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# UNIQUENESS OF EPIDEMIC WAVES IN A HOST-VECTOR DISEASE MODEL

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Abstract. A diffusive integro-differential equation which serves as a model for the evolution of a host-vector epidemic was extensively studied in literature. The traveling wave solutions of this model describe the spread of the disease from a disease-free state to an infective state, which are epidemic waves. It is a challenging problem if epidemic waves with the minimal propagation speed are unique up to translation. In this paper, we establish the uniqueness of all epidemic waves with any an admissible wave speed by the sliding method and solve this challenging problem completely.

## 1. Introduction

The spatial spread of disease is an important subject in mathematical epidemiology. One of the central goals of studying the spread is to predict disease transmission patterns in population and estimate the invasion speed of disease transferring from a disease-free state to an infective state. Hence, a topic that has received great attention is the existence and uniqueness of traveling wave solutions in epidemic models, that is, the study of epidemic waves in epidemic models; see [1], [14], [15], [19], [20], [22] and the references therein. For more works on the spatio-temporal dynamics of epidemiological models one can be referred to the surveys by Ruan [23], Gourley et. al [12], and the book by Murray [18].

Consider the initial growth of a host-vector epidemic such as malaria. Ruan and Xiao in [24] derived the following diffusive integro-differential equation based on the works of Busenberg and Cooke [4], Cooke [5], Marcati and Pozio [17], and Volz [28]:

$$\begin{aligned} w_t(t,x) =& d\Delta w(t,x) - aw(t,x) \\ &+ b[1-w(t,x)]\int_{-\infty}^{t}\int_{\Omega} F(t,s,x,y)w(s,y)dyds, \end{aligned} \tag{1}$$

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where $w(t,x)$ is the normalized spatial density of an infectious host at time $t$ and location $x$, i.e., $w(t,x) \geq 0$ for $t \geq 0$ and $x \in \Omega$. $d$ is the diffusion constant, $\Omega \subset \mathbb{R}^n (n \leq 3)$ is the spatial habitat, $\Delta$ is the Laplacian operator, $a$ is the cure/recovery rate of the infected host, $b$ is the host-vector contact rate, the convolution kernel $F(t,s,x,y)$ is a nonnegative continuous function with respect to its variables, which measures the probability of becoming infective at location $x$ and time $t$, and the integrals total up all possible infected positions for those individuals at location $x$ and time $t$ and all possible times that these individuals might have taken to become infective. More details on the model and the global stability of steady states for (1) with bounded domain $\Omega$ can be found in [24].

When the spatial domain $\Omega$ is unbounded with $\Omega = \mathbb{R}$ and the delay kernel assumes some special forms, the existence or uniqueness of traveling wave solutions for the models was extensively studied by different methods in literature. Here we list parts of them as follows.

If $F(t,s,x,y) = \delta(x-y)\frac{t-s}{\tau^2}e^{-\frac{(t-s)}{\tau}}$ with $\delta(\cdot)$ being the Dirac-delta function, then (1) reduces to the following integro-differential equation:

$$\begin{aligned} w_t(t,x) =& d\Delta w(t,x) - aw(t,x) \\ &+ b[1-w(t,x)]\int_{-\infty}^{t} \frac{t-s}{\tau^2}e^{-\frac{(t-s)}{\tau}} w(s,x)ds. \end{aligned} \tag{2}$$

Ruan and Xiao [24] showed that for any $c_0 \geq 2\sqrt{b-a}$, there exists a small number $\tau_0 = \tau_0(c_0)$ such that for any $\tau \in [0,\tau_0]$, the model (2) admits a traveling wave solution connecting the two steady states $E_0 \equiv 0$ and $E_1 \equiv 1 - \frac{a}{b}$ with the wave speed $c = c(\tau)$ close to $c_0$.

If $F(t,s,x,y) = \frac{1}{\tau}e^{-\frac{1}{\tau}(t-s)}\frac{1}{\sqrt{4\pi(t-s)}}e^{-\frac{(x-y)^2}{4(t-s)}}$, then (1) reduces to the following integro-differential equation:

$$\begin{aligned} w_t(t,x) =& d\Delta w(t,x) - aw(t,x) \\ &+ b[1-w(t,x)]\int_{-\infty}^{t}\int_{-\infty}^{\infty} \frac{1}{\tau}e^{-\frac{1}{\tau}(t-s)}\frac{1}{\sqrt{4\pi(t-s)}}e^{-\frac{(x-y)^2}{4(t-s)}} w(s,y)dyds. \end{aligned} \tag{3}$$

Lv and Wang [16] obtained similar existence results of traveling wave solutions as in [24] for equation (3) and discussed the uniqueness of traveling wave solutions with the large admissible wave speed.

