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A NOTE ON THE DISCRETE FOURIER RESTRICTION PROBLEM

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ABSTRACT. In this paper, we establish a general discrete Fourier restriction theorem. As an application, we make some progress on the discrete Fourier restriction problem associated with KdV equations.

1. INTRODUCTION

Recently, the Fourier restriction problem has been widely studied (for example, see [10], [11], [1], [5], [3]). In this paper, we investigate the discrete Fourier restriction problems. Let us first see the discrete Fourier restriction problem associated with KdV equations. More precisely, we are going to seek the best constant $A_{p,N}$ satisfying

$$\sum_{|n|\le N} |\hat{f}(n,n^3)|^2 \le A_{p,N}\|f\|^2_{L^{p'}(\mathbb{T}^2)} \tag{1.1}$$

where f is a periodic function on $\mathbb{T}^2$, $\hat{f}$ is the Fourier transform of f on $\mathbb{T}^2$, i.e., $\hat{f}(\xi)=\int_{\mathbb{T}^2} e^{-2\pi i x\cdot\xi} f(x)dx$, N is a sufficiently large integer, $p\ge 2$, and $\frac{1}{p}+\frac{1}{p'}=1$. For any $\varepsilon>0$, Bourgain [2] showed that $A_{6,N}\le N^{\varepsilon}$. Later Hu and Li [7] proved that $A_{p,N}\lesssim_\varepsilon N^{1-\frac{8}{p}+\varepsilon}$ for $p\ge 14$.

Bourgain [2] and Hu and Li [7] conjectured that

$$A_{p,N}\le \begin{cases} C_p & \text{for } \quad 2\le p<8,\\ C_{\varepsilon,p}N^{1-\frac{8}{p}+\varepsilon} & \text{for } \quad p\ge 8.\end{cases} \tag{1.2}$$

Clearly, $p=8$ is the critical number. In this paper, we will make slight progress for this conjecture. We will show that $A_{p,N}\lesssim_\varepsilon N^{1-\frac{8}{p}+\varepsilon}$ for $p\ge 12$.

It is easy to see that the study of $A_{p,N}$ is equivalent to the periodic Strichartz inequality associated with the KdV equation:

$$\Big\|\sum_{|n|\le N} a_n e^{2\pi i(xn+tn^3)}\Big\|_{L^p_{x,t}(\mathbb{T}^2)} \le K_{p,N}\Big(\sum_{|n|\le N}|a_n|^2\Big)^{\frac{1}{2}}. \tag{1.3}$$

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In fact, we have $A_{p,N} \approx K_{p,N}^2$ by using the dual method. Later while considering the Cauchy problem of the fifth-order KdV-type equations, Hu and Li [8] studied the following Strichartz inequality:

$$\Big\| \sum_{|n|\le N} a_n e^{2\pi i(xn+tn^k)} \Big\|_{L^p_{x,t}(\mathbb{T}^2)} \le \mathcal{K}_{p,N} \Big(\sum_{|n|\le N} |a_n|^2 \Big)^{\frac{1}{2}}, \tag{1.4}$$

where k is a positive integer and $k \ge 2$. They [8] proved that $\mathcal{K}_{6,N} \lesssim N^{\varepsilon}$ if k is odd and $\mathcal{K}_{p,N} \lesssim_\varepsilon N^{\frac{1}{2}(1-\frac{2(k+1)}{p})+\varepsilon}$ for $p \ge p_0$ where

$$p_0 = \begin{cases} (k-2)2^k + 6 & \text{if } k \text{ is odd,} \\ (k-1)2^k + 4 & \text{if } k \text{ is even.} \end{cases}$$

In (1.3) and (1.4), the discrete Fourier restriction problems are studied in two dimensions when the Fourier transform is indeed restricted to the curve (n, n^3) and (n, n^k). It is natural to consider a similar problem for higher dimensions when the Fourier transform is restricted to the general curve $(n^{k_1}, \cdots, n^{k_d})$, where $k_1, \cdots, k_d$ are positive integers. Let $K_{p,d,N}$ be the best constant in the following inequality:

$$\Big\| \sum_{|n|\le N} a_n e^{2\pi i(\alpha_1 n^{k_1}+\cdots+\alpha_d n^{k_d})} \Big\|_{L^p(\mathbb{T}^d)} \le K_{p,d,N} \Big(\sum_{|n|\le N} |a_n|^2 \Big)^{\frac{1}{2}}. \tag{1.5}$$

Our main result in the present paper is as follows.

