

PROPERTIES OF CARATHÉODORY MEASURE HYPERBOLIC UNIVERSAL COVERS OF COMPACT KÄHLER MANIFOLDS

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ABSTRACT. This article explores some properties of universal covers of compact Kähler manifolds under the assumption of Carathéodory measure hyperbolicity. In particular, by comparing invariant volume forms, an inequality is established between the volume of canonical bundle of a compact Kähler manifold and the Carathéodory measure of its universal cover (similar result as in [Proc. Amer. Math. Soc. 139 (2011), pp. 1411–1420]). Using a similar method, an inequality is established between the restricted volume of a canonical bundle of a compact Kähler manifold and the restricted Carathéodory measure of its covering, solving a conjecture in [Michigan Math. J. 62 (2013), pp. 259–292].

1. INTRODUCTION

It is interesting to study non-compact complex manifolds with the assumption that there exist bounded holomorphic functions on it, since they include the examples of bounded domains and, in particular, Hermitian symmetric spaces of non-compact type. In this article, we shall study properties of universal covers of compact Kähler manifolds under the assumption of Carathéodory measure hyperbolicity. In this case, there are abundant supplies of holomorphic functions, leading to interesting properties. In [Kikuta 10], it was established that the volume of the canonical line bundle of a compact Kähler manifold is bounded from below by a constant multiple of its Carathéodory measure. It is natural to ask the same question regarding restricted volumes and measures. It was stated as a conjecture in [Kikuta 13] that the restricted volume of the canonical line bundle of a compact Kähler manifold is also bounded from below by a constant multiple of its restricted Carathéodory measure. Inspired by the work of [Yeung], which established uniform estimates among invariant metrics, the author obtains uniform estimates among invariant volume forms, and the conjecture follows as an easy consequence.

2. INVARIANT VOLUME FORMS

Let μ_r^d be the **Poincaré volume form** for \mathbb{B}_r^d , the complex d -ball of radius r , i.e.,

$$\mu_r^d = \frac{r^2}{(r^2 - \|z\|^2)^{d+1}} \cdot \frac{\sqrt{-1}}{2} dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge \frac{\sqrt{-1}}{2} dz^d \wedge d\bar{z}^d.$$

For the unit ball \mathbb{B}_1^n , we write μ_1^d simply as μ^d .

Let M be an n -dimensional complex manifold:

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The **Bergman pseudo-volume form** v_M^B is defined by, for any $p \in M$,

$$v_M^B(p) := \sum_i (\varphi_i \wedge \bar{\varphi}_i)(p),$$

where $\{\varphi_i\}$ is an orthonormal basis for $L^2(M, K_M)$.

From [Hahn], we have

$$v_M^B(p) = \varphi_p \wedge \overline{\varphi_p},$$

where φ_p maximizes $(\varphi \wedge \bar{\varphi})(p)$ over the unit ball of $L^2(M, K_M)$.

The **Carathéodory pseudo-volume form** v_M^C is defined by ([Eisenman], p. 57), for any $p \in M$,

$$\begin{aligned} v_M^C(p) &:= \sup \{ (f^* \mu^n)(p); f : M \rightarrow \mathbb{B}_1^n \text{ holomorphic, } f(p) = 0 \} \\ &= \sup \{ |Jac(f)(p)|^2; f : M \rightarrow \mathbb{B}_1^d \text{ holomorphic, } f(p) = 0 \}. \end{aligned}$$

The **Kobayashi pseudo-volume form** v_M^K is defined by ([Eisenman], p. 57), for any $p \in M$,

$$v_M^K(p) := \inf \{ |Jac(f)(0)|^{-2}; f : \mathbb{B}_1^n \rightarrow M \text{ holomorphic, } f(0) = p \}.$$

Next let us consider the case when there is a d -dimensional subvariety Z of M . In an analogous manner to the above, we may have the following definitions:

The **restricted Carathéodory pseudo-volume form** $v_{M|Z}^C$ on the regular locus Z_{reg} of Z is defined by, for any $p \in Z_{reg}$,

$$v_{M|Z}^C(p) := \sup \{ |Jac(f|_Z)(p)|^2; f : M \rightarrow \mathbb{B}_1^d \text{ holomorphic, } f(p) = 0 \}.$$

The **restricted Kobayashi pseudo-volume form** $v_{M|Z}^K$ on the regular locus Z_{reg} of Z is defined by, for any $p \in Z_{reg}$,

$$v_{M|Z}^K(p) := \inf \{ |Jac(i_Z \circ f)(0)|^{-2}; f : \mathbb{B}_1^d \rightarrow M \text{ holomorphic, } f(0) = p \},$$

where $i_Z : M \rightarrow Z$ is the restriction map.

3. BASIC PROPERTIES

The following are the volume decreasing properties of the corresponding volume forms, which are included here for the convenience of the reader.

