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FOURTH ORDER PARTIAL DIFFERENTIAL EQUATIONS FOR KRALL-TYPE ORTHOGONAL POLYNOMIALS ON THE TRIANGLE

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Abstract. We construct bivariate polynomials orthogonal with respect to a Krall-type inner product on the triangle defined by adding Krall terms over the border and the vertexes to the classical inner product. We prove that these Krall-type orthogonal polynomials satisfy fourth order partial differential equations with polynomial coefficients, as an extension of the classical theory introduced by H. L. Krall in the 1940s.

1. Introduction

In 1938, H. L. Krall studied the problem of determining polynomial solutions of a linear differential equation of even order with polynomial coefficients, and he found necessary and sufficient conditions for this solutions to exist. In [12], he classified the fourth order equations with polynomial solutions. Afterwards, A. M. Krall studied these new polynomials in 1981 (see [13]) and he called them Legendre-type, Laguerre-type, and Jacobi-type polynomials. These polynomials are orthogonal with respect to modifications of classical weight functions by adding a Dirac delta at one point of the support of the measure. This kind of modification had been previously studied by V. B. Uvarov in [18]. In the main result, he expressed the polynomials orthogonal with respect to the new measure in terms of the polynomials orthogonal with respect to the classical one.

In some special cases of classical Laguerre and Jacobi measures, if the perturbations are given at the endpoints of the support of the measure, then the new polynomials are eigenfunctions of higher order differential operators with polynomial coefficients and are called *Krall polynomials* (see, for instance, [19] and the references therein).

In [11], T. H. Koornwinder focused his study on the so-called Jacobi-type polynomials, namely the case where the measure is the Jacobi weight function together with additional mass points at 1 and -1. He constructed the corresponding orthogonal polynomials and studied their properties. In fact, a relationship between

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Jacobi-type polynomials and the classical ones was established. This case was also studied by L. L. Littlejohn in [15], where he showed that the orthogonal polynomials satisfy a fourth order differential equation in the particular case when the parameters are equal to zero and the masses are the same in both points. In [9], Koekoek and Koekoek analysed the case when Jacobi-type polynomials satisfy a finite order differential equation. In a more general framework, it is possible to consider perturbations of regular functionals via the addition of Dirac deltas. In recent years, this kind of polynomial has been extensively studied (see [2] and the references therein).

In the multivariate case, orthogonal polynomials with respect to a moment functional obtained from a standard one by adding Dirac masses at some points were studied in [3, 7]. In particular, a Jacobi weight function on the simplex with mass points added at the vertices is considered in [3]. In [4], the authors presented a Uvarov modification of the two variable classical measure on the unit disk by adding a finite set of equally spaced mass points on the border. A general framework of Uvarov modifications in the multivariate case was studied in [5].

Besides Uvarov modifications by adding Dirac masses at a finite and discrete set of points, in the context of several variables it is possible to modify the moment functional with other moment functionals defined on lower-dimensional manifolds such as curves, surfaces, etc. Recently, a family of orthogonal polynomials with respect to such a Uvarov modification of the classical ball measure by means of a mass uniformly distributed over the sphere was introduced in [16]. The authors proved that, at least in the Legendre case, these multivariate orthogonal polynomials satisfy a fourth order partial differential equation, which constitutes a natural extension of Krall orthogonal polynomials ([13]) to the multivariate case.

The aim of this work is to examine a Krall-type inner product on the triangle defined by adding Krall terms over the border and the vertexes of the triangle. For general values of the parameters, we construct a mutually orthogonal basis obtained in terms of univariate Jacobi-type orthogonal polynomials. We show that, for particular values of the parameters, these Krall-type orthogonal polynomials satisfy fourth order partial differential equations with polynomial coefficients, as an extension of the classical theory introduced by H. L. Krall in the 1940s [12] and developed later in [13]. The restriction on the values of the parameters is inherited by the fact that we need to use the univariate results of [9] and [10] for classical Jacobi-type polynomials. We point out that the eigenvalues of the partial differential equation may depend on the partial degrees of the polynomial eigenfunctions. This phenomenon has already appeared in the literature, for example, for the tensor product of two Jacobi polynomials, and several other nontrivial examples (see [10]).

