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STRONGLY ERGODIC ACTIONS HAVE LOCAL SPECTRAL GAP

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Abstract. We show that an ergodic measure preserving action $\Gamma \curvearrowright (X,\mu)$ of a discrete group Γ on a σ-finite measure space (X,μ) satisfies the local spectral gap property introduced in Invent. Math. 208 (2017), 715–802, if and only if it is strongly ergodic. In fact, we prove a more general local spectral gap criterion in arbitrary von Neumann algebras. Using this criterion, we also obtain a short proof of Connes' spectral gap theorem for full II_1 factors as well as its recent generalization to full type III factors.

1. Introduction

The main goal of this paper is to show that the local spectral gap property introduced in [BISG15] is, in fact, equivalent to strong ergodicity for arbitrary ergodic measure preserving actions. Before we give a precise statement, let us first recall the notion of local spectral gap.

Definition. Let $\Gamma \curvearrowright (X,\mu)$ be an ergodic measure preserving action of a discrete group Γ on a σ-finite measure space (X,μ). Let $B \subset X$ be a measurable subset with $0 < \mu(B) < +\infty$. We say that $\Gamma \curvearrowright (X,\mu)$ has *local spectral gap* with respect to B if there exists a finite set $S \subset \Gamma$ and a constant $\kappa > 0$ such that

$$\|f\|_{2,B} \leq \kappa \sum_{g \in S} \|g \cdot f - f\|_{2,B}$$

for every function $f \in \mathrm{L}^2(X,\mu)$ such that $\int_B f \, \mathrm{d}\,\mu = 0$, where

$$\|f\|_{2,B} := \left(\int_B |f|^2 \, \mathrm{d}\,\mu \right)^{1/2}.$$

This local spectral gap property depends in general on the choice of the set B (see [BISG15, Remark 1.3.(4)]). When μ is a probability measure and $B = X$, we just get the usual notion of spectral gap for probability measure preserving actions.

It is well-known that, for probability measure preserving actions, spectral gap implies strong ergodicity while the converse is not true. However, our main result shows that the local spectral gap property introduced in [BISG15] is equivalent to strong ergodicity for any ergodic measure preserving action.

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Theorem A. *Let $\Gamma \curvearrowright (X,\mu)$ be an ergodic measure preserving action of a discrete group Γ on a σ-finite measure space (X,μ). Then the following are equivalent:*

1. *The action $\Gamma \curvearrowright (X,\mu)$ is strongly ergodic.*
2. *There exists a subset $B \subset X$ with $0 < \mu(B) < +\infty$ such that the action $\Gamma \curvearrowright (X,\mu)$ has local spectral gap with respect to B.*

Theorem A is deduced from the following more general local spectral gap criterion in tracial von Neumann algebras. For a non-tracial version, see Theorem 2.1.

Theorem B. *Let (M,τ) be a tracial von Neumann algebra. Let $N \subset M$ be any von Neumann subalgebra and let $\Sigma \subset M$ be any self-adjoint subset. Suppose that for every bounded net $(x_i)_{i\in I}$ in N which satisfies $\lim_i \|x_i a - a x_i\|_2 = 0$ for all $a \in \Sigma$, we have $\lim_i \|x_i - \tau(x_i)\|_2 = 0$.*

Then we can find a non-zero projection $p \in N$, a finite subset $S \subset \Sigma$, and a constant $\kappa > 0$ such that for every $x \in pNp$ with $\tau(x) = 0$, we have

$$\|x\|_2 \leq \kappa \sum_{a\in S} \|x(pap) - (pap)x\|_2.$$

Moreover, we can choose p to be arbitrarily close to 1.

We provide other applications of Theorem B in the last section of this article.

