PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 9, September 2018, Pages 3709–3713
http://dx.doi.org/10.1090/proc/14041
Article electronically published on June 13, 2018

COUNTEREXAMPLES ON SPECTRA OF SIGN PATTERNS

YAROSLAV SHITOV

(Communicated by Patricia L. Hersh)

Abstract. An $n \times n$ *sign pattern* S, which is a matrix with entries $0, +, -$, is called *spectrally arbitrary* if any monic real polynomial of degree n can be realized as a characteristic polynomial of a matrix obtained by replacing the nonzero elements of S by numbers of the corresponding signs. A sign pattern S is said to be a *superpattern* of those matrices that can be obtained from S by replacing some of the nonzero entries by zeros. We develop a new technique that allows us to prove spectral arbitrariness of sign patterns for which the previously known *Nilpotent Jacobian* method does not work. Our approach leads us to solutions of numerous open problems known in the literature. In particular, we provide an example of a sign pattern S and its superpattern S' such that S is spectrally arbitrary but S' is not, disproving a conjecture proposed in 2000 by Drew, Johnson, Olesky, and van den Driessche.

1. Conjectures

The study of the spectra of matrix patterns received a significant amount of attention in recent publications. The conjecture mentioned in the abstract appeared in one of the foundational papers on this topic ([10]), and many subsequent works proved it in different special cases ([3, 7, 12, 14, 15]). One of the known sufficient conditions for superpatterns to be spectrally arbitrary is the *Nilpotent Jacobian* condition ([2, 10]), which allowed for the solving of several intriguing problems on this topic ([4, 11, 19]). Despite these efforts, the *superpattern conjecture* has remained open until now, and we mention [5, 17, 18] as other recent work discussing this conjecture.

Conjecture 1 ([10, Conjecture 16]). *If S is a minimal spectrally arbitrary sign pattern, then any superpattern of S is spectrally arbitrary.*

We note that this conjecture involves the concept of a *minimal* spectrally arbitrary sign pattern, that is, a sign pattern S which is spectrally arbitrary but is not a superpattern of any other spectrally arbitrary sign pattern. In our paper, we construct a sign pattern S and its superpattern S' such that S is spectrally arbitrary but S' is not. We do not investigate the question of minimality of S, but S is a superpattern of some minimal spectrally arbitrary pattern S_0, and the pair (S_0, S') provides a counterexample to Conjecture 1 even if S is not minimal.

Received by the editors December 22, 2016, and, in revised form, October 31, 2017, and November 14, 2017.

2010 *Mathematics Subject Classification.* Primary 15A18, 15B35.

Key words and phrases. Matrix theory, eigenvalues, sign pattern.

As stated above, the *Nilpotent Jacobian* condition is sufficient for a zero pattern (and every superpattern of it) to be spectrally arbitrary. Our results show that this condition is not necessary, answering the questions posed explicitly in [1, 10, 17].

As a byproduct of our approach, we obtain solutions of two other related problems on the topic. Namely, we construct a sign pattern U such that $\mathrm{diag}(U, U)$ is spectrally arbitrary but U itself is not. This gives a solution to the problem posed in [8, Section 5] and an answer to [5, Question 3].

An $n \times n$ sign pattern S is said to allow *arbitrary refined inertias* if, for any family n_+, n_-, n_0, n_i of nonnegative integers such that $n_+ + n_- + n_0 + 2n_i = n$, there is a matrix with sign pattern S which has n_+ eigenvalues with positive real part, n_- eigenvalues with negative real part, n_0 zero eigenvalues, and n_i purely imaginary eigenvalues. We provide an example of a sign pattern that allows arbitrary refined inertias but is not spectrally arbitrary, which solves the problem asked in [8, Section 5] and in [13, Section 5].

2. Counterexamples

We define the sign patterns

$$T = \begin{pmatrix} + & + & 0 & 0 & 0 & 0 \\ - & - & + & 0 & 0 & 0 \\ 0 & 0 & 0 & + & 0 & 0 \\ 0 & 0 & 0 & 0 & + & 0 \\ - & - & 0 & 0 & 0 & + \\ + & + & + & 0 & - & 0 \end{pmatrix}, \quad T' = \begin{pmatrix} + & + & 0 & 0 & 0 & 0 \\ - & - & + & 0 & 0 & 0 \\ + & 0 & 0 & + & 0 & 0 \\ 0 & 0 & 0 & 0 & + & 0 \\ - & - & 0 & 0 & 0 & + \\ + & + & + & 0 & - & 0 \end{pmatrix},$$

which agree at every entry except $(3, 1)$, so T' is indeed a superpattern of T. Also, we fix any 2×2 spectrally arbitrary pattern[1] and denote it by D. Let us prove several observations which we will put together in the theorem below.

