CHARACTERIZATION OF COMPACTNESS OF COMMUTATORS OF BILINEAR SINGULAR INTEGRAL OPERATORS

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(Communicated by Svitlana Mayboroda)

ABSTRACT. The commutators of bilinear Calderón–Zygmund operators and pointwise multiplication with a symbol in CMO are bilinear compact operators on products of Lebesgue spaces. We show that, for certain non-degenerate Calderón–Zygmund operators, the symbol being in CMO is not only sufficient but actually necessary for the compactness of the commutators.

1. INTRODUCTION

In this note we resolve a problem that has been open for a while in the multilinear Calderón–Zygmund theory: namely, whether the compactness of the commutators of the bilinear Riesz transforms (see the next section for technical definitions) with pointwise multiplication can be used to characterize the space $\mathrm{CMO}(\mathbb{R}^n)$. For the purpose of this article, $\mathrm{CMO}(\mathbb{R}^n)$ is the closure in the John–Nirenberg $\mathrm{BMO}(\mathbb{R}^n)$, with its usual topology, of the space of infinitely differentiable functions with compact support. This problem has been motivated by the analogous situation in the classical (linear) Calderón–Zygmund theory and several preliminary existing results in the multilinear setting, which we summarize in what follows.

As is well-known, the first to study the commutator

$$[b, \mathcal{R}^k](f) := \mathcal{R}^k(bf) - b\mathcal{R}^k(f)$$

of the classical Riesz transforms $\mathcal{R}^k$ with pointwise multiplication by a function b were Coifman, Rochberg, and Weiss [5]. They showed that $[b, \mathcal{R}^k]$ is bounded on L^p for some p with $1 < p < \infty$ if and only if the symbol b is in $\mathrm{BMO}(\mathbb{R}^n)$. Their result was then extended to other non-degenerate Calderón–Zygmund operators by Janson [7] and Uchiyama [14]. Moreover, Uchiyama showed that $[b, \mathcal{R}^k]$ is compact on L^p for some (then for all) $1 < p < \infty$ if and only if the function b is not just in $\mathrm{BMO}(\mathbb{R}^n)$ but actually in $\mathrm{CMO}(\mathbb{R}^n)$.

In the multilinear setting, an interesting situation arises: multilinear Calderón–Zygmund operators, their commutators, and other related operators tend to be bounded also into L^p spaces outside the Banach space situation. For example, in the bilinear case a Calderón–Zygmund operator T in the sense of Grafakos and Torres [6] (see also the references therein) satisfies

$$T : L^{p_1} \times L^{p_2} \to L^p,$$

for all $1 < p_1 < \infty$, $1 < p_2 < \infty$, and $1/p_1 + 1/p_2 = 1/p < 2$. This creates complications when studying the case of $p < 1$ in the target space, as some analytic tools (often depending on duality) fail in this situation. For this reason the cases $p > 1$ and $p < 1$ have been occasionally treated separately in the literature and by different arguments. For example, the boundedness of the commutators

$$[b, T]_1(f, g) := T(bf, g) - bT(f, g),$$
$$[b, T]_2(f, g) := T(f, bg) - bT(f, g)$$

of a bilinear Calderón–Zygmund operator T with a BMO function b was first obtained by Pérez and Torres in [10] when $p > 1$, while the case of $p \leq 1$ was later studied independently by Tang [12] and Lerner et al. [8]. The compactness of the same commutators when $b \in \mathrm{CMO}(\mathbb{R}^n)$ was obtained by Bényi and Torres in [1] but only for $p \geq 1$. Nonetheless, it was recently observed by Torres, Xue, and Yan [13] that the result also holds for $1/2 < p < 1$. The partial converse fact that the boundedness of $[b, T]_1$ or $[b, T]_2$ for certain bilinear Calderón–Zygmund operators forces b to be in $\mathrm{BMO}(\mathbb{R}^n)$ was first proved by Chaffee [2] and was then also revisited by Li and Wick [9] using different techniques. In both cases the results are also under the assumption $p > 1$. Finally, in a very recent manuscript posted to the arXiv by Wang, Zhou, and Teng [15], the result of Chaffee [2] was extended to $1/2 < p \leq 1$.