The structure of the paper is as follows. Section 2 is devoted to introduce the Krall-type inner product and to study Krall-type orthogonal polynomials on the triangle. To this end, we recall some properties of classical bivariate orthogonal polynomials on the triangle. In particular, we construct a basis of orthogonal polynomials for this Krall-type inner product using a method developed by T. H. Koornwinder in [10]. The main results of this work are contained in Section 3. Here we study differential properties for Krall-type orthogonal polynomials, and we deduce fourth order partial differential equations satisfied by these orthogonal polynomials.

2. Krall-type orthogonal polynomials on the triangle

In this section, we study general low-dimensional Krall modifications of the classical triangle inner product, including masses along the borders and over the vertexes of the triangle. Moreover, we construct a basis of orthogonal polynomials following the ideas of Koornwinder ([10], revisited in [6]). We begin the section by recalling the classical bivariate inner product on the triangle as well as a base of orthogonal polynomials associated with it.

2.1. **Classical bivariate orthogonal polynomials on the triangle.** Let us consider the triangle in $\mathbb{R}^2$:

$$\Delta = \{(x,y) \in \mathbb{R}^2 : x \geqslant 0,\, y \geqslant 0\,, 1-x-y \geqslant 0\}.$$

For $\alpha, \beta, \gamma > -1$, classical orthogonal polynomials on the triangle are orthogonal with respect to the weighted inner product

$$\langle f, g\rangle_\Delta = \omega_{(\alpha,\beta,\gamma)} \iint_\Delta f(x,y)g(x,y)x^\alpha y^\beta (1-x-y)^\gamma dxdy,$$

where

$$\omega_{(\alpha,\beta,\gamma)} = \left(\iint_\Delta x^\alpha y^\beta (1-x-y)^\gamma dxdy\right)^{-1} = \frac{\Gamma(\alpha+\beta+\gamma+3)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}.$$

A base of orthogonal polynomials on the triangle can be found in [6] and is given by

$$P_{n,k}^{(\alpha,\beta,\gamma)}(x,y) = P_{n-k}^{(\beta_k,\alpha)}(x)\,(1-x)^k\, P_k^{(\gamma,\beta)}\Big(\frac{y}{1-x}\Big),\ 0 \leqslant k \leqslant n, \tag{2.1}$$

where

$$\beta_k = 2k+\beta+\gamma+1, \quad k \geqslant 0. \tag{2.2}$$

Here, $\{P_n^{(a,b)}(t)\}_{n\geqslant 0}$ denotes the sequence of classical Jacobi polynomials orthogonal on $[0,1]$ with respect to the inner product

$$\langle f, g\rangle_{(a,b)} = \omega_{(a,b)} \int_0^1 f(t)g(t)(1-t)^a t^b dt,$$

where

$$\omega_{(a,b)} = \left(\int_0^1 (1-t)^a t^b dt\right)^{-1} = \frac{\Gamma(a+b+2)}{\Gamma(a+1)\Gamma(b+1)}. \tag{2.3}$$

These polynomials are normalized as in [17, eq. (4.1.1)] in the form

$$P_n^{(a,b)}(1) = \binom{n+a}{n} = \frac{(a+1)_n}{n!}.$$

As a consequence, if we denote by

$$h_n^{(a,b)} = \langle P_n^{(a,b)}, P_n^{(a,b)}\rangle_{(a,b)}, \quad h_{n,k}^{(\alpha,\beta,\gamma)} = \langle P_{n,k}^{(\alpha,\beta,\gamma)}, P_{n,k}^{(\alpha,\beta,\gamma)}\rangle_\Delta$$

the corresponding squared norms, then they are related by

$$h_{n,k}^{(\alpha,\beta,\gamma)} = \frac{\omega_{(\beta_0,\alpha)}}{\omega_{(\beta_k,\alpha)}}\, h_{n-k}^{(\beta_k,\alpha)}\, h_k^{(\gamma,\beta)},$$

since $\omega_{(\alpha,\beta,\gamma)} = \omega_{(\gamma,\beta)}\omega_{(\beta_0,\alpha)}$.