Proposition 3.1 (Volume decreasing property for Carathéodory measure). *Let N be an n -dimensional complex manifold. Then, for all $\varphi \in Hol(M, N)$,*

$$\varphi^*(v_N^C) \leq v_M^C.$$

Proof. Any map from N to \mathbb{B}_1^n induces a map from M to \mathbb{B}_1^n . □

Proposition 3.2 (Volume decreasing property for Kobayashi measure). *Let N be an n -dimensional complex manifold. Then, for all $\varphi \in Hol(M, N)$,*

$$\varphi^*(v_N^K) \leq v_M^K.$$

Proof. Any map from \mathbb{B}_1^n to M induces a map from \mathbb{B}_1^n to N . □

Similarly, we have the following:

Proposition 3.3 (Volume decreasing property for restricted Carathéodory measure). *Let Y be an n -dimensional complex manifold and let W be its d -dimensional complex subvariety. Then, for all $\varphi \in Hol(X, Y)$, such that $\varphi(Z) \subset W$,*

$$(\varphi|_Z)^*(v_{Y|W}^C) \leq v_{X|Z}^C.$$

Proposition 3.4 (Volume decreasing property for restricted Kobayashi measure). *Let Y be an n -dimensional complex manifold and let W be its d -dimensional complex subvariety. Then, for all $\varphi \in \text{Hol}(X, Y)$, such that $\varphi(Z) \subset W$,*

$$(\varphi|_Z)^*(v_{Y|W}^K) \leq v_{X|Z}^K.$$

We also have continuity for the Carathéodory volume form.

Proposition 3.5. *v_M^C is locally Lipschitz.*

Proof. Let $o \in M$, and choose a local coordinate system U at o . Let \mathbb{B}_r^n be the coordinate ball centered at o . Let $p, q \in \mathbb{B}_{r/2}^n$ and suppose that

$$v_M^C(p) \geq v_M^C(q).$$

Then

$$\begin{aligned} v_M^C(p) - v_M^C(q) &= \sup |Jac(f)(p)|^2 - \sup \frac{|Jac(f)(q)|^2}{(1 - \|f(q)\|^2)^{n+1}} \\ &\leq \sup \{ |Jac(f)(p)|^2 - |Jac(f)(q)|^2 \} \\ &\leq \sup \{ (|Jac(f)(p) - Jac(f)(q)|)(|Jac(f)(p)| + |Jac(f)(q)|) \} \\ &\leq \frac{n^{3/2} 2^{n+2} n!}{r^{n+1}} \cdot \|p - q\| \cdot \frac{2^{n+1} n!}{r^n} \\ &= \frac{n^{3/2} 2^{2n+3} (n!)^2}{r^{2n+1}} \cdot \|p - q\|, \end{aligned}$$

where the supremum is taken over the set $\{f : M \rightarrow \mathbb{B}_1^n, f(p) = 0\}$. The last inequality above follows from Cauchy estimates. \square

Now let X be an n -dimensional compact Kähler manifold, and let \tilde{X} be its universal cover. The next result is a key property of Carathéodory measure hyperbolic universal covers.

Proposition 3.6 ([Kikuta 10]). *Suppose \tilde{X} is Carathéodory measure hyperbolic (i.e., $v_{\tilde{X}}^C > 0$ at every point). Then we have $c_1(K_X) > 0$.*

Proof. By the Arzela-Ascoli theorem, for any $p \in \tilde{X}$, there is a map $f_p : \tilde{X} \rightarrow \mathbb{B}_1^n$ such that $v_{\tilde{X}}^C(p) = (f_p^* \mu^n)(p) = |Jac(f_p)|^2(p)$.

For $f : \tilde{X} \rightarrow \mathbb{B}_1^n$ and $\zeta \in \mathbb{C}^n$, consider

$$\begin{aligned} &(\sqrt{-1} \partial \bar{\partial} \log(f^* \mu^n))(\zeta, \bar{\zeta}) \\ &= (n+1) \sum_{i,j,k,l} \left(\frac{(1 - \|f\|^2) \delta_{ij} + f_j \bar{f}_i}{(1 - \|f\|^2)^2} \right) \frac{\partial f_i}{\partial z_k} \frac{\bar{\partial} \bar{f}_j}{\partial z_l} \zeta^k \bar{\zeta}^l \\ &= (n+1) \left(\sum_{i,k,l} \frac{1}{1 - \|f\|^2} \frac{\partial f_i}{\partial z_k} \frac{\bar{\partial} \bar{f}_i}{\partial z_l} \zeta^k \bar{\zeta}^l + \left| \sum_{i,k} \frac{\partial f_i}{\partial z_k} \bar{f}_i \zeta^k \right|^2 \right) \\ &\geq (n+1) \cdot \frac{1}{1 - \|f\|^2} \sum_{i,k,l} \frac{\partial f_i}{\partial z_k} \frac{\bar{\partial} \bar{f}_i}{\partial z_l} \zeta^k \bar{\zeta}^l \\ &= (n+1) \cdot \frac{1}{1 - \|f\|^2} \cdot \zeta^*(Jac(f))^* \cdot Jac(f) \zeta. \end{aligned}$$

Now, for any $p \in \tilde{X}$, $|Jac(f_p)|^2(p) > 0$. Hence there is a neighborhood K of p and constants $c, \epsilon > 0$ such that

$$1 - \|f_p(q)\|^2 \geq c > 0$$

and

$$\zeta^*(Jac(f_p))^*(q) \cdot Jac(f_p)(q)\zeta \geq \epsilon|\zeta|^2$$

for all $q \in K$. Notice that $v_{\tilde{X}}^C$ is continuous and therefore

$$\log(v_{\tilde{X}}^C) = \sup_{p \in \tilde{X}} (f_p^* \mu^n)$$

is strictly plurisubharmonic.