We will show in Theorem 3.1 below that at least for the bilinear Riesz transforms, the compactness of the commutators forces the symbol b to be in $\mathrm{CMO}(\mathbb{R}^n)$. Our work follows ideas of Uchiyama [14] and Chen, Ding, and Wang [4] in the linear case, as well as modifications done in [3] for the bilinear operators. We note however that the main difference with respect to the work in [3], and a difficulty we overcome here, is that the operators in [3] are bilinear fractional integral operators which are hence positively defined, which is a property heavily used in [3] but which certainly completely fails for Calderón–Zygmund operators. We refer the reader to [3] and the references therein for more on commutators of fractional singular operators in both linear and multilinear settings.

2. Definitions

As mentioned in the introduction, the space $\mathrm{CMO}(\mathbb{R}^n)$ is the closure in the $\mathrm{BMO}(\mathbb{R}^n)$ topology of the space of infinitely differentiable functions with compact support, denoted here by $C_c^\infty(\mathbb{R}^n)$. For brevity, throughout the paper we denote $L^p(\mathbb{R}^n)$ by L^p and similarly for BMO, CMO, and C_c^∞. Also, for convenience, we will use the BMO norm (modulo constants) defined for a locally integrable function b by

$$\|b\|_{\mathrm{BMO}} := \sup_Q \fint_Q |b(x) - b_Q| \, dx < \infty,$$

with the supremum taken over all cubes $Q \subset \mathbb{R}^n$ with edges parallel to the coordinate axes, and where for any locally integrable function f we use the standard notation $f_Q = \fint_Q f := \frac{1}{|Q|} \int_Q f(x) \, dx$ for the average of f over Q. In addition, we recall (see [14]) that $b \in \mathrm{BMO}$ is in CMO if and only if

$$(1) \qquad \lim_{a \to 0} \sup_{|Q|=a} \frac{1}{|Q|} \int_Q |b(x) - b_Q| \, dx = 0,$$

$$(2) \qquad \lim_{a \to \infty} \sup_{|Q|=a} \frac{1}{|Q|} \int_Q |b(x) - b_Q| \, dx = 0, \qquad \text{and}$$

$$(3) \qquad \lim_{|y| \to \infty} \frac{1}{|Q+y|} \int_{Q+y} |b(x) - b_{Q+y}| \, dx = 0, \ \text{for each } Q.$$

For $x \in \mathbb{R}^n$ we will use the notation $x = (x^1, \ldots, x^n)$ and consider the $2n$ *bilinear Riesz transform operators* defined for $k = 1, \ldots, n$ by

$$\mathcal{R}_1^k(f,g)(x) := \mathrm{p.v.} \iint_{\mathbb{R}^{2n}} \frac{x^k - y^k}{(|x-y|^2 + |x-z|^2)^{n+1/2}} \, f(y)g(z) \, dydz,$$

$$\mathcal{R}_2^k(f,g)(x) := \mathrm{p.v.} \iint_{\mathbb{R}^{2n}} \frac{x^k - z^k}{(|x-y|^2 + |x-z|^2)^{n+1/2}} \, f(y)g(z) \, dydz.$$

The name of these operators is justified by the fact that they can be "obtained" by considering the linear Riesz transforms in $\mathbb{R}^{2n}$ defined by

$$\mathcal{R}^k(F)(u) := \mathrm{p.v.} \int_{\mathbb{R}^{2n}} \frac{u^k - v^k}{|u-v|^{2n+1}} \, F(v) \, dv,$$

where $u = (u^1, \ldots, u^{2n})$ and $v = (v^1, \ldots, v^{2n})$, $k = 1, \ldots, 2n$. Note that setting $u = (x,x)$, $v = (y,z)$ with $x, y, z \in \mathbb{R}^n$, and $F(y,z) = f(y)g(z)$ leads, formally, to the bilinear operators $\mathcal{R}_j^k$, $j = 1, 2$. For $k = 1, \ldots, n$, $\mathcal{R}_1^k(f,g)(x) = \mathcal{R}^k(fg)(x,x)$, while $\mathcal{R}_2^k(f,g)(x) = \mathcal{R}^{k+n}(fg)(x,x)$.

The boundedness of the $\mathcal{R}_j^k$ operators from $L^{p_1} \times L^{p_2}$ to L^p, for $1 < p_1 < \infty$, $1 < p_2 < \infty$, and $1/p_1 + 1/p_2 = 1/p < 2$, is by now well-known. See for example [6] and the references therein.