If $F(t,s,x,y) = \delta(t-s-\tau)\frac{1}{\sqrt{4\pi\tau}}e^{-\frac{(x-y)^2}{4\tau}}$, then (1) reduces to the following integro-differential equation:

$$\begin{aligned} w_t(t,x) =& d\Delta w(t,x) - aw(t,x) \\ &+ b[1-w(t,x)]\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\tau}}e^{-\frac{(x-y)^2}{4\tau}} w(t-\tau,y)dy. \end{aligned} \tag{4}$$

Peng et. al [21] approximately analyzed the existence of traveling wave solutions for equation (4).

If $F(t,s,x,y) = h(x-y)\delta(t-s-\tau)$ with $h$ a nonnegative, integrable even function, then (1) reduces to the following integro-differential equation:

$$\begin{aligned} w_t(t,x) =& d\Delta w(t,x) - aw(t,x) \\ &+ b[1-w(t,x)]\int_{-\infty}^{\infty} h(x-y)w(t-\tau,y)dy. \end{aligned} \tag{5}$$

Wang et. al [29] proved that there is a positive number $c^*$ such that (5) has a traveling wave solution for any $c \geq c^*$ and the traveling wave solutions with large wave speed $c > c^*$ are unique up to translation.

If $F(t,s,x,y) = \delta(x-y)\delta(t-s-\tau)$, then (1) reduces to the following differential equation with discrete delay:

$$w_t(t,x) = d\Delta w(t,x) - aw(t,x) + b[1-w(t,x)]w(t-\tau,x). \tag{6}$$

Schaaf [25] showed that for any $\tau \geq 0$, there exists a nonnegative constant $c^*(\tau)$ such that for every $c > c^*(\tau)$, (6) has a traveling wave solution and it is unique up to translation.

It should be pointed out that the above-mentioned works essentially deal with the traveling wave problem for equations with discrete delay or without delay by choosing special kernel function $F(t,s,x,y)$ in (1) (see the cited works for details). In [31] Zhao and Xiao considered a general kernel function $F(t,s,x,y) = F(t-s, x-y)$ for (1) and derived the following equation with *infinite distributed delay*:

$$\begin{aligned} w_t(t,x) =& d\Delta w(t,x) - aw(t,x) \\ &+ b[1-w(t,x)]\int_{-\infty}^{t}\int_{\mathbb{R}} F(t-s,x-y)w(s,y)dyds, \end{aligned} \tag{7}$$

which has two steady states $E_0 \equiv 0$ and $E_1 \equiv 1 - \frac{a}{b}$ for general unit kernel. It is clear that (2)–(6) are the special cases of (7).

Under the assumption that

(H1) $b > a > 0$, $F(s,y) = F(s,-y) \geq 0$, and $\int_0^\infty \int_{-\infty}^\infty F(s,y)dyds = 1$,

(H2) $\int_0^\infty \int_{-\infty}^\infty F(s,y)e^{\lambda(y-cs)}dyds < \infty$ for all $c \geq 0$ and $\lambda \geq 0$,

Zhao and Xiao in [31] and [32] established the existence of the traveling wave solutions of (7) in the study of the spatial spread and wave propagation dynamics for (7), which shows that there exists a positive constant $c^*$ (minimal wave speed) such that (7) has a traveling wave solution connecting the two steady states $E_0$ and $E_1$ if and only if its wave speed $c \geq c^*$. We state the result as follows.

**Proposition 1** (See [31])**.** *Assume that* (H1)-(H2). *Then there exists a* $c^* > 0$ *such that the following statements are valid:*

(i) *For any* $c \geq c^*$, (7) *admits a monotone traveling wave solution* $w(t,x) = U(x+ct)$ *connecting two steady states* 0 *and* $1 - \frac{a}{b}$.

(ii) *For any* $0 < c < c^*$, (7) *has no traveling wave solution* $w(t,x) = U(x+ct)$ *connecting two steady states* 0 *and* $1 - \frac{a}{b}$.