Theorem 1.1. *Let a_n be a complex number for all $|n| \le N$. Let $d > 1$ and $k_1, \cdots, k_d$ be positive integers with $1 \le k_1 < \cdots < k_d = k$. Set $\mathfrak{K} = \sum_{i=1}^d k_i$. Let $K_{p,d,N}$ be defined in* (1.5). *Suppose $p \ge k(k+1)$. Then for any $\varepsilon > 0$, we have*

$$K_{p,d,N} \lesssim_\varepsilon N^{\frac{1}{2}(1-\frac{2\mathfrak{K}}{p})+\varepsilon}, \tag{1.6}$$

where the implicit constant depends on $k_1, \cdots, k_d$, p, ε, but does not depend on N.

Remark 1.2. In [9], T. D. Wooley adapted the efficient congruencing method to prove that (1.6) holds for $p \ge 2k(k+1)$. And whenever $p > 2k(k+1)$, one may take $\varepsilon = 0$ in (1.6).

Remark 1.3. In Section 3, we will show the bound in (1.6) is sharp up to a constant ε. One may conjecture (1.6) holds for all $p \ge 2\mathfrak{K}$. Notice if $k_i = i$, $i = 1, \cdots, d$, then $2\mathfrak{K} = d(d+1)$. Thus (1.6) is valid for $p \ge 2\mathfrak{K}$ in this case.

By using Theorem 1.1, one could make some progress on the previous results. Applying Theorem 1.1 with $d = 2$, $k_1 = 1$, $k_2 = 3$ and $d = 2$, $k_1 = 1$, $k_2 = k$ (here $k \ge 2$), we may obtain the following corollaries.

Corollary 1.4. *Let $K_{p,N}$ be defined in* (1.3). *Suppose $p \ge 12$. Then for any $\varepsilon > 0$, we get*

$$K_{p,N} \lesssim_\varepsilon N^{\frac{1}{2}(1-\frac{8}{p})+\varepsilon},$$

where the implicit constant is independent of N. If $p > 24$, one may take $\varepsilon = 0$.

Corollary 1.5. *Let $\mathcal{K}_{p,N}$ be defined in* (1.4). *Suppose $p \ge k(k+1)$. Then for any $\varepsilon > 0$, we have*

$$\mathcal{K}_{p,N} \lesssim_\varepsilon N^{\frac{1}{2}(1-\frac{2(k+1)}{p})+\varepsilon},$$

where the implicit constant is independent of N. If $p > 2k(k+1)$, one may take $\varepsilon = 0$.

By setting $d = k$, $k_i = i$, for $i = 1, \cdots, k$, $a_n = 1$ for $n = 1, \cdots, N$, $a_n = 0$ for $n = 0, -1, \cdots, -N$ in Theorem 1.1, one obtains

$$\int_{\mathbb{T}^k} \Big| \sum_{n=1}^{N} e^{2\pi i(\alpha_1 n + \cdots + \alpha_k n^k)} \Big|^p d\alpha \lesssim_\varepsilon N^{p-k(k+1)+\varepsilon} \tag{1.7}$$

for $p \geq k(k+1)$, which is Vinogradov's mean value theorem proved by Bourgain, Demeter and Guth [4] recently. (1.5) can be regarded as a weighted version of (1.7) and (1.5) is apparently harder than (1.7). Notice that the curve $(t^{k_1}, t^{k_2}, \cdots, t^{k_d})$ may be degenerate, for example, the curve (t, t^3) has zero curvature at point $(0, 0)$. It seems to be difficult to use the method developed in [3] and [4] to prove (1.6) for $p \geq 2\mathfrak{K}$, since what they deal with are hypersurfaces with nonzero Gaussian curvature or nondegenerate curve. The proof of Theorem 1.1 is based on a key lemma from [4]. Bourgain et al. [4] used this lemma to prove (1.7).

Throughout this paper, the letter C stands for a positive constant and C_a denotes a constant depending on a. $A \lesssim_\varepsilon B$ means $A \leq C_\varepsilon B$ for some constant C_ε. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$. For a set $E \subset \mathbb{R}^d$, we denote the Lebesgue measure of E by $|E|$.

2. Proof of Theorem 1.1

Before giving the proof of Theorem 1.1, we first introduce some lemmas.