Interpret $v_{\tilde{X}}^C$ as a volume form over X with $\log(v_{\tilde{X}}^C)$ being strictly plurisubharmonic. By [Richberg], $v_{\tilde{X}}^C$ can be approximated uniformly over X by a smooth volume form whose logarithm is strictly plurisubharmonic. Hence, $c_1(K_X) > 0$. \square

In the case that the covering is a bounded domain in \mathbb{C}^n , we have the following extra geometric property.

Proposition 3.7 (Uniform squeezing property for \tilde{X}). *Suppose \tilde{X} is a bounded domain in \mathbb{C}^n which covers a compact Kähler manifold X . Then there exist a, b satisfying $0 < a < b < \infty$ such that for any $x \in \tilde{X}$, there exists an embedding $\varphi_x : \tilde{X} \rightarrow \mathbb{C}^n$ with $\varphi_x(x) = 0$ and $\mathbb{B}_a^n \subset \varphi_x(\tilde{X}) \subset \mathbb{B}_b^n$.*

Proof. Let A be any fundamental domain of X in \tilde{X} . For any $x \in A$, take $r_x = \inf_{y \in \partial \tilde{X}} |x - y| > 0$, $R_x = \sup_{y \in \partial \tilde{X}} |x - y| < \infty$ so that we have $\mathbb{B}_{r_x}^n(x) \subset \tilde{X} \subset \mathbb{B}_{R_x}^n(x)$.

Now since A is relatively compact in \tilde{X} and r_x, R_x are Lipschitz continuous in x , we have $\inf_A r_x := a > 0$ and $\sup_A R_x := b < \infty$.

For any $x \in \tilde{X} - A$, there is an automorphism of \tilde{X} which brings x to a point in A , and hence for all points in \tilde{X} , we have an embedding φ_x such that $\varphi_x(x) = 0$ and $\mathbb{B}_a^n \subset \varphi_x(\tilde{X}) \subset \mathbb{B}_b^n$. \square

Finally we include the statement of two Schwarz lemmas and a result on L^2 -estimate for the $\bar{\partial}$ -equation that will be used.

Proposition 3.8 (Schwarz lemma of [Mok-Yau]). *Let M be a complete Hermitian manifold with scalar curvature bounded from below by $-K_1$ and let N be a complex manifold of the same dimension with a volume form V_N (i.e., positive (n, n) form, $n = \dim N$) such that the Ricci form is negative definite and $(\frac{\sqrt{-1}}{2} \partial \bar{\partial} \log V_N)^n \geq K_2 V_N$. Suppose $f : M \rightarrow N$ is a holomorphic map and the Jacobian is non-vanishing at one point. Then $K_1 > 0$ and*

$$\sup \frac{f^* V_N}{V_M} \leq \frac{K_1^n}{n^n K_2}$$

Proposition 3.9 (Schwarz lemma of [Royden]). *Let (M, g) be a complete Kähler manifold with Ricci curvature bounded from below by $k \leq 0$, and let (N, h) be*

a Kähler manifold with holomorphic sectional curvature bounded from above by $K < 0$. Then for any holomorphic map $f : M \rightarrow N$ we have

$$\sum_{\alpha, \beta, i, j} g^{\alpha\bar{\beta}} f_{\alpha}^i \bar{f}_{\beta}^j h_{i\bar{j}} \leq \frac{2\nu}{\nu + 1} \frac{k}{K},$$

where ν is the maximal rank of df .

Proposition 3.10 ([Hörmander]). *Let M be a complete Kähler manifold and let φ be a smooth strictly psh function on M . Then for any $v \in L^2_{n,1}(M, \partial\bar{\partial}\varphi, \varphi)$ with $\bar{\partial}v = 0$, there is an $(n, 0)$ -form u on M such that $\bar{\partial}u = v$ and*

$$\left| \int_M u \wedge \bar{u} e^{-\varphi} \right| \leq \int_M |v|^2_{\partial\bar{\partial}\varphi} e^{-\varphi}.$$

4. UNIFORM ESTIMATES AMONG INVARIANT VOLUMES

Theorem 4.1. *Let X be an n -dimensional compact Kähler manifold, and let \tilde{X} be its universal cover. Then we have, at any $p \in \tilde{X}$,*

(a)

$$v_{\tilde{X}}^C \leq v_{\tilde{X}}^K,$$

(b)

$$v_{\tilde{X}}^C \leq \frac{1}{n!} \cdot v_{\tilde{X}}^{KE},$$

$$v_{\tilde{X}}^{KE} \leq \frac{1}{n!} \cdot v_{\tilde{X}}^K,$$

$$c_1 \cdot v_{\tilde{X}}^C \leq v_{\tilde{X}}^B \leq c_2 \cdot v_{\tilde{X}}^C$$

if we assume \tilde{X} is Carathéodory measure hyperbolic, where $b_1 = b_1(X) > 0$ and $b_2 = b_2(X) < \infty$.