In this paper we deal with the uniqueness problem of traveling wave solutions propagating at a given admissible speed for (7). As you know, the uniqueness of traveling wave solutions is a very important topic in the study of wave propagation dynamics; see [2,6,7,10] and the references therein. If the traveling wave solution of (7) is unique up to translation, then *all* the traveling wave solutions are monotone and are separated from zero as $x + ct \to +\infty$, and their waveforms at a given admissible speed are the same. This is very helpful to understand the spread of disease; see [20]. To the best of our knowledge, there have been few uniqueness results on the traveling wave solutions with large wave speed for special cases (i.e., (2)–(6)) of (7), and no results on the uniqueness of the traveling wave solutions with minimal wave speed for any special cases of (7). It is still a challenging problem, even though there have been many significant works on the uniqueness of traveling wave solutions for different equations such as integral equation, lattice equation, integro-differential equation, diffusive equation, etc. (see [2, 6–8, 10, 11, 13, 27]). In particular, the works [2, 10, 11] established the uniqueness theorem of traveling wave solutions for a class of nonlinear integral equations, which can be successfully applied to certain types of nonlocal reaction-diffusion equations that can be reduced to the prescribed integral equations. However, to our knowledge it is difficult to transform the nonlocal reaction-diffusion equation (7) into the prescribed integral form in [2, 10, 11]. Hence, we could not use the conclusions of [2, 10, 11] to equation (7) directly. Using the ideas of the sliding method [3, 8] and developing some analytical skills, we solve the uniqueness problem of traveling waves for equation (7), and obtain that all the traveling wave solutions of (7) with a given admissible speed $c \geq c^*$ are unique up to translation. Therefore, we completely establish the uniqueness of epidemic waves of (7) for given a wave speed $c$ with $c \geq c^*$.

## 2. Uniqueness of epidemic waves

Due to the biological meaning of equation (7), we are only interested in the nonnegative bounded traveling wave solution of (7) with the form $w(t, x) = U(x + ct)$, where $c > 0$ is the wave speed, $z := x + ct$ is the moving coordinate, and $U \in C^2(\mathbb{R}, \mathbb{R}_+)$ is the wave profile, where $\mathbb{R}_+ := [0, +\infty)$. For a traveling wave solution $w$ connecting the two steady states $E_0 \equiv 0$ and $E_1 \equiv 1 - \frac{a}{b}$, it is easy to see that the wave profile function $U(z)$ satisfies

$$\begin{aligned} cU'(z) =& dU''(z) - aU(z) \\ &+ b[1 - U(z)] \int_0^\infty \int_{-\infty}^\infty F(s, y)U(z - y - cs)dyds, \end{aligned} \tag{8}$$

and subject to the boundary conditions

$$U(-\infty) := \lim_{z\to-\infty} U(z) = 0, \quad U(+\infty) := \lim_{z\to+\infty} U(z) = 1 - \frac{a}{b}. \tag{9}$$

The characteristic equation of the linearization of (8) around zero equilibrium is

$$P(\lambda, c) = d\lambda^2 - c\lambda - a + b \int_0^\infty \int_{-\infty}^\infty F(s, y)e^{-\lambda(y+cs)}dyds,$$

which admits the following properties; see [31, Lemma 2.2].

**Lemma 2.** *There exists a unique $c^* > 0$ such that*

(i) *system $P(\lambda, c^*) = 0$, $\frac{\partial}{\partial\lambda}P(\lambda, c^*) = 0$ has a unique positive solution $\lambda = \lambda^* = \lambda(c^*)$;*

(ii) *for any $c > c^*$, $P(\lambda, c) = 0$ has two distinct positive real roots $\lambda(c)$ and $\Lambda(c)$ with $0 < \lambda(c) < \Lambda(c)$.*

In [31, 32], the authors proved that the $c^*$ defined above is the minimal wave speed of traveling wave solutions of (7) by showing the fact that a traveling wave solution exists if and only if its wave speed $c \geq c^*$ (see Proposition 1). In this paper, we shall establish the following main result, which shows that the traveling wave solutions with any a given admissible speed $c \geq c^*$ are unique up to translation.

**Theorem 3.** *Assume that* (H1) *and* (H2) *hold. For any* $c \geq c^*$, *suppose* $U(z)$ *and* $\hat{U}(z)$ *are two solutions of* (8) *satisfying* (9); *then there is a constant* $\eta$ *such that* $U(z) = \hat{U}(z+\eta)$.

To state the proof of Theorem 3 briefly, we first establish a series of lemmas under the assumptions of (H1) and (H2) in Theorem 3. For brevity, we don't repeat the assumptions (H1) and (H2) in the following lemmas.

Define an integro-differential operator $L : C^2(\mathbb{R}, \mathbb{R}_+) \longrightarrow C(\mathbb{R})$ by

$$\begin{aligned} L[U](z) :=& dU^{''}(z) - cU^{'}(z) - aU(z) \\ &+ b[1-U(z)] \int_0^\infty \int_{-\infty}^\infty F(s,y)U(z-y-cs)dyds. \end{aligned}$$

Following the ideas in [9, 26], we can obtain a strong maximum principle as follows.

