PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 9, September 2018, Pages 3715–3720
http://dx.doi.org/10.1090/proc/14038
Article electronically published on May 24, 2018

ON THE MATLIS DUALS OF LOCAL COHOMOLOGY MODULES

GENNADY LYUBEZNIK AND TUĞBA YILDIRIM

(Communicated by Irena Peeva)

Abstract. Let $(R, \mathfrak{m})$ be a Noetherian regular local ring of characteristic $p > 0$ and let I be a nonzero ideal of R. Let $D(-) = \mathrm{Hom}_R(-, E)$ be the Matlis dual functor, where $E = E_R(R/\mathfrak{m})$ is the injective hull of the residue field $R/\mathfrak{m}$. In this short note, we prove that if $H^i_I(R) \neq 0$, then $\mathrm{Supp}_R(D(H^i_I(R))) = \mathrm{Spec}(R)$.

1. Introduction

Let $(R, \mathfrak{m})$ be a Noetherian local commutative ring with unity, let I be an ideal of R, and let $E := E_R(R/\mathfrak{m})$ be an R-injective hull of the residue field $R/\mathfrak{m}$. Then for any R-module M, we denote by $H^i_I(M)$ the i-th local cohomology module of M supported in I and by $D(M) := \mathrm{Hom}_R(M, E)$ the Matlis dual of M.

Suppose now that $H^i_I(R) = 0$ for all $i \neq c$ and let $\boldsymbol{x}= \{x_1, x_2, ..., x_c\}$ be a regular sequence in I. Hellus [[3], Corollary 1.1.4] proved that I is a set theoretic complete intersection ideal defined by x_i if and only if x_i form a $D(H^c_I(R))$-regular sequence. Motivated by this result, Hellus studied the associated primes of Matlis duals of the top local cohomology modules and conjectured the following equality:

$$\mathrm{Ass}_R(D(H^c_{(x_1,x_2,\cdots,x_c)}(R))) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid H^c_{(x_1,x_2,\cdots,x_c)}(R/\mathfrak{p}) \neq 0\}.$$

It has been shown that this conjecture holds true in many cases; see, e.g., [2], [5], [6], [7].

Furthermore, Hellus proved that the above conjecture is equivalent to the following condition [[3], Theorem 1.2.3]:

- If $(R, \mathfrak{m})$ is a Noetherian local domain, $c \geq 1$, and $x_1, x_2, \cdots, x_c \in R$, then the implication
  $H^c_{(x_1,x_2,\cdots,x_c)}(R) \neq 0 \implies 0 \in \mathrm{Ass}_R(D(H^c_{(x_1,x_2,\cdots,x_c)}(R)))$
  holds.

We conjecture that if R is regular, then the above implication holds for all nonzero ideals independently of the number of generators, i.e.,

Conjecture 1. *Let $(R, \mathfrak{m})$ be a Noetherian regular local ring, let I be a nonzero ideal of R, and $i \geq 1$. If $H^i_I(R) \neq 0$, then $0 \in \mathrm{Ass}_R(D(H^i_I(R)))$.*

Received by the editors July 3, 2017, and, in revised form, July 6, 2017 and November 27, 2017.

2010 *Mathematics Subject Classification.* Primary 13D45, 13H05.

Key words and phrases. Local cohomology, Matlis duality, F-modules.

The first author gratefully acknowledges NSF support through grant DMS-1500264.

The second author was supported by TÜBİTAK 2214/A Grant Program: 1059B141501072.

Note that Conjecture 1 is not true for nonregular rings. For a concrete example of a Noetherian local ring $(A, \mathfrak{m})$ of dimension > 1 such that $H^1_{\mathfrak{m}}(A) = A/\mathfrak{m}$, hence $0 \notin \mathrm{Ass}_R(D(H^1_{\mathfrak{m}}(A)))$; see [[1], Example 2.4]. The authors would like to thank M. Asgarzadeh for bringing this example to our attention.

We prove the following:

Theorem 1.1. *Let $(R, \mathfrak{m})$ be a complete Noetherian regular local ring of characteristic $p > 0$ and let $\mathcal{M}$ be an F-finite F-module such that $0 \notin \mathrm{Ass}(\mathcal{M})$. Then $0 \in \mathrm{Ass}(D(\mathcal{M}))$.*

We would like to point out that $0 \notin \mathrm{Ass}(\mathcal{M})$ is a necessary condition of Theorem 1.1. Indeed, R itself is an F-finite F-module and $0 \in \mathrm{Ass}(R)$ but $0 \notin \mathrm{Ass}(D(R)) = \mathrm{Ass}(E) = \{\mathfrak{m}\}$.