2.2. **Krall-type orthogonal polynomials on the triangle.** We denote $\sigma = (\alpha,\beta,\gamma)$ and $\lambda = (\lambda_1,\lambda_2,\lambda_3,\mu_1,\mu_2,\mu_3)$, and asume that $\alpha,\beta,\gamma > -1$ and $\lambda_i \geqslant 0$, $\mu_i \geqslant 0$, for $i = 1,2,3$. Let us define the *low-dimensional Krall perturbation of the classical inner product on the triangle* in the form

$$\begin{aligned}\langle f,g\rangle_\sigma^\lambda =&\omega_{(\alpha,\beta,\gamma)} \iint_\Delta f(x,y)g(x,y)x^\alpha y^\beta (1-x-y)^\gamma dxdy\\ &+ \lambda_1\,\omega_{(\beta_0,\alpha)} \int_0^1 f(x,0)g(x,0)(1-x)^{\beta_0}x^\alpha dx\\ &+ \lambda_2\,\omega_{(\beta_0,\alpha)} \int_0^1 f(x,1-x)g(x,1-x)(1-x)^{\beta_0}x^\alpha dx\\ &+ \lambda_3\,\omega_{(\gamma,\beta)} \int_0^1 f(0,y)g(0,y)(1-y)^\gamma y^\beta dy\\ &+ \mu_1\, f(0,0)\,g(0,0) + \mu_2\, f(0,1)\,g(0,1) + \mu_3\, f(1,0)\,g(1,0),\end{aligned} \tag{2.4}$$

where $\beta_0 = \beta+\gamma+1$ was defined in (2.2).

Let us observe that the inner product adds low-dimensional terms over the borders of the triangle and Dirac masses over the vertexes. The respective weight functions on the borders are the natural restrictions of the classical bivariate weight function to the respective edges.

Now, we want to give a basis of orthogonal polynomials associated with the inner product (2.4) in a similar form as the usual basis for the classical inner product on the triangle given by (2.1), using Jacobi-type polynomials in one variable instead of classical Jacobi polynomials. Next, we recall that Jacobi-type polynomials are defined in the following way.

Let $a,b > -1$ and $M,N \geqslant 0$ be given parameters. Consider the univariate Jacobi-type inner product on $[0,1]$ defined by

$$\langle f,g\rangle_{(a,b)}^{(M,N)} = \omega_{(a,b)} \int_0^1 f(t)g(t)(1-t)^a t^b dt + Mf(0)g(0) + Nf(1)g(1),$$

where $\omega_{(a,b)}$ is the normalization constant given by (2.3).

An explicit expression for a basis of polynomials orthogonal with respect to this inner product can be obtained directly from [11] by doing a change of variable. This basis can be written as

$$\begin{aligned}P_n^{(a,b;M,N)}(t) &= (C_n^{(a,b)})^2\\ &\times \left[\frac{a_n^{(b,a;N)}M(1-t) - a_n^{(a,b;M)}Nt}{a+b+1}\frac{d}{dt} + a_n^{(a,b;M)}a_n^{(b,a;N)}\right] P_n^{(a,b)}(t),\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}C_n^{(a,b)} &= \frac{(a+b+1)_n}{n!},\\ a_n^{(a,b;M)} &= \frac{(a+1)_n\,n!}{(b+1)_n(a+b+1)_n} + \frac{n(n+a+b+1)M}{(b+1)(a+b+1)}.\end{aligned}$$

The squared norms of these polynomials will be denoted by $h_n^{(a,b;M,N)}$.

For $0 \leqslant k \leqslant n$, we define the sequence of two-variable polynomials

$$Q_{n,k}^{(\sigma;\lambda)}(x,y) = P_{n-k}^{(\beta_k,\alpha;M_k,N_k)}(x)\,(1-x)^k\,P_k^{(\gamma,\beta;\lambda_1,\lambda_2)}\Big(\frac{y}{1-x}\Big), \tag{2.6}$$

where $\beta_k = 2k+\beta+\gamma+1$, $M_k, N_k \geqslant 0, k \geqslant 0$, and $\lambda_1, \lambda_2 \geqslant 0$.

We prove the orthogonality of these polynomials with respect to a Krall-type inner product such as (2.4) for an adequate election of the parameters μ_i, for $i=1,2,3$, and M_k, N_k, for $k \geqslant 0$.