2. The local spectral gap property in arbitrary von Neumann algebras

In this section, we prove Theorem B. In fact, we prove the following more general version for arbitrary von Neumann algebras (not necessarily tracial). For any faithful normal state φ on a von Neumann algebra M, we will say that a vector $\xi \in \mathrm{L}^2(M)$ is *φ-bounded* if $\xi \in M\xi_\varphi$ where $\xi_\varphi \in \mathrm{L}^2(M)$ is the cyclic vector associated to φ.

Theorem 2.1. *Let M be any von Neumann algebra with a faithful normal state φ. Let $N \subset M$ be any von Neumann subalgebra and let $\Sigma = J\Sigma \subset \mathrm{L}^2(M)$ be any self-adjoint subset of φ-bounded elements. Suppose that for every bounded net $(x_i)_{i\in I}$ in N which satisfies $\lim_i \|x_i\xi - \xi x_i\| = 0$ for all $\xi \in \Sigma$, we have $\lim_i \|x_i - \varphi(x_i)\|_\varphi = 0$.*

Then we can find a non-zero projection $p \in N$, a finite subset $S \subset \Sigma$, and a constant $\kappa > 0$ such that for every $x \in pNp$ with $\varphi(x) = 0$ we have

$$\|x\|_\varphi \leq \kappa \sum_{\xi\in S} \|p(x\xi - \xi x)p\|.$$

Moreover, we can chose p to be arbitrarily close to 1.

Proposition 2.2. *Let M be any von Neumann algebra and let φ be any normal state. Take $S \subset \mathrm{L}^2(M)$ any finite self-adjoint subset of φ-bounded elements. Let N be any von Neumann subalgebra of M. Then the following are equivalent:*

1. *There exists a constant $\kappa > 0$ such that for any projection $p \in N$ we have*

$$\varphi(p)\varphi(1-p) \leq \kappa \sum_{\xi\in S} \|p\xi - \xi p\|^2.$$

(2) *There exists a constant* $\kappa' > 0$ *such that for all* $x \in N$ *we have*

$$\|x - \varphi(x)\|_\varphi \leq \kappa' \sum_{\xi \in S} \|x\xi - \xi x\|.$$

The proof of Proposition 2.2 relies on the following lemma, which is inspired by Namioka's trick. Our new input is item (2).

Lemma 2.3. *Let* M *be any von Neumann algebra. For* $a \geq 0$ *and* $x \in M^+$*, we use the notation* $E_a(x) = 1_{[a,+\infty)}(x)$*.*

(1) *For every positive elements* $x, y \in M^+$ *and every* $\xi \in \mathrm{L}^2(M)$*, we have*

$$\int_0^\infty \|E_a(x^2)\xi - \xi E_a(y^2)\|^2 \, \mathrm{d}\, a \leq \|x\xi + \xi y\| \cdot \|x\xi - \xi y\|.$$

(2) *For every positive element* $x \in M^+$ *and every state* $\varphi \in M_*^+$*, we have*

$$\|x - \varphi(x)\|_\varphi^2 \leq \int_0^\infty \varphi(E_a(x^2))\varphi\left(1 - E_a(x^2)\right) \mathrm{d}\, a.$$

Proof. For item (1), see the proof of [CS78, Theorem 2]. We will only prove (2). First, note that $\|x - \varphi(x)\|_\varphi^2 = \varphi(x^2) - \varphi(x)^2$ and that

$$\varphi(x^2) = \int_0^\infty \varphi(E_a(x^2)) \, \mathrm{d}\, a.$$

Therefore, we only have to show that

$$\int_0^\infty \varphi(E_a(x^2))^2 \, \mathrm{d}\, a \leq \varphi(x)^2.$$

On $M \mathbin{\overline{\otimes}} M$, we have $E_a(x^2) \otimes E_a(x^2) \leq E_a(x \otimes x)$. Hence, by applying $\varphi \otimes \varphi$ we get

$$\varphi(E_a(x^2))^2 \leq (\varphi \otimes \varphi)(E_a(x \otimes x)),$$

and thus, after integrating over a, we finally get

$$\int_0^\infty \varphi(E_a(x^2))^2 \, \mathrm{d}\, a \leq (\varphi \otimes \varphi)(x \otimes x) = \varphi(x)^2.$$

□

Proof of Proposition 2.2. (2) clearly implies (1) when applied to a projection.