For $j = 1, 2$ and $k = 1, \ldots, n$, the (first-order) commutators of the Riesz transform operators with a symbol b are given by

$$(4) \qquad \begin{aligned} [b, \mathcal{R}_j^k]_1(f,g) &:= \mathcal{R}_j^k(bf,g) - b\mathcal{R}_j^k(f,g), \\ [b, \mathcal{R}_j^k]_2(f,g) &:= \mathcal{R}_j^k(f,bg) - b\mathcal{R}_j^k(f,g). \end{aligned}$$

Notice that $b \in \mathrm{BMO}$ is consistent with the fact that, by linearity, for any complex number C,

$$[b - C, \mathcal{R}_j^k]_1(f,g) = [b, \mathcal{R}_j^k]_1(f,g),$$
$$[b - C, \mathcal{R}_j^k]_2(f,g) = [b, \mathcal{R}_j^k]_2(f,g),$$

a fact that we will later use.

By the results mentioned in the introduction the boundedness of any of these commutators from $L^{p_1} \times L^{p_2}$ to L^p, for the full range of exponents $1 < p_1 < \infty$, $1 < p_2 < \infty$, and $1/p_1 + 1/p_2 = 1/p < 2$ is equivalent to b being in BMO. It is also known that they are compact for the same range of exponents if in addition $b \in \mathrm{CMO}$. The new result we shall present is the converse of this last statement.

3. Characterization of compactness

Theorem 3.1. *Let $1 < p_1 < \infty$, $1 < p_2 < \infty$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} < 2$.[1] Then each of the commutators in (4) is a compact bilinear operator from $L^{p_1} \times L^{p_2} \to L^p$ if and only if $b \in \mathrm{CMO}$.*

Proof. We only need to establish the necessity of $b \in \mathrm{CMO}$ since the other direction was proved in [1] and [13] as noted in the introduction. Moreover, by symmetry and a change of variables it is enough to consider, for example, $\mathcal{R}_1^1$ and $[b, \mathcal{R}_1^1]_1$. To simplify notation we denote $\mathcal{R}_1^1$ by $\mathcal{R}$.

Fix exponents p_1, p_2, p as in the statement of the theorem. Since bilinear compact operators are bounded, if we assume $\mathcal{R}$ to be compact from $L^{p_1} \times L^{p_2} \to L^p$ we must have that $b \in \mathrm{BMO}$; see [2] for $p > 1$ and [15] for $1/2 < p \le 1$. So for convenience, by linearity, we may assume that b is real valued and with $\|b\|_{\mathrm{BMO}} = 1$.

We will follow very closely some arguments in [4, 14] and [3] to show that if b fails to satisfy one of the conditions (1)–(3), then one arrives at a contradiction with the compactness of the operator. So b must be in CMO. We notice, however, that a main difference in the arguments below, in particular with respect to [4] and [3], is the fact alluded to in the introduction that the fractional integral operators considered in those works are actually positive operators, while the singular integrals studied here are not. This requires a modification in the lower estimate (8) proved below.

Assume that $\{Q_j\}_j$ is a sequence of cubes such that

$$(5) \qquad \frac{1}{|Q_j|} \int_{Q_j} |b(x) - b_{Q_j}|\, dx \ge \varepsilon,$$

for some $\varepsilon > 0$ and all $j \in \mathbb{N}$. As in [4] and [3], define two sequences of functions $\{f_j\}$ and $\{g_j\}$ associated with the cubes Q_j in the following way. Let

$$c_0 := |Q_j|^{-1} \int_{Q_j} \mathrm{sgn}(b(y) - b_{Q_j})\, dy$$

and define

$$f_j(y) := |Q_j|^{-\frac{1}{p_1}} \left(\mathrm{sgn}(b(y) - b_{Q_j}) - c_0 \right) \chi_{Q_j}(y).$$

Here sgn denotes the usual signum function. Define also

$$g_j(y) := |Q_j|^{-\frac{1}{p_2}} \chi_{Q_j}(y).$$

These functions satisfy the following properties:

    (a) $\mathrm{supp}\, f_j \subset Q_j$ and $\mathrm{supp}\, g_j \subset Q_j$,
    (b) $f_j(y)(b(y) - b_{Q_j}) \ge 0$,
    (c) $\int f_j(y)\, dy = 0$,
    (d) $\int (b(y) - b_{Q_j}) f_j(y)\, dy = |Q_j|^{-\frac{1}{p_1}} \int_{Q_j} |b(y) - b_{Q_j}|\, dy$,
    (e) $|f_j(y)| \le 2|Q_j|^{-\frac{1}{p_1}}$ and $0 \le g_j(y) \le |Q_j|^{-\frac{1}{p_2}}$,

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[1]We note that in a first draft of this article we had stated Theorem 3.1 only for $p > 1$. Although the computations in the proof (the same presented here) work for all $1/2 < p < \infty$, it was not known at the time whether the boundedness of the commutators when $1/2 < p \le 1$ implies $b \in \mathrm{BMO}$, which is a condition needed to jump-start our arguments in the proof. Nothing else in the proof depends on the value of $p > 1/2$. The recent result in [15] allows us now to state Theorem 3.1 for the full range of exponents without altering its proof.