Theorem 2.1. *The set of polynomials*

$$\{Q_{n,k}^{(\sigma;\lambda)}(x,y):\ 0 \leqslant k \leqslant n\},$$

defined by (2.6)*, is a sequence of orthogonal polynomials with respect to the low-dimensional Krall inner product* (2.4) *if and only if*

$$\mu_1 = \lambda_1\,\lambda_3, \qquad \mu_2 = \lambda_2\,\lambda_3 \tag{2.7}$$

and

$$M_k = \frac{\omega_{(\beta_k,\alpha)}}{\omega_{(\beta_0,\alpha)}}\lambda_3, \qquad N_k = \frac{\lambda_4}{1+\lambda_1+\lambda_2}\,\delta_{k,0}, \tag{2.8}$$

where $\lambda_4 = \mu_3$ and $\delta_{k,0}$ denotes the usual Dirac delta.

Proof. First, we compute the inner product (2.4) for a pair of functions $f(x,y)$ and $g(x,y)$ such that their product can be expressed in the form

$$f(x,y)g(x,y) = F(x, y/(1-x)) = F_1(x)F_2(y/(1-x)). \tag{2.9}$$

The first integral in the inner product (2.4), afer the change of variable $t = 1/(1-x)$, can be written as

$$\begin{aligned}\iint_\Delta F\Big(x, \frac{y}{1-x}\Big) x^\alpha y^\beta (1-x-y)^\gamma dxdy& \\ = \int_0^1 F_1(x)(1-x)^{\beta+\gamma+1}x^\alpha dx& \int_0^1 F_2(t)(1-t)^\gamma t^\beta dt.\end{aligned}$$

Then, the first three integrals in (2.4) can be expressed as a product of one-dimensional integrals plus two mass points:

$$\Big[\omega_{(\beta_0,\alpha)}\int_0^1 F_1(x)(1-x)^{\beta_0}x^\alpha dx\Big]\Big[\omega_{(\gamma,\beta)}\int_0^1 F_2(t)(1-t)^\gamma t^\beta dt + \lambda_1 F_2(0) + \lambda_2 F_2(1)\Big].$$

Let us assume that the constants μ_i in the inner product (2.4) are given by (2.7). Summing the next three terms in (2.4) for $F(x, y/(1-x)) = F_1(x)F_2(y/(1-x))$, we get

$$\lambda_3 F_1(0)\Big[\omega_{(\gamma,\beta)}\int_0^1 F_2(y)(1-y)^\gamma y^\beta dy + \lambda_1 F_2(0) + \lambda_2 F_2(0)\Big].$$

Therefore, except for the last term, we can see (2.4) as a product of univariate inner products with mass points as in [11] for functions of type (2.9), that is,

$$\begin{aligned}\langle f(x,y), g(x,y)\rangle_\sigma^\lambda =& \Big[\omega_{(\beta_0,\alpha)}\int_0^1 F_1(x)(1-x)^{\beta_0}x^\alpha dx + \lambda_3 F_1(0)\Big] \\ &\times \Big[\omega_{(\gamma,\beta)}\int_0^1 F_2(t)(1-t)^\gamma t^\beta dt + \lambda_1 F_2(0) + \lambda_2 F_2(1)\Big] \\ &+ \lambda_4 f(1,0)g(1,0).\end{aligned}$$

Observe that, for this kind of function, the last term is not always well defined.

Using the above expression, we compute $\langle Q_{n,k}^{(\sigma;\lambda)}, Q_{m,j}^{(\sigma;\lambda)}\rangle_\sigma^\lambda$. Now, the functions involved in the integrals are of the form (2.9),

$$Q_{n,k}^{(\sigma;\lambda)}(x,y)Q_{m,j}^{(\sigma;\lambda)}(x,y) = F_1(x)F_2\Big(\frac{y}{1-x}\Big),$$

where

$$F_1(x) = P_{n-k}^{(\beta_k,\alpha;M_k,N_k)}(x)P_{m-j}^{(\beta_j,\alpha;M_j,N_j)}(x)(1-x)^{k+j}$$

and

$$F_2\Big(\frac{y}{1-x}\Big) = P_k^{(\gamma,\beta;\lambda_1,\lambda_2)}\Big(\frac{y}{1-x}\Big)P_j^{(\gamma,\beta;\lambda_1,\lambda_2)}\Big(\frac{y}{1-x}\Big).$$