Observation 2. Let R be a matrix obtained from T' by replacing the signs with nonzero real numbers. Then R is not nilpotent.

Proof. The coefficients of t^3 and t^5 in the characteristic polynomial of R are equal to $-r_{12}r_{23}r_{31} + (r_{11} + r_{22})r_{56}r_{65}$ and $-r_{11} - r_{22}$, respectively. These coefficients can vanish simultaneously only if $r_{12}r_{23}r_{31} = 0$. $\square$

Observation 3. Let R be a matrix obtained from T by replacing the signs with nonzero real numbers. Assume that $t^6 + a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0$ is the characteristic polynomial of R. Then $a_3 = 0$ if and only if $a_5 = 0$.

Proof. The coefficients of t^3 and t^5 in the characteristic polynomial of R are equal to $(r_{11} + r_{22})r_{56}r_{65}$ and $-r_{11} - r_{22}$, respectively. As we see, the former of these numbers is zero if and only if the latter one is zero. $\square$

Observation 4. The sign pattern $\mathrm{diag}(T, D)$ is not spectrally arbitrary.

Proof. If $f = (t^2 + t + 1)(t^2 - t + 2)(t^2 + 1)(t^2 - 1)$ is realizable as the characteristic polynomial of a matrix with sign pattern $\mathrm{diag}(T, D)$, then f has a divisor realizable as the characteristic polynomial of a matrix with sign pattern T. A straightforward checking of possible cases leads to a contradiction with Observation 3. $\square$

[1] In fact, spectrally arbitrary $n \times n$ sign patterns exist for all $n \geqslant 2$; see [16].

In order to proceed, we consider the matrix

$$X = \begin{pmatrix} x_1 & 1 & 0 & 0 & 0 & 0 \\ -x_4 & -x_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -x_6 & -x_5 & 0 & 0 & 0 & 1 \\ x_7 & x_8 & x_9 & 0 & -x_3 & 0 \end{pmatrix},$$

whose sign pattern is T whenever the x_i's take positive values.

Observation 5. For all $b, c, d \in \mathbb{R}$, there are positive values of the x_i's such that the characteristic polynomial of X equals $(t^2+b)(t^2+c)(t^2+d)$.

Proof. First, we assume that x_1, x_3, x_8, x_9 are arbitrary and check that the matrix X defined by $x_2 = x_1$, $x_4 = b+c+d+x_1^2-x_3$, $x_5 = bc+bd+cd-bx_3-cx_3-dx_3+x_3^2+x_9$, $x_6 = bcx_1 + bdx_1 + cdx_1 - bx_1x_3 - cx_1x_3 - dx_1x_3 + x_1x_3^2 + x_8 + x_1x_9$, $x_7 = -bcd + x_1x_8 - bx_9 - cx_9 - dx_9 + x_3x_9$ has a desired characteristic polynomial. Picking $x_3 = 1$ and defining x_1 as a sufficiently large positive number, we make x_2, x_3, x_4 positive regardless of the values of x_8, x_9. Finally, the choice of x_9 allows us to make x_5 positive, and now x_6, x_7 tend to $+\infty$ as x_8 gets large. □

Observation 6. If $a_3/a_5 > 0$, then there are positive values of the x_i's such that the characteristic polynomial of X equals $t^6 + a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0$.

Proof. Again, we assume that x_1, x_8, x_9 are arbitrary and check that the matrix X defined by $x_2 = a_5 + x_1$, $x_3 = a_3/a_5$, $x_4 = (-a_3 + a_4a_5 + a_5^2x_1 + a_5x_1^2)/a_5$, $x_5 = (a_3^2 - a_3a_4a_5 + a_2a_5^2 + a_5^2x_9)/a_5^2$, $x_6 = (a_1a_5^2 + a_3^2x_1 - a_3a_4a_5x_1 + a_2a_5^2x_1 + a_5^2x_8 + a_5^3x_9 + a_5^2x_1x_9)/a_5^2$, $x_7 = (-a_0a_5 + a_5x_1x_8 + a_3x_9 - a_4a_5x_9)/a_5$ has a desired characteristic polynomial. Defining x_1 as a large enough positive number, we make x_2, x_3, x_4 positive regardless of the values of x_8, x_9. As in the proof of the previous observation, the choice of x_9 allows us to make x_5 positive, and then x_6, x_7 tend to $+\infty$ as x_8 gets large. □

Observation 7. Let f be a monic real polynomial of degree 16. Then f has a divisor realizable as the characteristic polynomial of a matrix with sign pattern T.