(f) $\|f_j\|_{L^{p_1}} \leq 2$,

(g) $\|g_j\|_{L^{p_2}} = 1$.

Let $\{y_j\}$ be the collection of centers of the cubes $\{Q_j\}$. Then for all $x \in (2\sqrt{n}Q_j)^c$ the following standard pointwise estimates hold:

$$(6) \qquad |\mathcal{R}((b - b_{Q_j})f_j, g_j)(x)| \lesssim |Q_j|^{\frac{1}{p_1} + \frac{1}{p_2}} |x - y_j|^{-2n},$$

$$(7) \qquad |\mathcal{R}(f_j, g_j)(x)| \lesssim |Q_j|^{\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{n}} |x - y_j|^{-2n-1},$$

where the constants involved are independent of j, b, f_j, g_j, and ε. Indeed, for all such x and all $y \in Q_j$ we have $|x - y| \approx |x - y_j| > 0$, and hence by (a) and (e),

$$|\mathcal{R}((b - b_{Q_j})f_j, g_j)(x)| = \left| \iint_{\mathbb{R}^{2n}} \frac{(x^1 - y^1)(b(y) - b_{Q_j})f_j(y)g_j(z)}{(|x-y|^2 + |x-z|^2)^{n+1/2}} \, dy dz \right|$$

$$\lesssim \frac{1}{|Q_j|^{\frac{1}{p_1} + \frac{1}{p_2}}} |x - y_j|^{-2n} \int_{Q_j} \int_{Q_j} |b(y) - b_{Q_j}| \, dy dz$$

$$\lesssim |Q_j|^{\frac{1}{p_1'} + \frac{1}{p_2'}} |x - y_j|^{-2n} \|b\|_{\mathrm{BMO}}$$

$$\lesssim |Q_j|^{\frac{1}{p_1'} + \frac{1}{p_2'}} |x - y_j|^{-2n}.$$

On the other hand, using (a), (e), the cancellation property (c) of f_j, and the regularity of the kernel of the operator $\mathcal{R}$, we obtain

$$|\mathcal{R}(f_j, g_j)(x)| = \left| \iint_{\mathbb{R}^{2n}} \frac{(x^1 - y^1)f_j(y)g_j(z)}{(|x-y|^2 + |x-z|^2)^{n+1/2}} \, dy dz \right|$$

$$= \left| \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left(\frac{(x^1 - y^1)f_j(y)g_j(z)}{(|x-y|^2 + |x-z|^2)^{n+1/2}} \right. \right. \right.$$

$$\left. \left. \left. - \frac{(x^1 - y_j^1)f_j(y)g_j(z)}{(|x-y_j|^2 + |x-z|^2)^{n+1/2}} \right) dy \right) dz \right|$$

$$\lesssim \int_{Q_j} \int_{Q_j} \frac{|y - y_j||f_j(y)|g_j(z)}{(|x-y_j|^2 + |x-z|^2)^{n+1}} \, dy dz$$

$$\lesssim \frac{|Q_j|^{\frac{1}{n}}}{|x - y_j|^{2n+1}} \int_{Q_j} \int_{Q_j} |f_j(y)||g_j(z) \, dy dz$$

$$\lesssim |Q_j|^{\frac{1}{p_1'} + \frac{1}{p_2'} + \frac{1}{n}} |x - y_j|^{-2n-1}.$$