**Lemma 4.** *Suppose that* $U \in C^2(\mathbb{R}, \mathbb{R}_+)$ *satisfies* (9) *and*

$$L[U](z) \geq 0 \quad on \quad \mathbb{R} \quad (L[U](z) \leq 0 \quad on \quad \mathbb{R}, \quad respectively).$$

*Then* $U$ *cannot achieve a maximum (minimum, respectively) on* $\mathbb{R}$.

*Proof.* We argue by contradiction. Suppose that $U(z)$ achieves a maximum at a point $z_0 \in \mathbb{R}$, i.e., $U(z_0) = \max_{z\in\mathbb{R}} U(z)$. Since $U \in C^2(\mathbb{R}, \mathbb{R}_+)$ satisfies (9), we have $U^{'}(z_0) = 0$, $U^{''}(z_0) \leq 0$, and $U(z_0) \geq 1 - \frac{a}{b}$. By the assumption, we also have $L[U](z_0) \geq 0$. It then follows that $U(z_0) < 1$ and

$$\begin{aligned} 0 \leq& L[U](z_0) \\ =& dU^{''}(z_0) - cU^{'}(z_0) - aU(z_0) \\ &+ b[1-U(z_0)] \int_0^\infty \int_{-\infty}^\infty F(s,y)U(z_0-y-cs)dyds \\ \leq& -aU(z_0) + b[1-U(z_0)] \int_0^\infty \int_{-\infty}^\infty F(s,y)U(z_0-y-cs)dyds \\ =& U(z_0)[-a + b(1-U(z_0))] \\ &+ b[1-U(z_0)] \int_0^\infty \int_{-\infty}^\infty F(s,y)[U(z_0-y-cs) - U(z_0)]dyds \\ \leq& b[1-U(z_0)] \int_0^\infty \int_{-\infty}^\infty F(s,y)[U(z_0-y-cs) - U(z_0)]dyds \leq 0, \end{aligned}$$

which implies that $\int_0^\infty \int_{-\infty}^\infty F(s,y)[U(z_0-y-cs) - U(z_0)]dyds = 0$. Thus, $U(z_0 - y - cs) = U(z_0)$ for every $(s,y) \in \text{supp}(F)$.

If $\text{supp}(F) = \mathbb{R}_+ \times \mathbb{R}$, we immediately get a contradiction since $U$ satisfies (9). If not, since $F(s,y) = F(s,-y)$ for $(s,y) \in \mathbb{R}_+ \times \mathbb{R}$, we can choose $(\hat{s}, \hat{y}) \in \text{supp}(F)$ such that $\hat{y} + c\hat{s} > 0$. Let $\sigma := \hat{y} + c\hat{s}$. Repeating the above arguments for $U(z_0 - \sigma)$, we obtain $U(z_0) = U(z_0 - \sigma) = U(z_0 - 2\sigma)$. By induction arguments, it follows that $U(z_0) = U(z_0 - n\sigma)$ for any $n \in \mathbb{Z}_+$. Letting $n \to +\infty$, we get a contradiction since $U(z_0) \geq 1 - \frac{a}{b}$. This completes the proof. □

From the strong maximum principle established in Lemma 4, we can directly derive the following lemma.

**Lemma 5.** *For any $c \geq c^*$, let $U \in C^2(\mathbb{R}, \mathbb{R}_+)$ be a solution of* (8) *and* (9)*. Then $0 < U(z) < 1 - \frac{a}{b}$ for all $z \in \mathbb{R}$.*

We now study the size relationship between two solutions of (8) satisfying (9).

**Lemma 6.** *For any $c \geq c^*$, let $U_1, U_2 \in C^2(\mathbb{R}, \mathbb{R}_+)$ be two solutions of* (8) *satisfying* (9)*. If $U_1(z) \leq U_2(z)$ with $U_1(z) \not\equiv U_2(z)$ for $z \in \mathbb{R}$, then $U_1(z) < U_2(z)$.*

*Proof.* Suppose that $U_1(z) \leq U_2(z)$ with $U_1(z) \not\equiv U_2(z)$ on $\mathbb{R}$. Then, the function defined by

$$W(z) := U_2(z) - U_1(z)$$

satisfies $W(z) \geq 0$ and $W(z) \not\equiv 0$ for all $z \in \mathbb{R}$.

We claim that $W(z) > 0$ on $\mathbb{R}$. Otherwise, there exists a $z_0$ such that

$$0 = W(z_0) = \min_{z \in \mathbb{R}} W(z)$$

and $W'(z_0) = 0$. We will divide it two cases to show that $W''(z_0) \geq 0$ as follows.