Lemma 2.1 (See Theorem 4.1 in [4]). *For each $1 \leq n \leq N$, let t_n be a point in $(\frac{n-1}{N}, \frac{n}{N}]$. Suppose B_R is a ball in $\mathbb{R}^d$ with center c_B and radius R. Define $w_{B_R}(x) = \Big(1 + \frac{|x - c_B|}{R}\Big)^{-200}$. Then for each $R \gtrsim N^d$, each ball B_R in $\mathbb{R}^d$, each $a_n \in \mathbb{C}$, each $p \geq 2$ and $\varepsilon > 0$, we have*

$$\begin{aligned} &\Big(\frac{1}{|B_R|} \int \Big| \sum_{n=1}^{N} a_n e^{2\pi i(x_1 t_n + \cdots + x_d t_n^d)} \Big|^p w_{B_R}(x) dx\Big)^{\frac{1}{p}} \\ &\qquad \lesssim_\varepsilon \Big(N^\varepsilon + N^{\frac{1}{2}(1 - \frac{d(d+1)}{p}) + \varepsilon}\Big)\Big(\sum_{n=1}^{N} |a_n|^2\Big)^{\frac{1}{2}}, \end{aligned} \tag{2.1}$$

where the implicit constant does not depend on N, R, and a_n.

Lemma 2.2. *Suppose $a_n \in \mathbb{C}$ and $p \geq 2$. Then for any $\varepsilon > 0$, we get*

$$\begin{aligned} &\Big(\int_{\mathbb{T}^d} \Big| \sum_{|n| \leq N} a_n e^{2\pi i(x_1 n + x_2 n^2 + \cdots + x_d n^d)} \Big|^p dx\Big)^{\frac{1}{p}} \\ &\qquad \lesssim_\varepsilon \Big(N^\varepsilon + N^{\frac{1}{2}(1 - \frac{d(d+1)}{p}) + \varepsilon}\Big)\Big(\sum_{|n| \leq N} |a_n|^2\Big)^{\frac{1}{2}}, \end{aligned} \tag{2.2}$$

where the implicit constant is independent of N and a_n.

Proof. We first notice that the function

$$\sum_{|n| \leq N} a_n e^{2\pi i(x_1 n + x_2 n^2 + \cdots + x_d n^d)}$$

is periodic with period 1 in the variables $x_1, \cdots, x_d$. By using Minkowski's inequality, making a change of variables and the above periodic fact, one may get

$$\begin{aligned}&\Big(\int_{\mathbb{T}^d}\Big|\sum_{|n|\le N} a_n e^{2\pi i(x_1 n+x_2 n^2+\cdots+x_d n^d)}\Big|^p dx\Big)^{\frac{1}{p}}\\ &\le a_0+\Big(\int_{\mathbb{T}^d}\Big|\sum_{n=1}^{N} a_n e^{2\pi i(x_1 n+x_2 n^2+\cdots+x_d n^d)}\Big|^p dx\Big)^{\frac{1}{p}}\\ &\quad+\Big(\int_{\mathbb{T}^d}\Big|\sum_{n=1}^{N} a_{-n} e^{2\pi i(x_1 n+x_2 n^2+\cdots+x_d n^d)}\Big|^p dx\Big)^{\frac{1}{p}}.\end{aligned}$$

Hence, to prove (2.2), it suffices to show that

$$\Big(\int_{\mathbb{T}^d}\Big|\sum_{n=1}^{N} a_n e^{2\pi i(x_1 n+x_2 n^2+\cdots+x_d n^d)}\Big|^p dx\Big)^{\frac{1}{p}}$$

has the desired bound. Applying Lemma 2.1 with $R=\sqrt{d}N^d$, $t_n=\frac{n}{N}$ and $B_R=B(0,R)$ which is centred at 0, we may obtain

$$\begin{aligned}&\Big(N^{-d^2}\int\Big|\sum_{n=1}^{N} a_n e^{2\pi i(x_1\frac{n}{N}+\cdots+x_d(\frac{n}{N})^d)}\Big|^p w_{B_R}(x)dx\Big)^{\frac{1}{p}}\\ &\lesssim_\varepsilon \Big(N^\varepsilon+N^{\frac{1}{2}(1-\frac{d(d+1)}{p})+\varepsilon}\Big)\Big(\sum_{n=1}^{N}|a_n|^2\Big)^{\frac{1}{2}}.\end{aligned}\tag{2.3}$$