As an immediate consequence of Theorem 1.1, we obtained the main result of this paper which establishes Conjecture 1 in the equicharacteristic $p > 0$ case:

Corollary 1.2. *Let $(R, \mathfrak{m})$ be a Noetherian regular local ring containing a field of characteristic $p > 0$ and let I be a nonzero ideal of R. If $H^i_I(R) \neq 0$, then*

$$\mathrm{Supp}_R(D(H^i_I(R))) = \mathrm{Spec}(R).$$

2. Preliminaries

In this section, we collect some basic definitions and results about F-module theory and our main reference is [9].

Throughout, R is a commutative Noetherian regular ring of characteristic $p > 0$.

Let $R^{'}$ be the additive group of R regarded as an R-bi-module with the usual left action and with the right R-action defined by $r^{'}r = r^p r^{'}$ for all $r \in R$ and $r^{'} \in R^{'}$. The Frobenius functor

$$F : R - \mathrm{mod} \longrightarrow R - \mathrm{mod}$$

of Peskine-Szpiro [10] is defined by

$$F(M) = R^{'} \otimes_R M$$

$$F(M \xrightarrow{h} N) = (R^{'} \otimes_R M \xrightarrow{id \otimes_R h} R^{'} \otimes_R N)$$

for all R-modules M and all R-module homomorphisms h, where $F(M)$ acquires its R-module structure via the left R-module structure on $R^{'}$.

The iteration of a Frobenius functor on R leads one to the iterated Frobenius functors $F^i(-)$ which are defined for all $i \geq 1$ recursively by $F^1(-) = F(-)$ and $F^{i+1} = F \circ F^i(-)$ for all $i \geq 1$.

Note that the Frobenius functor $F(-)$ is exact [[8], Theorem 2.1]; $F(R) \cong R$ and for any ideal I of R, $F(R/I) = R/I^{[p]}$, where $I^{[p]}$ is the ideal of R generated by p-th powers of all elements of I [[10], I.1.3d].

Note also that if R is a complete local ring, then for any Artinian R-module N, $F(D(N)) = D(F(N))$ [[9], Lemma 4.1] and so $R = F(R) = F(D(E)) = D(F(E))$ implies $F(E) = E$. Then it follows from Remark 1.0.(f) of [9] that for any finitely generated R-module M, $F(D(M)) = D(F(M))$.

Now, for an R-module M, define a Frobenius map $\psi_M : M \longrightarrow F(M)$ on M by $\psi_M(m) := 1 \otimes m \in F(M)$ for all $m \in M$. It is worth pointing out that if $\mathrm{ann}(m) = I \subseteq R$, then $\mathrm{ann}(\psi_M(m)) = I^{[p]}$.

An F-module $\mathcal{M}$ is an R-module equipped with R-module isomorphism $\theta : \mathcal{M} \longrightarrow F(\mathcal{M})$ which we call the structure morphism.

A generating morphism of an F-module $\mathcal{M}$ is an R-module homomorphism $\beta : M \longrightarrow F(M)$, where M is some R-module, such that $\mathcal{M}$ is the limit of the inductive system in the top row of the commutative diagram

$$\begin{array}{ccccccc} M & \xrightarrow{\beta} & F(M) & \xrightarrow{F(\beta)} & F^2(M) & \xrightarrow{F^2(\beta)} & \cdots \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle F(\beta)} & & \downarrow{\scriptstyle F^2(\beta)} & & \\ F(M) & \xrightarrow{F(\beta)} & F^2(M) & \xrightarrow{F^2(\beta)} & F^3(M) & \xrightarrow{F^3(\beta)} & \cdots \end{array}$$

and $\theta : \mathcal{M} \longrightarrow F(\mathcal{M})$, the structure isomorphism of $\mathcal{M}$, is induced by the vertical arrows in this diagram.

If β is an injective map, then the exactness of F implies that all maps in the direct limit system are injective, so that M injects into $\mathcal{M}$. In this case, we shall refer to β as a root morphism of $\mathcal{M}$, and M as a root of $\mathcal{M}$. If $\mathcal{M}$ is an F-module possessing a root morphism $\beta : M \longrightarrow \mathcal{M}$ with M finitely generated, then we say that $\mathcal{M}$ is F-finite. In particular, R, with any F-module structure, is an F-finite module.