*Case I.* $z_0$ is an isolated zero point of $W(z)$. Then $W(z) > 0$ in some neighborhood of $z_0$ except $z_0$. We claim that $W''(z_0) \geq 0$. Otherwise, we have

$$W''(z_0) = \lim_{z \downarrow z_0} \frac{W'(z) - W'(z_0)}{z - z_0} < 0,$$

which implies that $W'(z) < 0$ for $z \in (z_0, z_0 + \delta)$ with some $\delta > 0$. Thus

$$W(z) = W(z) - W(z_0) = W'(\xi)(z - z_0) < 0, \quad z \in (z_0, z_0 + \delta),$$

which leads to a contradiction. Thus, $W''(z_0) \geq 0$.

*Case II.* There exists a sequence $\{z_n\}$, $\lim_{n \to +\infty} z_n = z_0$, such that $W(z_n) = 0, n = 0, 1, \ldots$. Thus, $W'(z_n) = 0$, and so $W''(z_0) = 0$.

Summarizing the above analysis, we obtain that $W''(z_0) \geq 0$ if there exists a $z_0$ such that $W(z_0) = 0$ and $W'(z_0) = 0$.

We now calculate $cW'(z_0)$ and obtain that

$$\begin{aligned}
0 =& cW'(z_0)\\
=& c[U_2'(z_0) - U_1'(z_0)]\\
=& d[U_2''(z_0) - U_1''(z_0)] - a[U_2(z_0) - U_1(z_0)]\\
&+ [b(1 - U_2(z_0)) - b(1 - U_1(z_0))] \int_0^\infty \int_{-\infty}^\infty F(s, y)U_2(z_0 - y - cs)dyds\\
&+ b(1 - U_1(z_0)) \int_0^\infty \int_{-\infty}^\infty F(s, y)[U_2(z_0 - y - cs) - U_1(z_0 - y - cs)]dyds\\
=& dW''(z_0) - aW(z_0) - bW(z_0) \int_0^\infty \int_{-\infty}^\infty F(s, y)U_2(z_0 - y - cs)dyds\\
&+ b(1 - U_1(z_0)) \int_0^\infty \int_{-\infty}^\infty F(s, y)W(z_0 - y - cs)dyds\\
\geq& b(1 - U_1(z_0)) \int_0^\infty \int_{-\infty}^\infty F(s, y)W(z_0 - y - cs)dyds \geq 0,
\end{aligned}$$

which implies that $\int_0^\infty \int_{-\infty}^\infty F(s, y)W(z_0-y-cs)dyds = 0$. Thus, $W(z_0-y-cs) = 0$ for $(s, y) \in \mathrm{supp}(F)$. If $\mathrm{supp}(F) = \mathbb{R}_+ \times \mathbb{R}$, we have $W(z) = 0$ on $\mathbb{R}$, which is a contradiction. If not, we can repeat the previous calculation for $W(z_0 - y - cs)$ with every $(s, y) \in \mathrm{supp}(F)$. Note that $F(s, y) = F(s, -y)$ for $(s, y) \in \mathbb{R}_+ \times \mathbb{R}$. By repeating the above calculation infinitely many times, we cover all values of $\mathbb{R}$ and thus end up with $W(z) = 0$ on $\mathbb{R}$, which is a contradiction. Therefore, $W(z) > 0$ on $\mathbb{R}$, namely, $U_2(z) > U_1(z)$ on $\mathbb{R}$. This completes the proof.

□

**Lemma 7.** *For any $c \geq c^*$, let $U \in C^2(\mathbb{R}, \mathbb{R}_+)$ be a solution of* (8) *satisfying* (9). *Then there exist some positive constants $\alpha, \omega$ such that*

(i) *for $c > c^*$, $\lim_{z\to-\infty} U(z)e^{-\lambda(c)z} = \alpha$ and $\lim_{z\to-\infty} U'(z)e^{-\lambda(c)z} = \alpha\lambda(c)$.*

(ii) *for $c = c^*$, $\lim_{z\to-\infty} U(z)|z|^{-1}e^{-\lambda(c^*)z} = \omega$ and $\lim_{z\to-\infty} U'(z)|z|^{-1}e^{-\lambda(c^*)z} = \omega\lambda(c^*)$.*

The above result can be deduced from [30, Proposition 2.3], which implies that for any $c \geq c^*$, every wave profile $U$ satisfies $U'(z) > 0$ at some neighborhood of $-\infty$. Then, we can choose $\zeta > 0$ sufficiently large so that

$$U(z) \text{ is strictly monotone on } (-\infty, -\zeta).$$

**Lemma 8.** *For any $c \geq c^*$, let $U_1, U_2 \in C^2(\mathbb{R}, \mathbb{R}_+)$ be two solutions of* (8) *satisfying* (9). *If there exists $\ell_1 > 0$ such that $U_1(z) > U_2(z)$ for $z \leq -\ell_1$, then there exists $\rho \geq 0$ such that $U_1(z + \rho) \geq U_2(z)$ for $z \in \mathbb{R}$.*

*Proof.* We divide the proof into two steps.