Next, we note that if d_j is the side-length of Q_j, then for all positive numbers $\widetilde{\gamma}_1, \widetilde{\gamma}_2$, with $\widetilde{\gamma}_2 = 8\widetilde{\gamma}_1 \gg 1$ there always exists a cube $\widetilde{Q}_j$ of side-length $\frac{\widetilde{\gamma}_2}{4\sqrt{n}} d_j$ contained in the annulus

$$A = \{x \in \mathbb{R}^n : \widetilde{\gamma}_1 d_j < |x - y_j| < \widetilde{\gamma}_2 d_j\}$$

and such that $|x - y| \approx |x - y_j| \approx x^1 - y_j^1 \approx x^1 - y^1 > 0$ for all $x \in \widetilde{Q}_j$ and all $y \in Q_j$. We claim that for all such x,

$$(8) \qquad |\mathcal{R}((b - b_{Q_j})f_j, g_j)(x)| \gtrsim \varepsilon |Q_j|^{\frac{1}{p_1'} + \frac{1}{p_2'}} |x - y_j|^{-2n},$$

where again the constant involved is independent of $j, b, f_j, g_j,$ and ε. To see (8), we use properties (b) and (d) of f_j to estimate

$$
\begin{aligned}
|\mathcal{R}((b - b_{Q_j})f_j, g_j)(x)| &= \left| \iint_{\mathbb{R}^{2n}} \frac{(x^1 - y^1)(b(y) - b_{Q_j})f_j(y)g_j(z)}{(|x-y|^2 + |x-z|^2)^{n+1/2}} \, dy \, dz \right| \\
&\gtrsim |Q_j|^{1 - \frac{1}{p_2}} |x - y_j|^{-2n} \int_{Q_j} (b(y) - b_{Q_j})f_j(y) \, dy \\
&= C_1 |Q_j|^{1 - \frac{1}{p_2}} |x - y_j|^{-2n} |Q_j|^{1 - \frac{1}{p_1}} \frac{1}{|Q_j|} \int_{Q_j} |(b(y) - b_{Q_j})| \, dy \\
&\geq C_1 |Q_j|^{\frac{1}{p_1'} + \frac{1}{p_2'}} |x - y_j|^{-2n} \varepsilon.
\end{aligned}
$$

We continue to follow the computations in [14], [4], and [3] and want to establish now that there exist constants γ_1, γ_2 with $\gamma_2 > \gamma_1 > 0$ and $\gamma_3 > 0$, depending only on p_1, p_2, n, and ε, such that the following estimates hold:

$$
(9) \qquad \left(\int_{\gamma_1 d_j < |x - y_j| < \gamma_2 d_j} |[b, \mathcal{R}]_1(f_j, g_j)(x)|^p \, dx \right)^{\frac{1}{p}} \geq \gamma_3,
$$

$$
(10) \qquad \left(\int_{|x - y_j| > \gamma_2 d_j} |[b, \mathcal{R}]_1(f_j, g_j)(x)|^p \, dx \right)^{\frac{1}{p}} \leq \frac{\gamma_3}{4}.
$$

In order to prove (9) and (10), we first observe that for every large enough number $\widetilde{\gamma}_1 > (\frac{1}{\ln \sqrt{2}})^2$, by properties (a) and (e) and the John–Nirenberg inequality,

$$
\begin{aligned}
&\int_{|x - y_j| > \widetilde{\gamma}_1 d_j} \left| (b(x) - b_{Q_j})\mathcal{R}(f_j, g_j)(x) \right|^p \, dx \\
&\lesssim |Q_j|^{\left(\frac{1}{p_1'} + \frac{1}{p_2'} + \frac{1}{n} \right)p} \sum_{s = \lfloor \log_2(\widetilde{\gamma}_1) \rfloor}^{\infty} \int_{2^s d_j < |x - y_j| < 2^{s+1} d_j} \frac{|b(x) - b_{Q_j}|^p}{|x - y_j|^{p(2n+1)}} \, dx \\
&\lesssim |Q_j|^{\left(\frac{1}{p_1'} + \frac{1}{p_2'} + \frac{1}{n} \right)p} \\
&\quad \times \sum_{s = \lfloor \log_2(\widetilde{\gamma}_1) \rfloor}^{\infty} 2^{-s(2n+1)p} |Q_j|^{-\left(2 + \frac{1}{n} \right)p} \int_{2^s d_j < |x - y_j| < 2^{s+1} d_j} |b(x) - b_{Q_j}|^p \, dx \\
&\lesssim |Q_j|^{\left(\frac{1}{p_1'} + \frac{1}{p_2'} - 2 \right)p} \sum_{s = \lfloor \log_2(\widetilde{\gamma}_1) \rfloor}^{\infty} 2^{-s(2n+1)p} s^p 2^{sn} |Q_j| \\
&\lesssim \sum_{s = \lfloor \log_2(\widetilde{\gamma}_1) \rfloor}^{\infty} 2^{-s\left(2n - \frac{n}{p} + \frac{1}{2} \right)p},
\end{aligned}
$$

and hence by $1/p < 2$,

$$
(11) \qquad \left(\int_{|x - y_j| > \widetilde{\gamma}_1 d_j} \left| (b(x) - b_{Q_j})\mathcal{R}(f_j, g_j)(x) \right|^p \, dx \right)^{\frac{1}{p}} \leq C_2 \widetilde{\gamma}_1^{-\left(2n - \frac{n}{p} + \frac{1}{2} \right)}.
$$