(c)

$$v_{\tilde{X}}^K \leq \frac{b^{2n}}{a^{2n}} \cdot v_{\tilde{X}}^C$$

if we assume \tilde{X} is a bounded domain in \mathbb{C}^n , where $a = a(X) > 0$, and $b = b(X) < \infty$ are as in Proposition 3.7.

Proof. (a) $v_{\tilde{X}}^C \leq v_{\tilde{X}}^K$:

Fix $\epsilon > 0$, $p \in \tilde{X}$. Let $f : \mathbb{B}_1^n \rightarrow \tilde{X}$ and $g : \tilde{X} \rightarrow \mathbb{B}_1^n$ be any holomorphic maps with $f(0) = p$ and $g(p) = 0$ respectively and such that they satisfy

$$0 < |Jac(f)(0)|^{-2} < v_{\tilde{X}}^K + \epsilon$$

and

$$|Jac(g)(p)|^2 > v_{\tilde{X}}^C - \epsilon.$$

Consider the composition $\mathbb{B}_1^n \xrightarrow{f} \tilde{X} \xrightarrow{g} \mathbb{B}_1^n$.

Applying the Ahlfors-Schwarz lemma gives

$$(g \circ f)^* \mu^n \leq \mu^n.$$

Expressed in terms of f and g this gives

$$|Jac(g)(p)|^2 \leq |Jac(f)(0)|^{-2},$$

and hence

$$v_{\tilde{X}}^C - \epsilon \leq v_{\tilde{X}}^K + \epsilon.$$

We are done since ϵ is arbitrary.

(b) $v_{\tilde{X}}^C \leq \frac{1}{n!} \cdot v_{\tilde{X}}^{KE}$:

By Proposition (3.6), we know that $c_1(K_X) > 0$.

By the Calabi-Yau theorem, there is a unique Kähler-Einstein metric g_X^{KE} on X such that

$$Ric(g_X^{KE}) = -2(n + 1) \cdot \omega_X^{KE}.$$

Pull back the metric by π so that we have, on \tilde{X} , the complete metric $g_{\tilde{X}}^{KE}$ satisfying

$$Ric(g_{\tilde{X}}^{KE}) = -2(n + 1) \cdot \omega_{\tilde{X}}^{KE}.$$

Hence the scalar curvature of \tilde{X} w.r.t this metric is $-n(n + 1)$.

On \mathbb{B}_1^n , we can construct a Kähler-Einstein metric $g_{\mathbb{B}_1^n}^{KE}$ such that

$$Ric(g_{\mathbb{B}_1^n}^{KE}) = -2(n + 1) \cdot \omega_{\mathbb{B}_1^n}^{KE}.$$

Hence

$$\left(\frac{\sqrt{-1}}{2} \partial \bar{\partial} \log \mu^n \right)^n = (n + 1)^n n! \mu^n.$$

For any map $f : \tilde{X} \rightarrow \mathbb{B}_1^n$, applying the Schwarz lemma of [Mok-Yau] yields

$$f^* \mu^n \leq \frac{(n(n + 1))^n}{n^n (n + 1)^n n!} \cdot v_{\tilde{X}}^{KE} = \frac{1}{n!} \cdot v_{\tilde{X}}^{KE}.$$

Hence we have

$$v_{\tilde{X}}^C \leq \frac{1}{n!} \cdot v_{\tilde{X}}^{KE}.$$

(b) $v_{\tilde{X}}^{KE} \leq \frac{1}{n!} \cdot v_{\tilde{X}}^K$:

For any $p \in \tilde{X}$, consider any map $f : \mathbb{B}_1^n \rightarrow \tilde{X}$ with $f(0) = p$ and $(f^{-1})^*$ well-defined at p . Applying the Schwarz lemma of [Mok-Yau] yields

$$f^* v_{\tilde{X}}^{KE} \leq \frac{(n(n + 1))^n}{n^n (n + 1)^n n!} \cdot \mu^n.$$

Hence we have

$$v_{\tilde{X}}^{KE} \leq \frac{1}{n!} (f^{-1})^* \mu^n.$$

Therefore

$$v_{\tilde{X}}^{KE} \leq \frac{1}{n!} \cdot v_{\tilde{X}}^K.$$

(b) $b_1 \cdot v_{\tilde{X}}^C \leq v_{\tilde{X}}^B \leq b_2 \cdot v_{\tilde{X}}^C$:

For any $p \in \tilde{X}$, there is an $f_p : \tilde{X} \rightarrow \mathbb{B}_1^n$ such that $f_p(p) = 0$ and $v_{\tilde{X}}^C = |Jac(f_p)|^2 > 0$. Let λ be a C^∞ real smooth cutoff function that is $\equiv 1$ in a neighborhood of p . Let $\{x_i\}$ be a countable dense subset of \tilde{X} , and set $\psi := \log(1 + \sum_i \frac{1}{2^i} \|f_{x_i}\|^2)$. A direct computation shows that ψ is strictly plurisubharmonic ([Kikuta 10]) and clearly ψ is bounded. Choose $k > 0$ such that $\varphi := k\psi + \lambda(n + 1) \log |z|^2$ is strictly psh on $\tilde{X} \setminus \{p\}$. By Proposition 3.10, we can solve the equation $\bar{\partial} u = \bar{\partial}(\lambda Jac(f_p))$ on $\tilde{X} \setminus \{p\}$ with estimate

$$\left| \int_{\tilde{X} \setminus \{p\}} u \wedge \bar{u} e^{-\varphi} \right| \leq \int_{\tilde{X} \setminus \{p\}} |\bar{\partial}(\lambda Jac(f_p))|_{\partial \bar{\partial} \varphi}^2 e^{-\varphi} \leq C.$$

This implies that the L^2 holomorphic n -form u satisfies

$$u(p) = Jac(f_p)(p) \neq 0.$$

Hence, $v_{\tilde{X}}^B(p) > 0$. Consider the ratio $v_{\tilde{X}}^B/v_{\tilde{X}}^C$ being continuous and invariant under automorphisms of \tilde{X} , and let A be a fundamental domain of X in \tilde{X} . We have

$$\inf_{\tilde{X}} v_{\tilde{X}}^B/v_{\tilde{X}}^C = \inf_A v_{\tilde{X}}^B/v_{\tilde{X}}^C := b_1(X) > 0$$

and

$$\sup_{\tilde{X}} v_{\tilde{X}}^B/v_{\tilde{X}}^C = \sup_A v_{\tilde{X}}^B/v_{\tilde{X}}^C := b_2(X) < \infty.$$

(c) $v_{\tilde{X}}^K \leq \frac{b^{2n}}{a^{2n}} \cdot v_{\tilde{X}}^C$:

Realize \tilde{X} as $\mathbb{B}_a^n \subset \tilde{X} \subset \mathbb{B}_b^n$ with $p = (0, \dots, 0)$. Consider the composition

$$\mathbb{B}_1^n \rightarrow \mathbb{B}_a^n \hookrightarrow \tilde{X} \hookrightarrow \mathbb{B}_b^n \rightarrow \mathbb{B}_1^n$$

in which the first and last maps are given, respectively, by

$$(w_1, \dots, w_n) \mapsto (aw_1, \dots, aw_n)$$

and

$$(z_1, \dots, z_n) \mapsto (z_1/b, \dots, z_n/b).$$

Hence,

$$v_{\tilde{X}}^C(p) \geq \frac{1}{b^{2n}} \frac{\sqrt{-1}}{2} dz^1 \wedge d\bar{z}^1 \wedge \dots \wedge \frac{\sqrt{-1}}{2} dz^n \wedge d\bar{z}^n$$

and

$$v_{\tilde{X}}^K(p) \leq \frac{1}{a^{2n}} \frac{\sqrt{-1}}{2} dz^1 \wedge d\bar{z}^1 \wedge \dots \wedge \frac{\sqrt{-1}}{2} dz^n \wedge d\bar{z}^n.$$

□

We know that the Carathéodory pseudo-volume form is invariant under a bi-holomorphic map. Hence, the **Carathéodory measure** $\mu_{\tilde{X}}^C(X)$ of X can thus be defined to be

$$\mu_{\tilde{X}}^C(X) := \int_A v_{\tilde{X}}^C,$$

where A can be any fundamental domain of X in \tilde{X} .

Next, let L be a holomorphic line bundle over X . The **volume** $\text{vol}_X(L)$ of L is defined as

$$\text{vol}_X(L) := \limsup_{m \rightarrow \infty} \frac{\dim H^0(X, \mathcal{O}(mL))}{m^n/n!}.$$

The following result will be needed to estimate the volume of K_X .

Proposition 4.2 ([Boucksom]). *Let X be an n -dimensional compact Kähler manifold, and let L be a holomorphic line bundle over X . Then*

$$\text{vol}_X(L) = \sup_T \left\{ \int_X (T_{ac})^n \right\},$$

where T is semi-positive $(1, 1)$ -current in $c_1(L)$ and T_{ac} is the absolutely continuous part of T in its Lebesgue decomposition $T = T_{ac} + T_{sg}$. T_{ac} is considered as a $(1, 1)$ -form with L^1_{loc} coefficients, so that $(T_{ac})^n$ is meant pointwise.