*Step I.* We first show that for any given $\ell > 0$, there exists $\rho \geq 0$ such that

$$U_1(z + \rho) > U_2(z) \quad \text{for} \quad z \in (-\infty, \ell].$$

Since $U_1(z) > 0$ and $U_1(+\infty) = 1 - \frac{a}{b}$, we have $\mu := \inf_{z\in[-\zeta,+\infty)} U_1(z) > 0$.

Note that $U_2(-\infty) = 0$. Thus, there exists $\ell_0 > 0$ such that $U_2(z) < \mu$ for $z \leq -\ell_0$.

Then for every $\rho \geq 0$, we have

$$U_1(z+\rho) > U_2(z) \quad \text{for} \quad z \in [-\zeta - \rho, -\ell_0].$$

Note that $\sup_{z\in[-\ell_0,\ell]} U_2(z) < 1 - \frac{a}{b}$ and $U_1(+\infty) = 1 - \frac{a}{b}$. One can choose a suitable $\rho \geq 0$ such that

$$U_1(z+\rho) > U_2(z) \quad \text{for} \quad z \in [-\ell_0, \ell].$$

It then follows that

$$U_1(z+\rho) > U_2(z) \quad \text{for} \quad z \in [-\zeta - \rho, \ell].$$

Since $U_1(z)$ is strictly monotone and $U_1(z) > U_2(z)$ on $(-\infty, -\zeta)$ $(\zeta \geq \ell_1)$, it follows that for $\rho \geq 0$,

$$U_1(z+\rho) \geq U_1(z) > U_2(z) \quad \text{for} \quad z \in (-\infty, -\zeta - \rho].$$

Thus, for any given $\ell > 0$, there exists $\rho \geq 0$ such that

$$U_1(z+\rho) > U_2(z) \quad \text{for} \quad z \in (-\infty, \ell]. \tag{10}$$

*Step II.* Choose $M, N > 0$ such that $\int_0^M \int_{-N}^N F(s,y)dyds > \frac{1}{2}$ by hypothesis (H1).

Since $U_i(+\infty) = 1 - \frac{a}{b}$, $i = 1, 2$, there exists $\delta > 0$ such that

$$U_1(z) \geq \frac{2}{3}\left(1 - \frac{a}{b}\right), \quad U_2(z) \geq \frac{2}{3}\left(1 - \frac{a}{b}\right) \quad \text{for any} \quad z \geq \delta. \tag{11}$$

We now fix $\ell \geq \delta + N + cM$ in (10), and define

$$\hat{\vartheta} = \inf\{\vartheta \geq 0 |\ U_1(z+\rho) + \vartheta \geq U_2(z) \quad \text{for} \quad z \in \mathbb{R}\}.$$

Clearly, $0 \leq \hat{\vartheta} < 1 - \frac{a}{b}$ since $U_2(z) < 1 - \frac{a}{b}$, and

$$U_1(z+\rho) + \hat{\vartheta} \geq U_2(z) \quad \text{for} \quad z \in \mathbb{R}.$$

If $\hat{\vartheta} = 0$, then the conclusion of the lemma follows.

Suppose that $\hat{\vartheta} > 0$. Define a function as follows:

$$\widehat{U}(z) = U_1(z+\rho) + \hat{\vartheta} - U_2(z), \quad z \in \mathbb{R}.$$

It is clear that $\widehat{U}(z) \geq 0$ and $\widehat{U}(\pm\infty) = \hat{\vartheta} > 0$. Thus, there exists $z^*$ such that

$$0 = \widehat{U}(z^*) = \min_{z\in\mathbb{R}} \widehat{U}(z).$$

Note that $U_1(z^*+\rho) - U_2(z^*) = -\hat{\vartheta} < 0$. It follows from (10) that $z^* > \ell \geq \delta + N + cM$. Then by $\widehat{U}'(z^*) = 0$, $\widehat{U}''(z^*) \geq 0$, and (11), we have