3. Proofs

Our aim in this section is to give the proof of Theorem 1.1. But we first need a series of lemmas.

Lemma 3.1. *Let $(R, \mathfrak{m})$ be a complete Noetherian regular local ring containing a field of characteristic $p > 0$ and let $\mathcal{M}$ be an F-finite F-module such that $0 \notin \operatorname{Ass}(\mathcal{M})$. Then the Matlis dual of $\mathcal{M}$, $D(\mathcal{M})$, can be expressed as*

$$D(\mathcal{M}) = \varprojlim(N \xleftarrow{\alpha} F(N) \xleftarrow{F(\alpha)} F^2(N) \xleftarrow{F^2(\alpha)} \cdots),$$

where N is an Artinian R-module and $\alpha : F(N) \longrightarrow N$ is a surjective map such that $\operatorname{Ker}(\alpha : F(N) \to N) \neq 0$.

Proof. Since $\mathcal{M}$ is an F-finite F-module, there exists a root morphism $\beta : M \to F(M)$ with a finitely generated R-module M such that

$$\mathcal{M} = \varinjlim(M \xrightarrow{\beta} F(M) \xrightarrow{F(\beta)} F^2(M) \xrightarrow{F^2(\beta)} \cdots).$$

Then applying Matlis dual functor $D(-) = \operatorname{Hom}_R(-, E_R(R/\mathfrak{m}))$ to $\mathcal{M}$, we obtain

$$D(\mathcal{M}) = \varprojlim(D(M) \xleftarrow{D(\beta)} D(F(M)) \xleftarrow{D(F(\beta))} D(F^2(M)) \xleftarrow{D(F^2(\beta))} \cdots).$$

But then since Frobenius functor commutes with $D(-)$, we can write $D(\mathcal{M})$ as

$$D(\mathcal{M}) = \varprojlim(N \xleftarrow{\alpha} F(N) \xleftarrow{F(\alpha)} F^2(N) \xleftarrow{F^2(\alpha)} \cdots),$$

where $N = D(M)$ and $\alpha = D(\beta)$. Then since β is injective and M is finitely generated, $\alpha = D(\beta)$ is surjective and $N = D(M)$ is Artinian.

On the other hand, since $0 \notin \operatorname{Ass}(\mathcal{M})$, $I = \operatorname{Ann}(M) = \operatorname{Ann}(N)$ is a nonzero ideal of R. Then it follows that $\operatorname{Ann}(F(N)) = I^{[p]}$ and so $\operatorname{Ker}(\alpha : F(N) \to N) \neq 0$, as desired. □

Lemma 3.2. *Let the notation be as in Lemma* 3.1. *Then, for each $k \geq 1$, there exists $b_k \in \operatorname{Ker}(F^{k-1}(\alpha))$ such that* $\operatorname{ann}(b_k) = \mathfrak{m}^{[p^{k-1}]}$.

Proof. Since $\mathrm{Ker}(\alpha : F(N) \to N) \neq 0$ is a nonzero Artinian R-module, there exists an element $b_1 \in \mathrm{Soc}(\mathrm{Ker}(\alpha)) \subseteq F(N)$, where $\mathrm{Soc}(\mathrm{Ker}(\alpha)) := \mathrm{Ann}_{\mathrm{Ker}(\alpha)}(\mathfrak{m})$ denotes the socle of $\mathrm{Ker}(\alpha)$ and define b_k, for all $k \geq 2$, inductively as the image of b_{k-1} under the Frobenius map (defined in the preceding section) on $F^{k-1}(N)$, that is, $b_k := \psi_{F^{k-1}(N)}(b_{k-1}) = 1 \otimes b_{k-1} \in F^k(N)$. Then by induction on k (considering that $\mathrm{ann}(b_1) = \mathfrak{m}$ and $\mathrm{ann}(x) = I$ implies $\mathrm{ann}(\psi(x)) = I^{[p]}$), we have $\mathrm{ann}(b_k) = \mathfrak{m}^{[p^{k-1}]}$. On the other hand, since $b_1 \in \mathrm{Ker}(\alpha) := \mathrm{Ker}(F^0(\alpha))$, an easy induction argument shows that $b_k \in \mathrm{Ker}(F^{k-1}(\alpha))$ for all $k \geq 0$. For if $b_{k-1} \in \mathrm{Ker}(F^{k-2}(\alpha))$, then $F^{k-1}(\alpha)(b_k) = F^{k-1}(\alpha)(1 \otimes b_{k-1}) = 1 \otimes F^{k-2}(\alpha)(b_{k-1}) = 0$. □

Lemma 3.3. *Let the notation be as in Lemma* 3.1 *and let b_k be defined as in Lemma* 3.2 *and $y \in \mathfrak{m} \setminus \mathfrak{m}^k$. Then* $\mathrm{ann}(yb_k) \subseteq \mathfrak{m}^{p^{k-1}-k}$. *In particular, if $k \geq 4$,* $\mathrm{ann}(yb_k) \subseteq \mathfrak{m}^k$.