Corollary 4.3. *Let X be an n -dimensional compact Kähler manifold such that \tilde{X} is Carathéodory measure hyperbolic. Then we have*

$$\frac{(n!)^2(n+1)^n}{(\pi)^n} \cdot \mu_{\tilde{X}}^C(X) \leq \text{vol}_X(K_X).$$

Proof. From

$$n! \cdot v_{\tilde{X}}^C \leq v_{\tilde{X}}^{KE} = \frac{1}{n!} (\omega_{\tilde{X}}^{KE})^n = \frac{1}{n!(2(n+1))^n} (-Ric(g_{\tilde{X}}^{KE}))^n$$

we have

$$\frac{(n!)^2(n+1)^n}{(\pi)^n} \cdot v_{\tilde{X}}^C \leq \left(-\frac{1}{2\pi} Ric(g_{\tilde{X}}^{KE}) \right)^n.$$

Let A be a fundamental domain of X in \tilde{X} . Integrating the above inequality over A , we have

$$\frac{(n!)^2(n+1)^n}{(\pi)^n} \int_A v_{\tilde{X}}^C \leq \int_A \left(-\frac{1}{2\pi} Ric(g_{\tilde{X}}^{KE}) \right)^n.$$

Applying Proposition 4.2, we get

$$\frac{(n!)^2(n+1)^n}{(\pi)^n} \cdot \mu_{\tilde{X}}^C(X) \leq \text{vol}_X(K_X).$$

□

A similar estimate is obtained in [Kikuta 10] by using a different method.

5. UNIFORM ESTIMATES AMONG RESTRICTED INVARIANT VOLUMES

Similar to the treatment of the last section, we have the following:

Theorem 5.1. *Let X be an n -dimensional compact Kähler manifold, let $\pi : \tilde{X} \rightarrow X$ be its universal cover, let Z be a d -dimensional subvariety of X , and let \tilde{Z}_{reg} be the regular part of the subvariety $\tilde{Z} := \pi^{-1}(Z)$. Then we have, at any $p \in \tilde{Z}_{reg}$,*

(a)

$$v_{\tilde{X}|\tilde{Z}}^C \leq v_{\tilde{X}|\tilde{Z}}^K,$$

(b)

$$v_{\tilde{X}|\tilde{Z}}^C \leq \frac{d^d(n+1)^d}{(d+1)^d} \cdot v_{\tilde{X}|\tilde{Z}}^{KE}$$

if we assume \tilde{X} is Carathéodory measure hyperbolic.

(c)

$$v_{\tilde{X}|\tilde{Z}}^K \leq \frac{b^{2d}}{a^{2d}} \cdot v_{\tilde{X}|\tilde{Z}}^C$$

if we assume \tilde{X} is a bounded domain in \mathbb{C}^n , where $a = a(X)$, and $b = b(X)$ are as in Proposition 3.7.

Proof. (a) $v_{\tilde{X}|\tilde{Z}}^C \leq v_{\tilde{X}|\tilde{Z}}^K$:

Fix $\epsilon > 0$, $p \in \tilde{Z}_{reg}$. Let $f : \mathbb{B}_1^d \rightarrow \tilde{X}$ and $g : \tilde{X} \rightarrow \mathbb{B}_1^d$ be any holomorphic maps with $f(0) = p$ and $g(p) = 0$ respectively and such that they satisfy

$$0 < |Jac(i_{\tilde{Z}_{reg}} \circ f)(0)|^{-2} < v_{\tilde{X}|\tilde{Z}}^K + \epsilon$$

and

$$|Jac(g|_{\tilde{Z}_{reg}})(p)|^2 > v_{\tilde{X}|\tilde{Z}}^C - \epsilon.$$

Consider the composition $\mathbb{B}_1^d \xrightarrow{f} \tilde{X} \xrightarrow{i_{\tilde{Z}_{reg}}} \tilde{Z}_{reg} \xrightarrow{g|_{\tilde{Z}_{reg}}} \mathbb{B}_1^d$.

Applying the Ahlfors-Schwarz lemma gives

$$(g \circ f)^* \mu^d \leq \mu^d.$$

Expressing this in terms of $i_{\tilde{Z}_{reg}} \circ f$ and $g|_{\tilde{Z}_{reg}}$ gives

$$|Jac(g|_{\tilde{Z}_{reg}})(p)|^2 \leq |Jac(i_{\tilde{Z}_{reg}} \circ f)(0)|^{-2},$$

and hence

$$v_{\tilde{X}|\tilde{Z}}^C - \epsilon \leq v_{\tilde{X}|\tilde{Z}}^K + \epsilon.$$

We are done since ϵ is arbitrary.

(b) $v_{\tilde{X}|\tilde{Z}}^C \leq \frac{d^d(n+1)^d}{(d+1)^d} \cdot v_{\tilde{X}|\tilde{Z}}^{KE}$:

By Proposition 3.6, we know that $c_1(K_X) > 0$.