Since $w_{B_R}(x)\approx 1$ on $B(0,R)$ and $[0,N^d]^d\subset B(0,R)$, the left side of (2.3) is larger than

$$\Big(N^{-d^2}\int_{[0,N^d]^d}\Big|\sum_{n=1}^{N} a_n e^{2\pi i(x_1\frac{n}{N}+\cdots+x_d(\frac{n}{N})^d)}\Big|^p dx\Big)^{\frac{1}{p}}.$$

By making a change of variables, $x_1=N\alpha_1,\cdots,x_d=N^d\alpha_d$, the above integral equals

$$\Big(N^{-d^2+\frac{d(d+1)}{2}}\int_{A_N}\Big|\sum_{n=1}^{N} a_n e^{2\pi i(\alpha_1 n+\cdots+\alpha_d n^d)}\Big|^p d\alpha\Big)^{\frac{1}{p}},\tag{2.4}$$

where $A_N=[0,N^{d-1}]\times[0,N^{d-2}]\times\cdots\times[0,1]$. Notice that the function

$$K_N(\alpha)=\sum_{n=1}^{N} a_n e^{2\pi i(\alpha_1 n+\alpha_2 n^2+\cdots+\alpha_d n^d)}$$

is periodic with period 1 in the variables $\alpha_1,\cdots,\alpha_d$. Since A_N has $N^{\frac{d(d-1)}{2}}$ number of unit cubes, by the periodic fact of $K_N(\alpha)$, it follows that (2.4) is equal to

$$\Big(\int_{\mathbb{T}^d}\Big|\sum_{n=1}^{N} a_n e^{2\pi i(\alpha_1 n+\cdots+\alpha_d n^d)}\Big|^p d\alpha\Big)^{\frac{1}{p}},$$

which completes the proof. □

Now we begin with the proof of Theorem 1.1. We first show that the proof can be reduced to the case $p_k = k(k+1)$, that is,

$$\Big\| \sum_{|n|\le N} a_n e^{2\pi i(\alpha_1 n^{k_1}+\cdots+\alpha_d n^{k_d})}\Big\|_{L^{p_k}(\mathbb{T}^d)} \lesssim_\varepsilon N^{\frac{1}{2}(1-\frac{2\mathfrak{K}}{p_k})+\varepsilon}\Big(\sum_{|n|\le N}|a_n|^2\Big)^{\frac{1}{2}}. \tag{2.5}$$

Suppose (2.5) is true. Utilizing the Cauchy-Schwarz inequality, we get

$$\Big\| \sum_{|n|\le N} a_n e^{2\pi i(\alpha_1 n^{k_1}+\cdots+\alpha_d n^{k_d})}\Big\|_{L^{\infty}(\mathbb{T}^d)} \lesssim N^{\frac{1}{2}}\Big(\sum_{|n|\le N}|a_n|^2\Big)^{\frac{1}{2}}.$$

By using the Riesz-Thorin interpolation theorem (see, for example, [6]) to interpolate (2.5) and the above L^∞ estimate, one could easily get the required bound of the L^p estimate for $p \ge k(k+1)$ in Theorem 1.1. Therefore, it remains to show (2.5). Consider positive integers $k_1, \cdots, k_d$ with $1 \le k_1 < \cdots < k_d = k$ and denote by $\{l_1, \cdots, l_s\}$ the complement set of $\{k_1, \cdots, k_d\}$ in $\{1, 2, \cdots, k\}$. Set $\mathfrak{K} = \sum_{n=1}^d k_n$. Then we may see

$$\sum_{i=1}^{s} l_i = \frac{1}{2}k(k+1) - \mathfrak{K}. \tag{2.6}$$

Note that $p_k = k(k+1)$ is an even integer; therefore, we may set $p_k = 2u$. By using the simple fact $\int_0^1 e^{2\pi i x y}dy = \delta(x)$, here δ is a Dirac measure at 0 and we have

$$\begin{aligned}
\Lambda &:= \int_{\mathbb{T}^d}\Big|\sum_{|n|\le N} a_n e^{2\pi i(\alpha_1 n^{k_1}+\cdots+\alpha_d n^{k_d})}\Big|^{2u} d\alpha \\
&= \int_{\mathbb{T}^d}\Big(\sum_{|n|\le N} a_n e^{2\pi i(\alpha_1 n^{k_1}+\cdots+\alpha_d n^{k_d})}\cdot \sum_{|m|\le N}\overline{a_m} e^{-2\pi i(\alpha_1 m^{k_1}+\cdots+\alpha_d m^{k_d})}\Big)^u d\alpha \\
&= \sum_{|n_1|,\cdots,|n_u|\le N,}\ \sum_{|m_1|,\cdots,|m_u|\le N} a_{n_1}\cdots a_{n_u}\overline{a_{m_1}}\cdots\overline{a_{m_u}} \\
&\qquad\times \delta\Big(\sum_{i=1}^{u}(n_i^{k_1}-m_i^{k_1})\Big)\cdots\delta\Big(\sum_{i=1}^{u}(n_i^{k_d}-m_i^{k_d})\Big).
\end{aligned} \tag{2.7}$$