Proof. Clearly, f is the product of eight quadratics of the form $t^2 + a_it + b_i$. If b_i is negative, then such a quadratic has two roots of different signs, so we can assume without loss of generality that at least seven of the initial quadratics have their b_i's nonnegative. By the pigeonhole principle, among these seven quadratics there are three that either have all a_i's positive, or all a_i's negative, or all a_i's zero. In the first two cases, the product of these three quadratics is a polynomial as in Observation 6, and the case of zero a_i's corresponds to Observation 5. □

Observation 8. Let $V = \operatorname{diag}(T, \ldots, T, D, \ldots, D)$ be a sign pattern of size $(6t+2d)$. (T occurs t times, D occurs d times.) If $d \geqslant 5$, then V is spectrally arbitrary.

Proof. The result is true for $t = 0$ because D is spectrally arbitrary (see also in [9, Proposition 2.1]). Now let $t > 0$ and let f be a monic real polynomial of degree $6t + 2d$ (which is at least 16). We apply Observation 7 and find a polynomial h that divides f and arises as the characteristic polynomial of a matrix M_1 with sign pattern T. Using the inductive assumption, we find a matrix M_2 with characteristic polynomial f/h and sign pattern that has the same form as V but with one T-block removed. Now the matrix $\operatorname{diag}(M_1, M_2)$ has sign pattern V and characteristic polynomial f. $\square$

Observation 9. For any family $\nu = (n_+, n_-, n_0, n_i)$ of nonnegative integers such that $n_+ + n_- + n_0 + 2n_i = 8$, there is a family $\mu \leqslant \nu$ and a matrix M with sign pattern T and refined inertia μ.

Proof. If $n_0 + 2n_i \geqslant 6$, then we are done because of Observation 5. Otherwise, we have $n_+ + n_- \geqslant 3$, and it suffices to check that any tuple $\mu = (m_+, m_-, m_0, m_i)$ with $m_+ + m_- \geqslant 3$ arises as a refined inertia of a matrix with sign pattern T.

Now we see that one of the tuples $\mu - (3,0,0,0)$, $\mu - (0,3,0,0)$, $\mu - (2,1,0,0)$, $\mu - (1,2,0,0)$ consists of nonnegative integers, and this tuple corresponds to some monic polynomial h of degree 3. We note that, for a sufficiently large positive N, the polynomials $(t-N)^3h$, $(t+N)^3h$, $(t+3N)(t-N)^2h$, $(t-3N)(t+N)^2h$ satisfy the condition as in Observation 6. As said above, one of these polynomials has μ as a refined inertia. $\square$

Observation 10. The sign pattern $\operatorname{diag}(T, D)$ allows arbitrary refined inertias.

Proof. Let $\nu = (n_+, n_-, n_0, n_i)$ be a family of nonnegative integers such that $n_+ + n_- + n_0 + 2n_i = 8$. By Observation 9, there is a family $\mu \leqslant \nu$ and a matrix M_1 with sign pattern T and refined inertia μ. Since D is spectrally arbitrary, it allows a matrix M_2 with refined inertia $\nu - \mu$, and then the matrix $\operatorname{diag}(M_1, M_2)$ has sign pattern $\operatorname{diag}(T, D)$ and refined inertia ν. $\square$

Now we put all the observations together and conclude the paper.

Theorem 11. *Let T, T', D be as above. Then*

(1) *the sign pattern $S = \operatorname{diag}(T, D, D, D, D, D)$ is spectrally arbitrary, but its superpattern $S' = \operatorname{diag}(T', D, D, D, D, D)$ is not spectrally arbitrary;*

(2) $\operatorname{diag}(T, D)$ *allows arbitrary refined inertias but is not spectrally arbitrary;*

(3) *there is a sign pattern U such that* $\operatorname{diag}(U, U)$ *is spectrally arbitrary but U is not.*

Proof. The definition of T and T' immediately shows that S' is a superpattern of S. By Observation 2, S' does not allow a nilpotent matrix, so it is not spectrally arbitrary. Observation 8 shows that S is spectrally arbitrary and completes the proof of (1).

The sign pattern $\operatorname{diag}(T, D)$ is not spectrally arbitrary by Observation 4, and it allows arbitrary refined inertias by Observation 10. This proves (2).