By the Calabi-Yau theorem ([Yau]), there is a unique Kähler-Einstein metric g_X^{KE} on X such that

$$Ric(g_X^{KE}) = -2(n+1) \cdot \omega_X^{KE}.$$

Pull back the metric by π so that we have, on \tilde{X} , the complete metric $g_{\tilde{X}}^{KE}$ satisfying

$$Ric(g_{\tilde{X}}^{KE}) = -2(n+1) \cdot \omega_{\tilde{X}}^{KE}.$$

On \mathbb{B}_1^d , we can construct a Kähler-Einstein metric $g_{\mathbb{B}_1^d}^{KE}$ such that

$$Ric(g_{\mathbb{B}_1^d}^{KE}) = -2(d+1) \cdot \omega_{\mathbb{B}_1^d}^{KE}.$$

For any map $f : \tilde{X} \rightarrow \mathbb{B}_1^d$, applying the Schwarz lemma of [Royden] yields

$$f^* \omega_{\mathbb{B}_1^d}^{KE} \leq \frac{2d}{d+1} \frac{n+1}{2} \cdot \omega_{\tilde{X}}^{KE}.$$

Taking the d -th power on both sides yields

$$f^* \mu^d \leq \frac{d^d(n+1)^d}{(d+1)^d} \cdot \frac{(\omega_{\tilde{X}}^{KE})^d}{d!}.$$

Restricting to \tilde{Z}_{reg} gives

$$v_{\tilde{X}|\tilde{Z}}^C \leq \frac{d^d(n+1)^d}{(d+1)^d} \cdot v_{\tilde{X}|\tilde{Z}}^{KE}.$$

(c) $v_{\tilde{X}|\tilde{Z}}^K \leq \frac{b^{2d}}{a^{2d}} \cdot v_{\tilde{X}|\tilde{Z}}^C$:

Realize \tilde{X} as $\mathbb{B}_a^n \subset \tilde{X} \subset \mathbb{B}_b^n$ with $p = (0, \dots, 0)$, and locally at p , \tilde{Z} is given by $z_{d+1} = \dots = z_n = 0$. Consider the composition

$$\mathbb{B}_1^d \rightarrow \mathbb{B}_a^n \hookrightarrow \tilde{X} \hookrightarrow \mathbb{B}_b^n \rightarrow \mathbb{B}_1^d$$

in which the first and last maps are given, respectively, by

$$(w_1, \dots, w_d) \mapsto (aw_1, \dots, aw_d, 0, \dots, 0)$$

and

$$(z_1, \dots, z_n) \mapsto (z_1/b, \dots, z_d/b).$$

Hence,

$$v_{\tilde{X}|\tilde{Z}}^C(p) \geq \frac{1}{b^{2d}} \frac{\sqrt{-1}}{2} dz^1 \wedge d\bar{z}^1 \wedge \dots \wedge \frac{\sqrt{-1}}{2} dz^d \wedge d\bar{z}^d$$

and

$$v_{\tilde{X}|\tilde{Z}}^K(p) \leq \frac{1}{a^{2d}} \frac{\sqrt{-1}}{2} dz^1 \wedge d\bar{z}^1 \wedge \dots \wedge \frac{\sqrt{-1}}{2} dz^d \wedge d\bar{z}^d.$$

□

We know that the restricted Carathéodory pseudo-volume form is invariant under biholomorphic map preserving Z . Hence, the **restricted Carathéodory measure** $\mu_{\tilde{X}|\tilde{Z}}^C(Z)$ of Z can thus be defined to be

$$\mu_{\tilde{X}|\tilde{Z}}^C(Z) := \int_{A \cap \tilde{Z}_{reg}} v_{\tilde{X}|\tilde{Z}}^C,$$

where A can be any fundamental domain of X in the covering $\pi : \tilde{X} \rightarrow X$ and \tilde{Z}_{reg} is the regular part of the subvariety $\tilde{Z} := \pi^{-1}(Z)$.

Next, let L be a holomorphic line bundle over X , the **restricted volume** $vol_{X|Z}(L)$ of L along Z is defined as

$$vol_{X|Z}(L) := \limsup_{m \rightarrow \infty} \frac{\dim H^0(X|Z, \mathcal{O}(mL))}{m^d/d!}$$

The following result will be needed to estimate the restricted volume of K_X .

Proposition 5.2 ([Boucksom],[Hisamoto],[Matsumura]). *Let X be an n -dimensional projective manifold, let L be a big line bundle over X , and let Z be an irreducible d -dimensional subvariety of X . Furthermore, suppose that $Z \not\subset \mathbb{B}_+(L)$. Then*

$$vol_{X|Z}(L) = \sup_T \left\{ \int_{Z_{reg}} (T|_{Z_{reg}})_{ac}^d \right\},$$

where T is semi-positive $(1, 1)$ -currents in $c_1(L)$ that have small unbounded loci and whose unbounded loci do not contain Z . Here we denote by $T|_{Z_{reg}}$ the restriction of T to the regular locus of Z and by $(T|_{Z_{reg}})_{ac}$ its absolutely continuous part.