Proof. To prove the fact that $\mathrm{ann}(yb_k) \subseteq \mathfrak{m}^{p^{k-1}-k}$, suppose on the contrary that there exists an element $z \in \mathrm{ann}(yb_k)$ such that $z \notin \mathfrak{m}^{p^{k-1}-k}$. Then clearly, $yz \in \mathrm{ann}\, b_k$. On the other hand as $R \cong \kappa[[X_1, ..., X_n]]$, $\kappa \cong R/\mathfrak{m}$ a field of characteristic $p > 0$, and $y \notin \mathfrak{m}^k$ and $z \notin \mathfrak{m}^{p^{k-1}-k}$, we may write

$$y = f + f',$$
$$z = g + g',$$

where f (resp., g) is a nonzero polynomial in $\kappa[[X_1, X_2, ..., X_n]]$ of degree at most $k-1$(resp., $p^{k-1} - k - 1$) and f'(resp., g') is either zero or a formal power series in $\kappa[[X_1, X_2, ..., X_n]]$ in which each summand has degree at least k (resp., $p^{k-1} - k$). Then $yz = fg + fg' + gf' + g'f'$. Note that since $\kappa[[X_1, ..., X_n]]$ is an integral domain and f and g are nonzero elements in $\kappa[[X_1, ..., X_n]]$, so is fg. Note also that since fg', gf', and $g'f'$ are either zero or contain terms of degrees strictly larger than the smallest degree of fg, they cannot cancel any terms of smallest degree. But then since the degree of the smallest term of fg is less than or equal to $0 \neq \deg(fg) \leq p^{k-1} - k - 1 + k - 1 = p^{k-1} - 2$, $yz \notin \mathfrak{m}^{p^{k-1}}$ which contradicts the fact that $yz \in \mathrm{ann}(b_k) = \mathfrak{m}^{[p^{k-1}]}$. Hence $\mathrm{ann}(yb_k) \subseteq \mathfrak{m}^{p^{k-1}-k}$, as desired.

If, in particular, $k \geq 4$, then $p^{k-1} - k \geq k$ and so $\mathrm{ann}(yb_k) \subseteq \mathfrak{m}^{p^{k-1}-k} \subseteq \mathfrak{m}^k$. □

Now we are ready to give the proof of Theorem 1.1:

Proof of Theorem 1.1. Since $\mathcal{M}$ is an F-finite F-module such that $0 \notin \mathrm{Ass}(\mathcal{M})$, it follows from Lemma 3.1 that

$$D(\mathcal{M}) = \varprojlim(N \xleftarrow{\alpha} F(N) \xleftarrow{F(\alpha)} F^2(N) \xleftarrow{F^2(\alpha)} \cdots),$$

for some Artinian R-module N and surjective map $\alpha : F(N) \longrightarrow N$. It is worth noting that the exactness of the functor $F^k(-)$ implies that $F^k(\alpha)$ is surjective for all $k \geq 0$.

Now we claim that there exists a nonzero element $n' = (n'_0, n'_1, \cdots, n'_k, \cdots) \in D(\mathcal{M})$ such that $\mathrm{ann}(n'_k) \subseteq \mathfrak{m}^k$ for all $k \geq 4$, where n'_k is the image of n' in $F^k(N)$.