Now assume that (1) holds for some constant $\kappa > 0$, and let us show that (2) also holds for some constant $\kappa' > 0$. A direct application of Lemma 2.3 shows that for every element $x \in N^+$, we have

$$\|x - \varphi(x)\|_\varphi^2 \leq \kappa \sum_{\xi \in S} \|x\xi + \xi x\| \cdot \|x\xi - \xi x\|.$$

Since the elements of S are φ-bounded and S is self-adjoint, we can find a constant $C > 0$ such that for all $x \in N^+$ and $\xi \in S$, we have $\|x\xi + \xi x\| \leq \|x\xi\| + \|\xi x\| \leq C\|x\|_\varphi$. Hence, we get

$$\forall x \in N^+, \quad \|x - \varphi(x)\|_\varphi^2 \leq C\kappa \|x\|_\varphi \sum_{\xi \in S} \|x\xi - \xi x\|.$$

Now, for every $x = x^* \in N$ with $\varphi(x) = 0$, write $x = x_+ - x_-$ where $x_+, x_- \in N^+$ and $x_+ x_- = 0$. Then we have

$$\begin{aligned}\|x\|_\varphi^2 &\leq 2(\|x_+ - \varphi(x_+)\|_\varphi^2 + \|x_- - \varphi(x_-)\|_\varphi^2) \\ &\leq 2C\kappa \|x\|_\varphi \sum_{\xi \in S} (\|x_+\xi - \xi x_+\| + \|x_-\xi - \xi x_-\|),\end{aligned}$$

and since $\|x_\pm \xi - \xi x_\pm\| \leq \|x\xi - \xi x\|$, we obtain

$$\|x\|_\varphi \leq 4C\kappa \sum_{\xi \in S} \|x\xi - \xi x\|.$$

By applying this inequality to $x - \varphi(x)$, we obtain (2) for every self-adjoint $x \in N$. Finally, since S is self-adjoint, it is easy to obtain (2) for every element $x \in N$ by decomposing it into its real and imaginary parts. □

Proof of Theorem 2.1. Fix $0 < \varepsilon < \frac{1}{4}$. Then, there exists a finite self-adjoint subset $S \subset \Sigma$ and $\eta > 0$ such that for every projection $p \in N$ we have

$$\sum_{\xi \in S} \|p\xi - \xi p\|^2 \leq \eta \implies \min(\varphi(p), \varphi(1-p)) \leq \varepsilon.$$

Consider the set Λ of all projections e in N such that $\varphi(e) \leq \varepsilon$ and

$$\sum_{\xi \in S} \|e\xi - \xi e\|^2 \leq \eta \varphi(e).$$

Then Λ is closed for the strong topology; hence it is an inductive set. Therefore, by Zorn's lemma, we can choose e a maximal element of Λ. Let $p = 1 - e$. Take a projection $f \in pNp$. Suppose that

$$\varphi(f)\varphi(p-f) > \frac{1}{\eta} \sum_{\xi \in S} \|f(p\xi p) - (p\xi p)f\|^2.$$

Then up to replacing f by $p - f$, we can suppose that $\varphi(f) \leq \frac{1}{2}$. Now let $q = e + f$. Then we can check that

$$\sum_{\xi \in S} \|q\xi - \xi q\|^2 \leq \sum_{\xi \in S} \|e\xi - \xi e\|^2 + \sum_{\xi \in S} \|f(p\xi p) - (p\xi p)f\|^2 \leq \eta\varphi(e) + \eta\varphi(f) = \eta\varphi(q).$$