Thus (2.7) equals the number of integral solutions of the system of equations

$$\begin{cases} \sum_{i=1}^{u}(n_i^{k_1}-m_i^{k_1}) = 0, \\ \quad\cdots \\ \sum_{i=1}^{u}(n_i^{k_d}-m_i^{k_d}) = 0, \\ |n_i| \le N, |m_i| \le N, i = 1, \cdots, u, \end{cases} \tag{2.8}$$

with each solution counted with weight $a_{n_1}\cdots a_{n_u}\overline{a_{m_1}}\cdots\overline{a_{m_u}}$.

For each solution $(n_1, \cdots, n_u, m_1, \cdots, m_u)$ of (2.8), there exist integers $h_j, j = 1, \cdots, k$, such that $(n_1, \cdots, n_u, m_1, \cdots, m_u)$ is an integral solution of the following system of equations:

$$
\begin{cases}
\sum_{i=1}^{u}(n_i - m_i) = h_1, \\
\sum_{i=1}^{u}(n_i^2 - m_i^2) = h_2, \\
\cdots \\
\sum_{i=1}^{u}(n_i^k - m_i^k) = h_k, \\
|n_i| \le N, |m_i| \le N, i = 1, \cdots, u,
\end{cases}
\tag{2.9}
$$

where $h_j = 0$ if $j = k_i$ for some $i = 1, \cdots, d$. By the last condition of (2.9), it is easy to see that $|h_j| \le 2uN^j$ for $j = 1, \cdots, k$.

On the other hand, for each integral solution $(n_1, \cdots, n_u, m_1, \cdots, m_u)$ of (2.9) with $|h_j| \le 2uN^j$ for $j = 1, \cdots, k$ and $h_j = 0$ if $j = k_i$ for some $1 \le i \le d$, $(n_1, \cdots, n_u, m_1, \cdots, m_u)$ is also an integral solution of (2.8). Now we define

$$\Lambda(h) = \int_{\mathbb{T}^k} \Big| \sum_{|n| \le N} a_n e^{2\pi i(\alpha_1 n + \alpha_2 n^2 + \cdots + \alpha_k n^k)} \Big|^{2u} e^{2\pi i(-\alpha_1 h_1 - \cdots - \alpha_k h_k)} d\alpha.$$

By using orthogonality, the above term is equal to

$$\sum_{|n_1|, \cdots, |n_u| \le N} \sum_{|m_1|, \cdots, |m_u| \le N} a_{n_1} \cdots a_{n_u} \overline{a_{m_1}} \cdots \overline{a_{m_u}}$$
$$\times \delta\Big(\sum_{i=1}^{u} (n_i - m_i) - h_1 \Big) \cdots \delta\Big(\sum_{i=1}^{u} (n_i^k - m_i^k) - h_k \Big),$$

which counts the number of integral solution of (2.9) with each solution counted with weight $a_{n_1} \cdots a_{n_u} \overline{a_{m_1}} \cdots \overline{a_{m_u}}$. Combining the above arguments, we conclude that

$$\Lambda = \sum_{|h_{l_1}| \le 2uN^{l_1}} \cdots \sum_{|h_{l_s}| \le 2uN^{l_s}} \Lambda(h),$$

where h in the sum also satisfies $h_j = 0$ if $j = k_i$ for some $i = 1, \cdots, d$. Obviously, $|\Lambda(h)| \le \Lambda(0)$. Hence we obtain

$$|\Lambda| \le \sum_{|h_{l_1}| \le 2uN^{l_1}} \cdots \sum_{|h_{l_s}| \le 2uN^{l_s}} \Lambda(0) \le (2u)^s N^{l_1 + \cdots + l_s} \Lambda(0)$$
$$\lesssim N^{\frac{1}{2}k(k+1) - \mathfrak{K}} N^{p_k \varepsilon} \Big(\sum_{|n| \le N} |a_n|^2 \Big)^{\frac{p_k}{2}},$$

where in the last inequality we use (2.6) and apply Lemma 2.2 with $p = k(k+1)$. Hence we establish (2.5) which completes the proof of Theorem 1.1.

