $(u_\alpha)_{\alpha\in\mathbb{N}} \subset M$ such that

$$u_\alpha = u_\alpha^*, \quad u_\alpha^2 = 1, \quad u_\alpha\Omega \to 0 \text{ weakly in } \mathcal{F}_q(D).$$

We may show that for any vector $\xi \in H^{\otimes n}$ with $n \geq 1$ which is orthogonal to $\mathcal{F}_q(D)$, and for $w \in \Gamma_q(H_\mathbb{R}, U_t)$, if one of the following conditions is satisfied:

(a) $w \in C_q^*(H_\mathbb{R}, U_t)$;

(b) the operator $z\Omega \mapsto zu_\alpha\Omega$ is uniformly bounded on $\mathcal{F}_q(H)$;

then

$$\varphi(u_\alpha w^* u_\alpha W(\xi)) = \langle W_r(\xi)u_\alpha\Omega, wu_\alpha\Omega\rangle \to 0. \tag{4.2}$$

Indeed, we note that the anti-linear functional $z \mapsto \varphi(u_\alpha z^* u_\alpha W(\xi))$ is uniformly bounded on $C_q^*(H_\mathbb{R}, U_t)$ with respect to α, and if (b) is satisfied, the anti-linear functional $z\Omega \mapsto \varphi(u_\alpha z^* u_\alpha W(\xi))$ is uniformly bounded on $\mathcal{F}_q(H)$ with respect to α. So if any one of (a) and (b) is satisfied, we may find a sequence of vectors $(\eta_k)_{k=1}^\infty$ in the algebraic span of $\{H^{\otimes n} : n \geq 1\}$ such that we have the convergence

$$\varphi(u_\alpha W(\eta_k)^* u_\alpha W(\xi)) \to \varphi(u_\alpha w^* u_\alpha W(\xi)), \quad k \to \infty,$$

which is uniform with respect to α. This means that in order to see (4.2) under the condition (a) or (b), it suffices to assume that w belongs to the algebraic span of $\{H^{\otimes n} : n \geq 1\}$. On the other hand, recall that $\xi \perp \mathcal{F}_q(D)$, which means that ξ is the combination of words of the form

$$e_{m_1} \otimes \cdots \otimes e_{m_n}, \quad e_{m_1}, \ldots, e_{m_n} \in H \cup D^\perp \exists 1 \leq k \leq n,\ e_{m_k} \in D^\perp.$$

Thus by the Wick formula in Lemma 4, it suffices to prove the convergence

$$\langle r(e_{i_1}) \cdots r(e_{i_m}) r^*(I_r e_{i_{m+1}}) \cdots r^*(I_r e_{i_n}) u_\alpha\Omega, l(e_{j_1}) \cdots l(e_{j_s}) l^*(I e_{j_{s+1}}) \cdots l^*(I e_{j_p}) u_\alpha\Omega\rangle \to 0,$$

where there is $1 \leq k \leq n$ such that $e_{i_k} \in D^\perp$, $e_{i_{k'}} \in H$ for $1 \leq k < k'$. By Lemma 4, $I_r e_{i_k} \in D^\perp$ holds as well. Consequently, if $k \geq m+1$, then $r^*(I_r e_{i_k}) \cdots r^*(I_r e_{i_n}) u_\alpha\Omega = 0$ and the above convergence is trivial. Hence we assume $k \leq m$. Recall that

$$l^*(f)r^*(g) - r^*(g)l^*(f) = 0, \quad l(f)r^*(g) - r^*(g)l(f) = \langle f, g\rangle q^k \left(\oplus_{k\geq 0} \mathrm{id}_{H^{\otimes k}}\right), \quad f, g \in H.$$

Now applying Lemma 1 as in Theorem 3, we obtain the desired convergence (4.2).

Now the conclusion of the theorem is immediate. Take $x \in \Gamma_q(H_{\mathbb{R}}, U_t) \cap M'$. We have for all $\alpha \geq 1$ and every $\xi \in H^{\otimes n}$ with $n \geq 1$ which is orthogonal to $\mathcal{F}_q(D)$,

$$\langle \xi, x\Omega \rangle = \varphi(x^* W(\xi)) = \varphi(x^* u_\alpha^2 W(\xi)) = \varphi(u_\alpha x^* u_\alpha W(\xi)) = \langle W_r(\xi) u_\alpha \Omega, x u_\alpha \Omega \rangle.$$

If now the assumption of (1) holds, then by (a) and (4.2) we see that

$$\langle \xi, x\Omega \rangle = \langle W_r(\xi) u_\alpha \Omega, x u_\alpha \Omega \rangle \to 0.$$

Similarly if the assumption of (2) holds, then the u_α's belong to the centralizer of $\Gamma_q(H_{\mathbb{R}}, U_t)$, and hence

$$\|z u_\alpha \Omega\|^2 = \varphi(u_\alpha z^* z u_\alpha) = \varphi(z^* z u_\alpha^2) = \varphi(z^* z) = \|z\Omega\|^2,$$

so (b) is satisfied. By (4.2) this yields that

$$\langle \xi, x\Omega \rangle = \langle W_r(\xi) u_\alpha \Omega, x u_\alpha \Omega \rangle \to 0.$$

So $\langle \xi, x\Omega \rangle = 0$ for all words $\xi \in \mathcal{F}_q(D)^\perp$ and hence $x\Omega \in \mathcal{F}_q(D)$. □

We are ready to state the second main result of this article.

Theorem 7. *Assume* $\dim H_{\mathbb{R}} \geq 2$.