But, since e is maximal in Λ, we know that $q \notin \Lambda$. Hence we must have $\varphi(q) > \varepsilon$. Therefore, by the choice of S and η, we must have $\varphi(q) \geq 1 - \varepsilon$. But $\varphi(q) = \varphi(e) + \varphi(f) \leq \varepsilon + \frac{1}{2}$. Since $\varepsilon < \frac{1}{4}$, this is a contradiction. Hence, for all projections $f \in pNp$, we have

$$\varphi(f)\varphi(p-f) \leq \frac{1}{\eta} \sum_{\xi \in S} \|f(p\xi p) - (p\xi p)f\|^2.$$

Finally, for $\varphi' = \frac{1}{\varphi(p)} p\varphi p$ and $S' = pSp$, we can use Proposition 2.2 to conclude that there exists $\kappa' > 0$ such that

$$\forall x \in pNp, \quad \|x - \varphi'(x)\|_{\varphi'} \leq \kappa' \sum_{\xi \in S} \|x(p\xi p) - (p\xi p)x\|.$$

□

3. Applications

We first prove our main theorem, which motivated Theorem B.

Proof of Theorem A. We only have to prove that (1) implies (2), since the converse is contained in [BISG15, Theorem 7.6]. Let $A \subset X$ be a subset with $\mu(A) = 1$. Let $q = \mathbf{1}_A \in \mathrm{L}^\infty(X,\mu)$. Let $M = q(\mathrm{L}^\infty(X,\mu) \rtimes \Gamma)q$, $N = q\mathrm{L}^\infty(X,\mu)$, and $\Sigma = \{qu_g q \mid g \in \Gamma\} \subset M$. By strong ergodicity of the action, we know that every bounded Σ-central net in N is trivial. Hence, by Theorem B, we can find a non-zero projection $p \in N$, a finite subset $S \subset \Gamma$, and a constant $\kappa > 0$ such that for all $x \in pNp$ with $\tau(x) = 0$ we have

$$\|x\|_2 \leq \kappa \sum_{g \in S} \|x(pu_g p) - (pu_g p)x\|_2.$$

Now, take $x \in \mathrm{L}^\infty(X,\mu)$ with $\tau(px) = 0$. By applying the previous inequality to $px \in pNp$ we obtain

$$\|px\|_2 \leq \kappa \sum_{g \in S} \|p(xu_g - u_g x)p\|_2 \leq \kappa \sum_{g \in S} \|p(x - u_g x u_g^*)\|_2,$$

and this is exactly the local spectral gap property with respect to $B \subset X$ where $p = \mathbf{1}_B$. □

Our next application is a new proof and a generalization of [Co75b, Theorem 2.1]. Note that the original proof of [Co75b, Theorem 2.1] can be simplified so that it does not rely on singular states anymore (see [PoAD, Theorem 15.2.4]). By using Theorem B, we obtain an even shorter proof which does not rely on ultraproducts.

Theorem 3.1. *Let N be a II_1 factor and let $\sigma : \Gamma \curvearrowright N$ be an action of a discrete group Γ. Suppose that for every bounded net $(x_i)_{i \in I}$ in N which satisfies $\lim_i \|x_i a - a x_i\|_2$ for all $a \in N$ and $\lim_i \|\sigma_g(x_i) - x_i\|_2 = 0$ for all $g \in \Gamma$, we have $\lim_i \|x_i - \tau(x_i)\|_2 = 0$.*

Then there exists a finite set of unitaries $S \subset \mathcal{U}(N)$, a finite set $K \subset \Gamma$, and a constant $\kappa > 0$ such that for all $x \in N$ we have

$$\|x - \tau(x)\|_2 \leq \kappa \left(\sum_{u \in S} \|ux - xu\|_2 + \sum_{g \in K} \|\sigma_g(x) - x\|_2 \right).$$