To construct such an element, let n'_0 be an element of N and, for every $1 \leq k \leq 3$, choose $n'_k \in F^k(N)$ such that $n'_{k-1} = F^{k-1}(\alpha)(n'_k)$. For $k \geq 4$, let $b_k \in \mathrm{Ker}(F^{k-1}(\alpha))$ be as defined in Lemma 3.2 and define n_k in such a way that $F^{k-1}(\alpha)(n_k) = n'_{k-1}$. Then, either $\mathrm{ann}(n_k) \subseteq \mathfrak{m}^k$ or $\mathrm{ann}(n_k + b_k) \subseteq \mathfrak{m}^k$. Indeed, if

$\mathrm{ann}(n_k + b_k) \nsubseteq \mathfrak{m}^k$, there exists an element $y \in \mathfrak{m} \setminus \mathfrak{m}^k$ such that $y(n_k + b_k) = 0$ and so $\mathrm{ann}(n_k) \subseteq \mathrm{ann}(yn_k) = \mathrm{ann}(yb_k)$. But then it follows from Lemma 3.3 that $\mathrm{ann}(n_k) \subseteq \mathrm{ann}(yb_k) \subseteq \mathfrak{m}^k$.

Now, for $k \geq 4$, define

$$n_k^{'} = \begin{cases} n_k & \text{if } \mathrm{ann}(n_k) \subseteq \mathfrak{m}^k, \\ n_k + b_k & \text{otherwise.} \end{cases}$$

Clearly, $n^{'} = (n_0^{'}, n_1^{'}, \cdots, n_k^{'}, \cdots) \in \mathrm{D}(\mathcal{M})$ and $\mathrm{ann}(n_k') \subseteq \mathfrak{m}^k$ for all $k \geq 4$. This proves the claim.

Finally, $\mathrm{ann}(n^{'}) = 0$ for if $z \in \mathrm{ann}(n^{'})$, then $z \in \mathrm{ann}(n_k^{'}) \subseteq \mathfrak{m}^k$ for all $k \geq 4$ which then implies that $z \in \bigcap_{n \in \mathbb{N}} \mathfrak{m}^n = \{0\}$. This completes the proof of Theorem 1.1. □

The proof of Corollary 1.2 is an immediate consequence of Theorem 1.1:

Proof of Corollary 1.2. Without loss of generality, we may, and do, assume that R is complete [[3], Remark 4.1.1]. Since R is an F-finite F-module, so are its all local cohomology modules and since $0 \notin \mathrm{Ass}_R(H_I^i(R))$ for any nonzero ideal I of R, the result follows from Theorem 1.1. □

Acknowledgments

The results in this paper were obtained while the second author visited the School of Mathematics at the University of Minnesota. The second author would like to thank TÜBİTAK for their support and also all members of the Mathematics Department of the University of Minnesota for their hospitality. Both authors would like to thank the referee for valuable comments that have improved the exposition.

References

[1] Shiro Goto and Tsukane Ogawa, *A note on rings with finite local cohomology*, Tokyo J. Math. **6** (1983), no. 2, 403–411. MR732093

[2] M. Hellus, *On the associated primes of Matlis duals of top local cohomology modules*, Comm. Algebra **33** (2005), no. 11, 3997–4009. MR2183976

[3] M. Hellus, *Local Cohomology and Matlis Duality*, habilitation dissertation,University of Leipzig (2007), arXiv:math./0703124.

[4] Michael Hellus, *Matlis duals of top local cohomology modules and the arithmetic rank of an ideal*, Comm. Algebra **35** (2007), no. 4, 1421–1432. MR2313677

[5] Michael Hellus, *On the associated primes of Matlis duals of local cohomology modules II*, Comm. Algebra **39** (2011), no. 7, 2615–2621. MR2821736

[6] Michael Hellus and Peter Schenzel, *Notes on local cohomology and duality*, J. Algebra **401** (2014), 48–61. MR3151247

[7] Michael Hellus and Jürgen Stückrad, *Matlis duals of top local cohomology modules*, Proc. Amer. Math. Soc. **136** (2008), no. 2, 489–498. MR2358488

[8] Ernst Kunz, *Characterizations of regular local rings for characteristic* p, Amer. J. Math. **91** (1969), 772–784. MR0252389

[9] Gennady Lyubeznik, *F-modules: applications to local cohomology and D-modules in characteristic* $p > 0$, J. Reine Angew. Math. **491** (1997), 65–130. MR1476089

[10] C. Peskine and L. Szpiro, *Dimension Projective Finie et Cohomologie Locale*, I.H.E.S. Publ. Math.,**42** (1973), 47-119.

Department of Mathematics, University of Minnesota, 127 Vincent Hall, 206 Church Street, Minneapolis, Minnesota 55455
Email address: gennady@math.umn.edu

Department of Mathematics, Istanbul Technical University, Maslak, 34469, Istanbul, Turkey
Email address: tugbayildirim@itu.edu.tr