Next, for $\widetilde{\gamma}_2 = 8\widetilde{\gamma}_1$, using (8) and (11), we obtain the following estimates: for $p \geq 1$,

$$\left(\int_{\widetilde{\gamma}_1 d_j < |x-y_j| < \widetilde{\gamma}_2 d_j} |[b, \mathcal{R}]_1(f_j, g_j)(x)|^p \, dx \right)^{\frac{1}{p}}$$

$$\geq \left(\int_{\widetilde{\gamma}_1 d_j < |x-y_j| < \widetilde{\gamma}_2 d_j} |\mathcal{R}\left((b - b_Q)f_j, g_j\right)(x)|^p \, dx \right)^{\frac{1}{p}}$$

$$- \left(\int_{\widetilde{\gamma}_1 d_j < |x-y_j|} |(b(x) - b_Q)\mathcal{R}(f_j, g_j)(x)|^p \, dx \right)^{\frac{1}{p}}$$

$$\geq C_1 \varepsilon |Q_j|^{\frac{1}{p_1} + \frac{1}{p_2}} \left(\int_{\widetilde{Q}_j} |x - y_j|^{-2np} \, dx \right)^{\frac{1}{p}} - C_2 \widetilde{\gamma}_1^{-\left(2n - \frac{n}{p} + \frac{1}{2}\right)}$$

$$\geq C_1 \varepsilon |Q_j|^{\frac{1}{p_1} + \frac{1}{p_2}} |\widetilde{Q}_j|^{\frac{1}{p}} \widetilde{\gamma}_2^{-2n} |Q_j|^{-2} - C_2 \widetilde{\gamma}_1^{-\left(2n - \frac{n}{p} + \frac{1}{2}\right)}$$

$$(12) \qquad \geq C_1 \varepsilon (4\sqrt{n})^{-\frac{n}{p}} \widetilde{\gamma}_2^{-2n + \frac{n}{p}} - C_2 8^{\left(2n - \frac{n}{p} + \frac{1}{2}\right)} \widetilde{\gamma}_2^{-\left(2n - \frac{n}{p} + \frac{1}{2}\right)},$$

and for $1/2 < p < 1$,

$$\int_{\widetilde{\gamma}_1 d_j < |x-y_j| < \widetilde{\gamma}_2 d_j} |[b, \mathcal{R}]_1(f_j, g_j)(x)|^p \, dx$$

$$\geq \int_{\widetilde{\gamma}_1 d_j < |x-y_j| < \widetilde{\gamma}_2 d_j} |\mathcal{R}\left((b - b_Q)f_j, g_j\right)(x)|^p \, dx$$

$$- \int_{\widetilde{\gamma}_1 d_j < |x-y_j|} |(b(x) - b_Q)\mathcal{R}(f_j, g_j)(x)|^p \, dx$$

$$\geq C_1 \varepsilon^p |Q_j|^{\left(\frac{1}{p_1} + \frac{1}{p_2}\right)p} \int_{\widetilde{Q}_j} |x - y_j|^{-2np} \, dx - C_2 \widetilde{\gamma}_1^{-\left(2n - \frac{n}{p} + \frac{1}{2}\right)p}$$

$$\geq C_1 \varepsilon^p |Q_j|^{\left(\frac{1}{p_1} + \frac{1}{p_2}\right)p} |\widetilde{Q}_j| \widetilde{\gamma}_2^{-2np} |Q_j|^{-2p} - C_2 \widetilde{\gamma}_1^{-\left(2n - \frac{n}{p} + \frac{1}{2}\right)p}$$

$$(13) \qquad \geq C_1 \varepsilon^p (4\sqrt{n})^{-n} \widetilde{\gamma}_2^{-2np + n} - C_2 8^{\left(2n - \frac{n}{p} + \frac{1}{2}\right)p} \widetilde{\gamma}_2^{-\left(2n - \frac{n}{p} + \frac{1}{2}\right)p}.$$

We can now use (11) and (12) or (13) to replace $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ with γ_1 sufficiently large and $\gamma_2 = 8\gamma_1$, so that (9) and (10) are verified for some $\gamma_3 > 0$.