Proof. Let $M = N \rtimes_\sigma \Gamma$. Then M is finite since σ preserves the unique trace of N. Let $\Sigma = \mathcal{U}(N) \cup \{u_g \mid g \in \Gamma\}$. A direct application of Theorem B shows that we can find a projection $p \in N$ with $\tau(p) > \frac{1}{2}$, finite sets $S \subset \mathcal{U}(N)$ and $K \subset \Gamma$, and a constant $\kappa > 0$ such that for all $x \in pNp$ with $\tau(x) = 0$ we have

$$\|x\|_2 \leq \kappa \left(\sum_{u \in S} \|p(ux - xu)p\|_2 + \sum_{g \in K} \|p(u_g x - x u_g)p\|_2 \right).$$

Let $v = 2p - 1 \in \mathcal{U}(N)$ and let $w \in \mathcal{U}(N)$ be any unitary which satisfies $w(1-p)w^* \leq p$. Let $S' = S \cup \{v, w\}$. Then it is not hard to check that there exists some constant $\kappa' > 0$ such that for all $x \in N$ we have

$$\|x - \tau(x)\|_2 \leq \kappa' \left(\sum_{u \in S'} \|ux - xu\|_2 + \sum_{g \in K} \|\sigma_g(x) - x\|_2 \right).$$

□

We also obtain a new proof of [Ma16, Theorem A], with a more precise conclusion regarding the choice of the state.

Theorem 3.2. *Let M be a full σ-finite type* III *factor. Then there exists a faithful normal state φ on M, a finite set of positive φ-bounded elements $S \subset \mathrm{L}^2(M)$, and a constant $\kappa > 0$ such that*

$$\forall x \in M, \quad \|x - \varphi(x)\|_\varphi \leq \kappa \sum_{\xi \in S} \|x\xi - \xi x\|.$$

Moreover, if ψ is any given faithful normal state on M, we can choose φ to be of the form $\varphi = \frac{1}{\psi(vv^*)} v^* \psi v$ where $v \in M$ is an isometry which is arbitrarily close to 1.

Proof. Let ψ be any faithful normal state on M. Let $\Sigma \subset \mathrm{L}^2(M)$ be the set of all positive ψ-bounded vectors. Note that Σ spans a dense linear subspace of $\mathrm{L}^2(M)$ (use for example the density of ψ-analytic elements in M). Hence, since M is full, we know that every uniformly bounded net $(x_i)_{i \in I}$ which centralizes Σ is trivial. Therefore, by Theorem 2.1, we can find a non-zero projection $p \in M$ and a finite subset $T \subset \Sigma$ such that pTp has spectral gap in $(pMp, \frac{1}{\psi(p)} p\psi p)$. But since M is of type III, we can find an isometry $v \in M$ such that $vv^* = p$. Then, it is easy to see that for $\varphi := \frac{1}{\psi(p)} v^* \psi v$, the set $S = v^* T v$ is φ-bounded and has spectral gap in (M, φ). Finally, since p can be chosen arbitrarily close to 1, v can also be chosen arbitrarily close to 1. □

Finally, we conclude with a proposition which emphasizes the difference between the local spectral gap and true spectral gap in II_1 factors. Note that if Γ is not inner amenable, then $\mathcal{L}(\Gamma)$ is full, [Ef75] but the converse is not true [Va09].

Proposition 3.3. *Let Γ be a group with infinite conjugacy classes. Let $M = \mathcal{L}(\Gamma)$ be the associated II_1 factor. If M is full, then we can find a projection $p \in M$, arbitrarily close to* 1*, a finite set $K \subset \Gamma$, and a constant $\kappa > 0$ such that for all $x \in pMp$ with $\tau(x) = 0$ we have*

$$\|x\|_2 \leq \kappa \sum_{g \in K} \|p(xu_g - u_g x)p\|_2.$$

Moreover, one can choose $p = 1$ if and only if Γ is not inner amenable.

Proof. It is a direct application of Theorem B with $\Sigma = \{u_g \mid g \in \Gamma\}$. □

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