$$\begin{aligned}
0 =& c[U_1'(z^*+\rho) - U_2'(z^*)] \\
=& d[U_1''(z^*+\rho) - U_2''(z^*)] - a[U_1(z^*+\rho) - U_2(z^*)] \\
& + [b(1 - U_1(z^*+\rho)) - b(1 - U_2(z^*))] \int_0^\infty \int_{-\infty}^\infty F(s,y) U_1(z^*+\rho-y-cs)dyds \\
& + b(1 - U_2(z^*)) \int_0^\infty \int_{-\infty}^\infty F(s,y)[U_1(z^*+\rho-y-cs) - U_2(z^*-y-cs)]dyds \\
\geq& a\hat{\vartheta} + b\hat{\vartheta} \int_0^M \int_{-N}^N F(s,y) U_1(z^*+\rho-y-cs)dyds - b\hat{\vartheta}(1 - U_2(z^*)) \\
>& a\hat{\vartheta} + b\hat{\vartheta} \times \frac{1}{2} \times \frac{2}{3}\left(1 - \frac{a}{b}\right) - b\hat{\vartheta}\left[1 - \frac{2}{3}(1 - \frac{a}{b})\right] = 0,
\end{aligned}$$

which is a contradiction. Thus, we have $\hat{\vartheta} = 0$, and then $U_1(z+\rho) \geq U_2(z)$ for $z \in \mathbb{R}$. This completes the proof.

□

**Lemma 9.** *For any $c \geq c^*$, let $U_1, U_2 \in C^2(\mathbb{R}, \mathbb{R}_+)$ be two solutions of* (8) *satisfying* (9). *If there exists $\ell_1 > 0$ such that $U_1(z) > U_2(z)$ for $z \leq -\ell_1$, then $U_1(z) > U_2(z)$ for $z \in \mathbb{R}$.*

*Proof.* By Lemma 8, there exists $\rho \geq 0$ such that

$$U_1(z+\rho) \geq U_2(z) \quad \text{for} \quad z \in \mathbb{R}.$$

Define

$$\hat{\rho} = \inf\{\rho \geq 0 |\ U_1(z+\rho) \geq U_2(z) \quad \text{for} \quad z \in \mathbb{R}\},$$

which is well defined, and

$$U_1(z+\hat{\rho}) \geq U_2(z) \quad \text{for} \quad z \in \mathbb{R}.$$

We claim that $\hat{\rho} = 0$. Suppose, by contradiction, that $\hat{\rho} > 0$. Since $U_1(z)$ is strictly monotone on $(-\infty, -\zeta)$ and $U_1(z) > U_2(z)$ on $(-\infty, -\ell_1)$, it follows that $U_1(z+\hat{\rho}) \not\equiv U_2(z)$ on $\mathbb{R}$. By Lemma 6, we have

$$U_1(z+\hat{\rho}) > U_2(z) \quad \text{for} \quad z \in \mathbb{R}.$$

Since $U_1(z)$ is monotone on $(-\infty, -\zeta)$ and is uniformly continuous on a compact set, we have that for any $\ell > 0$, there exists small $\epsilon > 0$ such that

$$U_1(z+(\hat{\rho}-\epsilon)) > U_2(z) \quad \text{for} \quad z \in (-\infty, \ell].$$

Proceeding with the arguments as in the second step of the proof of Lemma 8, we have

$$U_1(z+(\hat{\rho}-\epsilon)) \geq U_2(z) \quad \text{for} \quad z \in \mathbb{R}.$$

This is a contradiction according to the definition of $\hat{\rho}$. Thus, we have $\hat{\rho} = 0$, and then

$$U_1(z) \geq U_2(z) \quad \text{for} \quad z \in \mathbb{R}.$$

By Lemma 6 and the assumption $U_1(z) > U_2(z)$ on $(-\infty, -\ell_1)$, we have $U_1(z) > U_2(z)$ for $z \in \mathbb{R}$. This completes the proof. □

Now we are in the position to prove the main result.

*Proof of Theorem* 3. Let $(c, \hat{U})$ and $(c, U)$ be two wave profiles with the same speed $c \geq c^*$. Since the proof is similar for both of the cases $c > c^*$ and $c = c^*$, we next focus on the case that $c = c^*$. By Lemma 7 and the translation invariance of solutions of (8), we have that there exists a $\eta \in \mathbb{R}$ such that

$$\lim_{z\to-\infty} \hat{U}(z+\eta)|z|^{-1}e^{-\lambda(c^*)z} = \lim_{z\to-\infty} U(z)|z|^{-1}e^{-\lambda(c^*)z} = \alpha$$

for some $\alpha > 0$.