From here the arguments used in [3], which in turn followed the ones in [4], can be repeated without any changes. Namely, it is possible to construct sequences of cubes $\{Q_j\}$ and functions $\{f_j\}$, $\{g_j\}$ in exactly the same way as in [3] so that if any one of the conditions (1)–(3) were to be violated by b, then we would arrive at a contradiction with the compactness of $[b, \mathcal{R}]_1$. The reader can easily follow the argument in [3, pp. 491–493], simply replacing $[b, I_\alpha]_1$ therein by $[b, \mathcal{R}]_1$. To make our paper more self-contained, we now sketch an outline of the argument.

Using (6) and (7) it can be shown that given γ_1, γ_2, and γ_3 from (9) and (10), there exists a β with $0 < \beta \ll \gamma_2$, depending on p_1, p_2, n, and ε, such that for each measurable set

$$E \subset \{x : \gamma_1 d_j < |x - y_j| < \gamma_2 d_j\}$$

with $|E|/|Q_j| < \beta^n$, we get

$$(14) \qquad \|[b, \mathcal{R}]_1(f_j, g_j)\|_{L^p(E)} \leq \frac{\gamma_3}{4}.$$

This estimate relies on the fact that the result of Lemma 3.17(1) of [11], which is stated there for $p = 1$, also holds for all $p > 0$ and hence also applies in our case, where $p > 1/2$. In [4], the estimate corresponding to our (14) was obtained using the case $p \geq 1$ of this lemma.

With this in hand, if we suppose that any one of the conditions (1)–(3) on b fails, we can construct a sequence of functions that will lead us to a contradiction with the compactness of $[b, \mathcal{R}]_1$. For instance, if b does not satisfy (1), then there exist some $\varepsilon > 0$ and a sequence $\{Q_j\}$ of cubes with $|Q_j| \to 0$ as $j \to \infty$ such that

$$\frac{1}{|Q_j|} \int_{Q_j} |b(y) - b_{Q_j}| \, dy \geq \varepsilon,$$

for every j. First, select a subsequence, denoted by $\{Q_j^{(i)}\}$, so that the side-lengths satisfy

$$\frac{d_{j+1}^{(i)}}{d_j^{(i)}} < \frac{\beta}{2\gamma_2}.$$

Next, let $f_j^{(i)}$ and $g_j^{(i)}$, as defined before, be the functions associated to the selected cubes $Q_j^{(i)}$. Finally, for each $k, m \in \mathbb{N}$, consider the sets

$$G := \{x : \gamma_1 d_k^{(i)} < |x - y_k^{(i)}| < \gamma_2 d_k^{(i)}\},$$
$$G_1 := G \setminus \{x : |x - y_{k+m}^{(i)}| \leq \gamma_2 d_{k+m}^{(i)}\},$$
$$G_2 := \{x : |x - y_{k+m}^{(i)}| > \gamma_2 d_{k+m}^{(i)}\}.$$

The choice of the $Q_j^{(i)}$s implies that

$$\frac{|G_2^c \cap G|}{|Q_k^{(i)}|} \leq \beta^n;$$

see again [4, p. 307]. For $p \geq 1$, we can then estimate

$$\|[b, \mathcal{R}]_1(f_k^{(i)}, g_k^{(i)}) - [b, \mathcal{R}]_1(f_{k+m}^{(i)}, g_{k+m}^{(i)})\|_{L^p}$$

$$(15) \qquad \geq \left(\int_G \left| [b, \mathcal{R}]_1(f_k^{(i)}, g_k^{(i)}) \right|^p - \int_{G_2^c \cap G} \left| [b, \mathcal{R}]_1(f_k^{(i)}, g_k^{(i)}) \right|^p \right)^{\frac{1}{p}}$$

$$- \left(\int_{G_2} \left| [b, \mathcal{R}]_1(f_{k+m}^{(i)}, g_{k+m}^{(i)}) \right|^p \right)^{\frac{1}{p}}.$$

Applying (9), (14), and (10) respectively to the three terms on the right-hand side of (15), we conclude that

$$\|[b, \mathcal{R}]_1(f_k^{(i)}, g_k^{(i)}) - [b, \mathcal{R}]_1(f_{k+m}^{(i)}, g_{k+m}^{(i)})\|_{L^p} \geq \left(\gamma_3^p - \frac{\gamma_3^p}{4^p} \right)^{\frac{1}{p}} - \frac{\gamma_3}{4}$$

$$\geq \frac{\gamma_3}{2},$$

at least for $p \geq 1$.