(1) *If there exists* $\xi_0 \in H_{\mathbb{R}}$ *such that* $U_t \xi_0 = \xi_0$, *then* $\Gamma_q(H_{\mathbb{R}}, U_t)$ *is a factor.*

(2) *Let* $H_{\mathbb{R}}^{(1)}, H_{\mathbb{R}}^{(2)}$ *be two finite-dimensional Hilbert subspaces of* $H_{\mathbb{R}}$ *which are invariant under* U_t *and are orthogonal with respect to the real inner product of* $H_{\mathbb{R}}$. *Assume that for* $k = 1, 2$ *the centralizer of* $\Gamma_q(H_{\mathbb{R}}^{(k)}, U_t|_{H_{\mathbb{R}}^{(k)}})$ *contains a diffuse element. Then* $\Gamma_q(H_{\mathbb{R}}, U_t)$ *is a factor.*

(3) $\Gamma_q(H_{\mathbb{R}}, U_t)' \cap C_q^*(H_{\mathbb{R}}, U_t) = \mathbb{C}1$.

Proof. (1) Since $\dim H_{\mathbb{R}} \geq 2$ and $U_t \xi_0 = \xi_0$, the subspace $(\mathbb{C}\xi_0)^\perp \subset H$ is invariant under U_t, and we may find a vector $\eta \in (\mathbb{C}\xi_0)^\perp$ such that $\eta \in H'_{\mathbb{R}}$, $\eta \perp \xi_0$. Note that in this case $W_r(\eta) = W_r(\eta)^*$ and $I\eta \perp \xi_0$. Take $x \in \Gamma_q(H_{\mathbb{R}}, U_t)' \cap \Gamma_q(H_{\mathbb{R}}, U_t)$ and denote $\xi = x\Omega$. Note that $W(\xi_0)$ belongs to the centralizer of $\Gamma_q(H_{\mathbb{R}}, U_t)$ by the assumption $U_t \xi_0 = \xi_0$, and that the spectral measure of $W(\xi_0)$ is q-semicircular ([Nou06, Remarks pp. 298-299]) and hence $W(\xi_0)$ generates a diffuse abelian von Neumann subalgebra. So by Proposition 6(2), we have

$$\xi \in \mathcal{F}_q(\mathbb{C}\xi_0), \quad \eta \perp \xi, I\eta \perp \xi.$$

Then we see that

$$\begin{aligned} W(\xi)\eta &= x W(\eta)\Omega = W(\eta) x \Omega = W(\eta)\xi \\ &= l(\eta)\xi + l^*(I\eta)\xi = \eta \otimes \xi. \end{aligned}$$

As a result, writing

$$\lambda = \langle \xi, \Omega \rangle, \quad \zeta = \xi - \lambda\Omega,$$

we have

$$\begin{aligned} \|\eta \otimes \xi\|^2 &= \langle \eta \otimes \xi, W(\xi)\eta \rangle = \langle \eta \otimes \xi, W_r(\eta)\xi \rangle = \langle W_r(\eta)(\eta \otimes \xi), \xi \rangle \\ &= \lambda \langle W_r(\eta)\eta, \xi \rangle + \langle W_r(\eta)(\eta \otimes \zeta), \xi \rangle \\ &= \lambda \langle \|\eta\|^2 \Omega, \xi \rangle + \lambda \langle \eta \otimes \eta, \xi \rangle + \langle \eta \otimes \zeta \otimes \eta, \xi \rangle \\ &= |\lambda|^2 \|\eta\|^2, \end{aligned}$$

where we have used the relation $\eta\perp\xi_0$ in the last equality. However

$$\|\eta\otimes\xi\|^2 = \|\eta\otimes(\lambda\Omega+\zeta)\|^2 = |\lambda|^2\|\eta\|^2 + \|\eta\otimes\zeta\|^2.$$

Thus the above two equalities yield that $\eta\otimes\zeta = 0$. Therefore $\zeta = 0$ and $x\Omega = \xi = \lambda\Omega$. This proves that

$$\Gamma_q(H_{\mathbb{R}}, U_t)' \cap \Gamma_q(H_{\mathbb{R}}, U_t) = \mathbb{C}1.$$

(2) This assertion follows directly from Proposition 6(2) since according to that result any $x \in \Gamma_q(H_{\mathbb{R}}, U_t)' \cap \Gamma_q(H_{\mathbb{R}}, U_t)$ should satisfy

$$x\Omega \in \mathcal{F}_q(H^{(1)}) \cap \mathcal{F}_q(H^{(2)}) (= \mathbb{C}\Omega).$$

(3) Since $\dim H_{\mathbb{R}} \geq 2$, we may find two vectors $e_1, e_2 \in H_{\mathbb{R}}$ which are orthogonal with respect to the real inner product of $H_{\mathbb{R}}$. Then $W(e_1)$ and $W(e_2)$ are self-adjoint diffuse elements as discussed before, and $\mathcal{F}_q(\mathbb{C}e_1) \cap \mathcal{F}_q(\mathbb{C}e_2) = \mathbb{C}\Omega$. Then according to Proposition 6(1), any $x \in \Gamma_q(H_{\mathbb{R}}, U_t)' \cap C_q^*(H_{\mathbb{R}}, U_t)$ should satisfy

$$x\Omega \in \mathcal{F}_q(\mathbb{C}e_1) \cap \mathcal{F}_q(\mathbb{C}e_2) (= \mathbb{C}\Omega).$$

Therefore the assertion is proved. □

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