Let $\bar{U}(z) = \hat{U}(z+\eta)$. It then follows that

$$\bar{U}(z) = \alpha|z|e^{\lambda(c^*)z} + o\left(|z|e^{\lambda(c^*)z}\right) \quad \text{as} \quad z \to -\infty \tag{12}$$

and

$$U(z) = \alpha|z|e^{\lambda(c^*)z} + o\left(|z|e^{\lambda(c^*)z}\right) \quad \text{as} \quad z \to -\infty. \tag{13}$$

Let $U^*(z) = \bar{U}(z+\rho)$ for any given $\rho > 0$. Then, we have

$$U^*(z) = \alpha|z|e^{\lambda(c^*)(z+\rho)} + o\left(|z|e^{\lambda(c^*)z}\right) \quad \text{as} \quad z \to -\infty. \tag{14}$$

The equalities (13) and (14) imply that there exists an $\ell_1 > 0$ such that

$$U^*(z) > U(z) \quad \text{for} \quad z \le -\ell_1.$$

By Lemma 9, we have $U^*(z) > U_2(z)$ for $z \in \mathbb{R}$, that is, $\bar{U}(z+\rho) > U(z)$ for $z \in \mathbb{R}$. Define

$$\rho^* = \inf\{\rho > 0 |\ \bar{U}(z+\rho) \ge U(z) \quad \text{for} \quad z \in \mathbb{R}\},$$

which is well defined, and

$$\bar{U}(z+\rho^*) \ge U(z) \quad \text{for} \quad z \in \mathbb{R}. \tag{15}$$

We next show that $\rho^* = 0$. Suppose, by contradiction, that $\rho^* > 0$. By (12) and (13), it follows that $\bar{U}(z+\rho^*) \not\equiv U(z)$ on $\mathbb{R}$. Then by Lemma 6, we have $\bar{U}(z+\rho^*) > U(z)$ on $\mathbb{R}$. Since $\rho^* > 0$, we obtain from (12) that

$$\bar{U}(z+\rho^*) = \alpha|z|e^{\lambda(c^*)(z+\rho^*)} + o\left(|z|e^{\lambda(c^*)z}\right) \quad \text{as} \quad z \to -\infty. \tag{16}$$

In view of (13) and (16), we obtain that for a small $\epsilon > 0$ there also exists an $\ell_1 > 0$ such that

$$\bar{U}(z + (\rho^* - \epsilon)) > U(z) \quad \text{for} \quad z \le -\ell_1.$$

Using Lemma 9, we have

$$\bar{U}(z + (\rho^* - \epsilon)) > U(z) \quad \text{for} \quad z \in \mathbb{R},$$

which contradicts the definition of $\rho^*$. Thus, $\rho^* = 0$, and then $\bar{U}(z) \ge U(z)$ on $\mathbb{R}$. Note that the role of $\bar{U}$ and $U$ above can be interchanged. So, we also have $\bar{U}(z) \le U(z)$ on $\mathbb{R}$ by repeating the above arguments. Thus, we have $\bar{U} \equiv U$ on $\mathbb{R}$, namely, $\hat{U}(z+\eta) = U(z)$ for $z \in \mathbb{R}$. This completes the proof. □

## 3. Concluding remarks

In this paper, we consider the uniqueness of epidemic waves for a vector-disease model. The important feature of this model is that the reflection of the current density of infectious vectors is related to the number of infectious hosts at earlier times. We show that all the epidemic waves of this vector-disease model with a given admissible speed are unique up to translation. This generalizes and completes the earlier results in the literature.

It is worthwhile to note that a combination of the uniqueness of traveling waves in Theorem 3 and the existence of monotone traveling waves in Proposition 1 can directly lead to the conclusion that all the traveling wave solutions of the vector-disease model (7) are monotone increasing and they are separated from zero as $x + ct \to +\infty$. This reveals the characteristics of the spatial spread of the disease.

As one can see, a basic assumption of the original vector-disease model (7) is that a susceptible vector can be infected only by the infectious host. However, in reality, there are some infectious diseases whose susceptible vector can receive the infection not only from the infectious host but also from the infectious vector in the transmission process of the disease. Taking this fact into account, recently Zhang in [33] extended the original vector-disease model (7) to the form

$$\begin{aligned} w_t(t,x) =& d\Delta w(t,x) - aw(t,x) + \beta[1-w(t,x)]w(t,x) \\ &+ b[1-w(t,x)] \int_{-\infty}^{t} \int_{\mathbb{R}} F(t-s, x-y) w(s,y) dy ds, \end{aligned} \tag{17}$$

and established the existence of epidemic waves for the modified vector-disease model (17). It can be checked that the developed techniques in this paper can also be applied to this modified vector-disease model for the uniqueness of epidemic waves.

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