In such a case, for the last term in (2.4) it is easy to see that

$$Q_{n,k}^{(\sigma;\lambda)}(1,0) = P_{n-k}^{(\beta_k,\alpha;M_k,N_k)}(1)l_k^{(\gamma,\beta;\lambda_1,\lambda_2)}\delta_{k,0} = P_{n-k}^{(\beta_k,\alpha;M_k,N_k)}(1)\delta_{k,0},$$

where $l_k^{(\gamma,\beta;\lambda_1,\lambda_2)}$ is the leading coefficient of $P_k^{(\gamma,\beta;\lambda_1,\lambda_2)}(t)$. Thus, the last term in the inner product reads

$$\lambda_4\, P_{n-k}^{(\beta_k,\alpha;M_k,N_k)}(1)P_{m-j}^{(\beta_j,\alpha;M_j,N_j)}(1)\delta_{0,k}\delta_{0,j}.$$

Then, by using the orthogonality of the univariate polynomials $\{P_k^{(\gamma,\beta;\lambda_1,\lambda_2)}(t)\}_{k\geqslant 0}$, we get

$$\begin{aligned}
\langle Q_{n,k}^{(\sigma;\lambda)}, &Q_{m,j}^{(\sigma;\lambda)}\rangle_\sigma^\lambda \\
&= \frac{\omega_{(\beta_0,\alpha)}}{\omega_{(\beta_k,\alpha)}}\, h_k^{(\gamma,\beta;\lambda_1,\lambda_2)}\delta_{k,j} \\
&\quad\times\Bigg[\omega_{(\beta_k,\alpha)}\int_0^1 P_{n-k}^{(\beta_k,\alpha;M_k,N_k)}(x)P_{m-k}^{(\beta_k,\alpha;M_k,N_k)}(x)(1-x)^{\beta_k}x^\alpha dx \\
&\quad + \lambda_3\frac{\omega_{(\beta_k,\alpha)}}{\omega_{(\beta_0,\alpha)}}P_{n-k}^{(\beta_k,\alpha;M_k,N_k)}(0)P_{m-k}^{(\beta_k,\alpha;M_k,N_k)}(0) \\
&\quad + \frac{\lambda_4}{h_k^{(\gamma,\beta;\lambda_1,\lambda_2)}}\frac{\omega_{(\beta_k,\alpha)}}{\omega_{(\beta_0,\alpha)}}P_{n-k}^{(\beta_k,\alpha;M_k,N_k)}(1)P_{m-k}^{(\beta_k,\alpha;M_k,N_k)}(1)\delta_{0,k}\Bigg].
\end{aligned}$$

We can observe that the last term in the above expression is different from zero only when $k=0$. Then, we use that $h_0^{(\gamma,\beta;\lambda_1,\lambda_2)} = 1+\lambda_1+\lambda_2$, and therefore,

$$\langle Q_{n,k}^{(\sigma;\lambda)}, Q_{m,j}^{(\sigma;\lambda)}\rangle_\sigma^\lambda = \frac{\omega_{(\beta_0,\alpha)}}{\omega_{(\beta_k,\alpha)}}h_{n-k}^{(\beta_k,\alpha;M_k,N_k)}\, h_k^{(\gamma,\beta;\lambda_1,\lambda_2)}\delta_{n,m}\delta_{k,j}.$$

A similar reasoning shows the direct implication. □

From now on, we will refer to the basis

$$\{Q_{n,k}^{(\sigma;\lambda)}(x,y):\ 0\leqslant k\leqslant n,\ n\geqslant 0\}$$

given in (2.6) as *Krall-type orthogonal polynomials on the triangle.*

As a consequence, if we denote it by

$$h_{n,k}^{(\sigma;\lambda)} = \langle Q_{n,k}^{(\sigma;\lambda)}, Q_{n,k}^{(\sigma;\lambda)}\rangle_\sigma^\lambda,$$

we obtain a relation between these norms and the norms of Jacobi-type polynomials.