In the case of $1/2 < p < 1$, a similar argument using the reverse triangle inequality applied to the pth power of the left-hand side of (15) leads to the lower bound

$$\|[b,\mathcal{R}]_1(f_k^{(i)}, g_k^{(i)}) - [b,\mathcal{R}]_1(f_{k+m}^{(i)}, g_{k+m}^{(i)})\|_{L^p}^p \geq \left(1 - \frac{2}{4^p}\right)\gamma_3^p.$$

This means that the image of the bounded set $\{(f_j, g_j)\}_j$ is not precompact, which contradicts our assumption on $[b,\mathcal{R}]_1$. The cases where b does not satisfy condition (2) or condition (3) are handled similarly, and we conclude our proof here. □

Remark 3.2. We observe that the arguments used for the Riesz transforms $\mathcal{R}_j^k$ in Theorem 3.1 also go through in more generality. In order to get the lower bound (as in formulas (8) and (9) above), one usually uses the assumption that the kernel of the operator is positive, if not in the whole space, then at least in a substantial portion of the space. For the Riesz transforms $\mathcal{R}_j^k$, although the kernel is not positive, for each cube Q_j we can find another cube $\widetilde{Q}_j$ such that $\widetilde{Q}_j$ lies in some large annulus centered at the center y_j of Q_j, and for all $x \in \widetilde{Q}_j$ and $y, z \in Q_j$,

$$K(x-y, x-z) > 0 \qquad \text{and} \qquad |x-y| \approx |x-z| \approx |x-y_j|.$$

This condition together with the Calderón–Zygmund conditions on the size and regularity of the kernel suffice to obtain the lower bound. This idea applies to certain other bounded convolution-type singular operators, as we now discuss.

In the linear case, as is shown in Uchiyama's paper [14], the Riesz transform can be replaced by convolution-type singular integral operators with kernel of the form

$$K(x) = \frac{\Omega(x)}{|x|^n},$$

where Ω is a homogeneous function of degree zero defined on the unit sphere in $\mathbb{R}^n$ and is sufficiently smooth. Such a kernel is *locally positive* in the sense that there is some spherical cap A in the unit sphere S^{n-1} such that $\Omega(x) > c_0 > 0$ for all $x \in A$.

Turning to the bilinear case, the arguments used for the bilinear Riesz transforms $\mathcal{R}_j^k$ in Theorem 3.1 can be repeated for bounded convolution bilinear operators with kernel of the form

$$K(y, z) = \frac{\Omega\left(\frac{(y,z)}{|(y,z)|}\right)}{(|y|^2 + |z|^2)^n},$$

where Ω is a homogeneous function of degree zero defined on the unit sphere in $\mathbb{R}^n \times \mathbb{R}^n$ and is sufficiently smooth. We need more assumptions on this kernel than in the linear case.

First, we assume that $1/K$ has an absolutely convergent Fourier series in some ball in $\mathbb{R}^{2n}$. This assumption guarantees that the boundedness of the commutator operator with a function b implies that $b \in \text{BMO}$, by the main result of [2].

Second, we assume that there is some spherical cap A on the unit sphere S^{n-1} such that $\Omega\left(\frac{(y,z)}{|(y,z)|}\right) > c_0 > 0$ for all $y, z \in A$. This assumption enables us to get the lower bound estimate (8). Indeed, given a cube Q_j centered at y_j, we can find another cube $\widetilde{Q}_j$ such that $\widetilde{Q}_j$ lies in some large annulus centered at y_j, and for all $x \in \widetilde{Q}_j$ and all $y, z \in Q_j$, $x - y$ and $x - z$ lie in an infinite cone in $\mathbb{R}^n$ whose

vertex is at the origin and which passes through the cap A. From our assumption, it follows that

$$K(x - y, x - z) > 0 \qquad \text{and} \qquad |x - y| \approx |x - z| \approx |x - y_j|$$

for all $x \in \widetilde{Q}_j$ and $y, z \in Q_j$. The computations in the proof of Theorem 3.1 can now be repeated. We leave the details to the interested reader.

ACKNOWLEDGMENT

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