3. Sharpness of Theorem 1.1

In this section, we show that $N^{\frac{1}{2}(1 - \frac{2\mathfrak{K}}{p})}$ is the best upper bound for $K_{p,d,N}$ when $p \ge 2\mathfrak{K}$. Therefore Theorem 1.1 is sharp up to a factor of N^{ε}.

Proposition 3.1. *Let $K_{p,d,N}$ be defined in (1.5). Suppose p is an even integer. Then there exist constants C_1, C_2 such that*

$$K_{p,d,N} \ge \max \Big\{ C_1, C_2 N^{\frac{1}{2}(1 - \frac{2\mathfrak{K}}{p})} \Big\}.$$

Proof. Set $p = 2u$. Let $1 \le k_1 < k_2 < \cdots < k_d$ and $\mathfrak{K} = k_1 + \cdots + k_d$. Define

$$\Lambda(N, 2u) = \int_{\mathbb{T}^d} \Big| \sum_{|n|\le N} e^{2\pi i(\alpha_1 n^{k_1} + \cdots + \alpha_d n^{k_d})} \Big|^{2u} d\alpha.$$

By using orthogonality, $\Lambda(N, 2u)$ counts the number of integral solutions of the following system of equations:

$$\begin{cases} \sum_{i=1}^{u} (n_i^{k_1} - m_i^{k_1}) = 0, \\ \cdots \\ \sum_{i=1}^{u} (n_i^{k_d} - m_i^{k_d}) = 0, \\ |n_i| \le N, |m_i| \le N, i = 1, \cdots, u, \end{cases} \tag{3.1}$$

Notice that the system of equations (3.1) has $(2N+1)^u$ number of trivial solutions. In fact, for each $(n_1, \cdots, n_u)$ with $|n_i| \le N$, $i = 1, 2, \cdots, u$, one may choose $(m_1, \cdots, m_u) = (n_1, \cdots, n_u)$. Hence we have

$$\Lambda(N, 2u) \ge CN^{\frac{p}{2}}. \tag{3.2}$$

Define the set Ω_N as

$$\Omega_N = \Big\{ \alpha \in \mathbb{T}^d : |\alpha_i| \le \frac{1}{8dN^{k_i}}, i = 1, \cdots, d \Big\}.$$

Then we have $|\Omega_N| \approx N^{-\mathfrak{K}}$. If $\alpha \in \Omega_N$ and $|n| \le N$, then

$$\begin{aligned} \Big| \sum_{|n|\le N} e^{2\pi i(\alpha_1 n^{k_1} + \cdots + \alpha_d n^{k_d})} \Big| &\ge \Big| Re \sum_{|n|\le N} e^{2\pi i(\alpha_1 n^{k_1} + \cdots + \alpha_d n^{k_d})} \Big| \\ &\ge \sum_{|n|\le N} \cos(2\pi(\alpha_1 n^{k_1} + \cdots + \alpha_d n^{k_d})) \ge CN. \end{aligned}$$

Now we conclude that

$$\Lambda(N, 2u) \ge \int_{\Omega_N} \Big| \sum_{|n|\le N} e^{2\pi i(\alpha_1 n^{k_1} + \cdots + \alpha_d n^{k_d})} \Big|^{2u} d\alpha \ge CN^p |\Omega_N| \ge CN^{p-\mathfrak{K}}. \tag{3.3}$$

Recall $K_{p,d,N}$ is the best constant for the following inequality:

$$\Big\| \sum_{|n|\le N} a_n e^{2\pi i(\alpha_1 n^{k_1} + \cdots + \alpha_d n^{k_d})} \Big\|_{L^p(\mathbb{T}^d)} \le K_{p,d,N} \Big(\sum_{|n|\le N} |a_n|^2 \Big)^{\frac{1}{2}}.$$

Choosing $a_n = 1$ for all $|n| \le N$, then we have $K_{p,d,N} \ge N^{-\frac{1}{2}} (\Lambda(N, p))^{\frac{1}{p}}$. Combining the estimates (3.2) and (3.3), we may get

$$K_{p,d,N} \ge \max \Big\{ C_1, C_2 N^{\frac{1}{2}(1 - \frac{2\mathfrak{K}}{p})} \Big\}$$

which completes the proof. ∎

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