Finally, let $U_1 = \operatorname{diag}(T, D)$. If $U_2 = \operatorname{diag}(U_1, U_1)$ is spectrally arbitrary, then the proof of (3) is complete. Otherwise, we define $U_3 = \operatorname{diag}(U_2, U_2)$, and we are done if U_3 is spectrally arbitrary. If this is still not the case, we complete the proof because $\operatorname{diag}(U_3, U_3)$ is spectrally arbitrary by Observation 8. $\square$

References

[1] Hannah Bergsma, Kevin N. Vander Meulen, and Adam Van Tuyl, *Potentially nilpotent patterns and the nilpotent-Jacobian method*, Linear Algebra Appl. **436** (2012), no. 12, 4433–4445. MR2917420

[2] T. Britz, J. J. McDonald, D. D. Olesky, and P. van den Driessche, *Minimal spectrally arbitrary sign patterns*, SIAM J. Matrix Anal. Appl. **26** (2004), no. 1, 257–271. MR2112860

[3] Michael S. Cavers and Shaun M. Fallat, *Allow problems concerning spectral properties of patterns*, Electron. J. Linear Algebra **23** (2012), 731–754. MR2966802

[4] Michael S. Cavers, In Jae Kim, Bryan L. Shader, and Kevin N. Vander Meulen, *On determining minimal spectrally arbitrary patterns*, Electron. J. Linear Algebra **13** (2005), 240–248. MR2181880

[5] M. Catral, D. D. Olesky, and P. van den Driessche, *Allow problems concerning spectral properties of sign pattern matrices: a survey*, Linear Algebra Appl. **430** (2009), no. 11-12, 3080–3094. MR2517861

[6] Michael S. Cavers and Kevin N. Vander Meulen, *Spectrally and inertially arbitrary sign patterns*, Linear Algebra Appl. **394** (2005), 53–72. MR2100576

[7] L. Corpuz and J. J. McDonald, *Spectrally arbitrary zero-nonzero patterns of order 4*, Linear Multilinear Algebra **55** (2007), no. 3, 249–273. MR2286686

[8] L. Deaett, D. D. Olesky, and P. van den Driessche, *Refined inertially and spectrally arbitrary zero-nonzero patterns*, Electron. J. Linear Algebra **20** (2010), 449–467. MR2735967

[9] Luz M. DeAlba, Irvin R. Hentzel, Leslie Hogben, Judith McDonald, Rana Mikkelson, Olga Pryporova, Bryan Shader, and Kevin N. Vander Meulen, *Spectrally arbitrary patterns: reducibility and the $2n$ conjecture for $n = 5$*, Linear Algebra Appl. **423** (2007), no. 2-3, 262–276. MR2312406

[10] J. H. Drew, C. R. Johnson, D. D. Olesky, and P. van den Driessche, *Spectrally arbitrary patterns*, Linear Algebra Appl. **308** (2000), no. 1-3, 121–137. MR1751135

[11] Colin Garnett and Bryan L. Shader, *A proof of the T_n conjecture: centralizers, Jacobians and spectrally arbitrary sign patterns*, Linear Algebra Appl. **436** (2012), no. 12, 4451–4458. MR2917422

[12] Colin Garnett and Bryan L. Shader, *The nilpotent-centralizer method for spectrally arbitrary patterns*, Linear Algebra Appl. **438** (2013), no. 10, 3836–3850. MR3034502

[13] I.-J. Kim, D. D. Olesky, B. L. Shader, P. van den Driessche, H. van der Holst, and K. N. Vander Meulen, *Generating potentially nilpotent full sign patterns*, Electron. J. Linear Algebra **18** (2009), 162–175. MR2491653

[14] In-Jae Kim, Bryan L. Shader, Kevin N. Vander Meulen, and Matthew West, *Spectrally arbitrary pattern extensions*, Linear Algebra Appl. **517** (2017), 120–128. MR3592013

[15] G. MacGillivray, R. M. Tifenbach, and P. van den Driessche, *Spectrally arbitrary star sign patterns*, Linear Algebra Appl. **400** (2005), 99–119. MR2131919

[16] J. J. McDonald, D. D. Olesky, M. J. Tsatsomeros, and P. van den Driessche, *On the spectra of striped sign patterns*, Linear Multilinear Algebra **51** (2003), no. 1, 39–48. MR1950412

[17] J. J. McDonald and A. A. Yielding, *Complex spectrally arbitrary zero-nonzero patterns*, Linear Multilinear Algebra **60** (2012), no. 1, 11–26. MR2869669

[18] Timothy C. Melvin, *Spectrally arbitrary zero-nonzero patterns of matrices over a variety of fields*, ProQuest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)–Washington State University. MR3211725

[19] Rajesh Pereira, *Nilpotent matrices and spectrally arbitrary sign patterns*, Electron. J. Linear Algebra **16** (2007), 232–236. MR2338347

129346 Russia, Moscow, Izumrudnaya ulitsa, dom 65, kvartira 4
Email address: yaroslav-shitov@yandex.ru