Here we arrive at our promised result, settling a conjecture in [Kikuta 13].

Corollary 5.3. *Let X be an n -dimensional compact Kähler manifold such that \tilde{X} is Carathéodory measure hyperbolic, and let Z be a d -dimensional subvariety of X . Then we have*

$$\frac{d!(d+1)^d}{\pi^d d^d} \cdot \mu_{\tilde{X}|\tilde{Z}}^C(Z) \leq vol_{X|Z}(K_X).$$

Proof. From

$$\frac{(d+1)^d}{d^d(n+1)^d} \cdot v_{\tilde{X}|\tilde{Z}}^C \leq v_{\tilde{X}|\tilde{Z}}^{KE} = \frac{1}{d!} ((\omega_{\tilde{X}}^{KE})|_{\tilde{Z}_{reg}})^d = \frac{1}{d!(2(n+1))^d} \left(-Ric(g_{\tilde{X}}^{KE})|_{\tilde{Z}_{reg}} \right)^d$$

we have

$$\frac{d!(d+1)^d}{\pi^d d^d} \cdot v_{\tilde{X}|\tilde{Z}}^C \leq \left(-\frac{1}{2\pi} Ric(g_{\tilde{X}}^{KE})|_{\tilde{Z}_{reg}} \right)^d.$$

Let A be a fundamental domain of X in \tilde{X} . Integrating the above inequality over $A \cap \tilde{Z}_{reg}$ gives

$$\frac{d!(d+1)^d}{\pi^d d^d} \int_{A \cap \tilde{Z}_{reg}} v_{\tilde{X}|\tilde{Z}}^C \leq \int_{A \cap \tilde{Z}_{reg}} \left(-\frac{1}{2\pi} Ric(g_{\tilde{X}}^{KE})|_{\tilde{Z}} \right)^d.$$

Applying Proposition 5.2, we get

$$\frac{d!(d+1)^d}{\pi^d d^d} \cdot \mu_{\tilde{X}|\tilde{Z}}^C(Z) \leq vol_{X|Z}(K_X).$$

□

6. COMPLETENESS OF BERGMAN METRIC

Suppose that $v_M^B > 0$ at every point. The **Bergman pseudo-metric** g_M^B on M is defined by

$$(g_M^B)_{i\bar{j}} := \frac{\partial^2 \log(v_M^B)}{\partial z_i \partial \bar{z}_j}.$$

From [Hahn], we have

$$g_M^B(p, v) = \frac{\partial_v g_{p,v} \wedge \overline{\partial_v g_{p,v}}}{\varphi_p \wedge \overline{\varphi_p}},$$

where φ_p maximizes $(\varphi \wedge \bar{\varphi})(p)$ and $g_{p,v}$ maximizes $(\partial_v g \wedge \overline{\partial_v g})(p)$ with $g(p) = 0$ over the unit ball of $L^2(M, K_M)$.

Theorem 6.1. *Let X be an n -dimensional compact Kähler manifold such that \tilde{X} is Carathéodory measure hyperbolic. Then $g_{\tilde{X}}^B$ is complete.*

Proof. By Theorem 4.1, we know that $v_{\tilde{X}}^B > 0$ at every point, so $g_{\tilde{X}}^B$ is well-defined. By Proposition 3.6, we know that $c_1(K_X) > 0$; hence there exists a complete Kähler-Einstein metric on \tilde{X} . By Proposition 2.3 of [Chen], to show that the Bergman metric is positive definite, it suffices to construct a bounded smooth strictly plurisubharmonic function ψ on \tilde{X} . Let $\{x_i\}$ be a countable dense subset of \tilde{X} , set $\psi := \log(1 + \sum_i \frac{1}{2^i} \|f_{x_i}\|^2)$. A direct computation shows that ψ is strictly plurisubharmonic ([Kikuta 10]), and clearly ψ is bounded.

Now let $\{y_i\}$ be a Cauchy sequence in \tilde{X} w.r.t. $g_{\tilde{X}}^B$. As $g_{\tilde{X}}^B$ is an invariant metric, the push-forward of $g_{\tilde{X}}^B$ by $\pi : \tilde{X} \rightarrow X$ onto X is a well-defined, positive definite metric, denoted by g_X . Hence, $\{\pi(y_i)\}$ is Cauchy w.r.t. g_X on the compact manifold X . Hence, $\{\pi(y_i)\}$ converges to, say, $z \in X$. Let $\pi^{-1}\{z\} = \{x_i\}_{i \in I}$. By the discreteness of the action of deck transformations on $\{x_i\}_{i \in I}$, there exists ϵ -neighborhoods of the x_i 's that are mutually non-intersecting. Hence, there exists x_{i_0} in $\{x_i\}_{i \in I}$ that is the limit of $\{y_i\}$. □

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