Corollary 2.2. *For $0\leqslant k\leqslant n$, we get*

$$h_{n,k}^{(\sigma;\lambda)} = \frac{\omega_{(\beta_0,\alpha)}}{\omega_{(\beta_k,\alpha)}}\, h_{n-k}^{(\beta_k,\alpha;M_k,N_k)}\, h_k^{(\gamma,\beta;\lambda_1,\lambda_2)}.$$

Remark 2.3. Using (2.5) and the relations (22.7.17) and (22.7.20) on p. 782 of [1], we can give a connection formula between Krall-type and classical polynomials on the triangle. For $n \geqslant 0$, we have

$$Q_{n,0}^{(\sigma;\lambda)}(x,y) = (C_n^{(\beta_0,\alpha)})^2 \times \left[d_1 P_{n,0}^{(\alpha,\beta,\gamma)}(x,y) + (d_2(1-x)+d_3x)P_{n-1,0}^{(\alpha+1,\beta+1,\gamma)}(x,y)\right],$$

and, for $1 \leqslant k \leqslant n$,

$$\begin{aligned} Q_{n,k}^{(\sigma;\lambda)}(x,y) = (C_{n-k}^{(\beta_k,\alpha)})^2 (C_k^{(\gamma,\beta)})^2 \\ \times \big[e_1\, P_{n,k}^{(\alpha,\beta,\gamma)}(x,y) + (e_2(1-x-y)+e_3y)P_{n-1,k}^{(\alpha+1,\beta+1,\gamma)}(x,y) \\ + (e_4(1-x-y)+e_5y)P_{n-1,k}^{(\alpha+1,\beta,\gamma+1)}(x,y) \\ + (e_6(1-x-y)+e_7y)\, P_{n-1,k-1}^{(\alpha,\beta+1,\gamma+1)}(x,y)\big], \end{aligned}$$

where the coefficients appearing in the identities depend on n, k, and the parameters involved in the inner product (2.4).

3. Partial differential equations for Krall-type polynomials

When the second family of Jacobi-type polynomials in the definition (2.6) of the Krall-type orthogonal polynomials satisfies a finite order differential equation, then this property is inherited by the bivariate polynomials. In [9] there is an exhaustive description of this situation, as we will see later.

Proposition 3.1. *If the Jacobi-type polynomials $\{P_k^{(\gamma,\beta;\lambda_1,\lambda_2)}(t)\}_{k\geqslant 0}$ satisfy a differential equation of order m, that is,*

$$\sum_{j=0}^{m} a_j(t)\,\frac{d^j\,u}{dt^j} = 0,$$

then the Krall-type orthogonal polynomials $\{Q_{n,k}^{(\sigma;\lambda)}(x,y)\}_{n\geqslant k\geqslant 0}$ satisfy the following partial differential equation of order m:

$$\sum_{j=0}^{m}(1-x)^j a_j\Big(\frac{y}{1-x}\Big)\frac{\partial^j}{\partial y^j} Q_{n,k}^{(\sigma;\lambda)}(x,y) = 0.$$

Proof. Using the explicit expression (2.6), it is clear that

$$\frac{\partial^j}{\partial y^j} Q_{n,k}^{(\sigma;\lambda)}(x,y) = \frac{1}{(1-x)^j} P_{n-k}^{(\beta_k,\alpha;M_k,N_k)}(x)\,(1-x)^k\,(P_k^{(\gamma,\beta;\lambda_1,\lambda_2)})^{j)}\Big(\frac{y}{1-x}\Big),$$

where $p^{j)}(t)$ denotes the jth derivative of the polynomial $p(t)$.

If $P_k^{(\gamma,\beta;\lambda_1,\lambda_2)}(t)$ satisfies a differential equation of order m, that is,

$$\sum_{j=0}^{m} a_j(t)(P_k^{(\gamma,\beta;\lambda_1,\lambda_2)})^{j)}(t) = 0,$$

then

$$\begin{aligned}
&\sum_{j=0}^{m}(1-x)^j a_j\Big(\frac{y}{1-x}\Big)\frac{\partial^j}{\partial y^j}Q_{n,k}^{(\sigma;\lambda)}(x,y)\\
&=\sum_{j=0}^{m} a_j\Big(\frac{y}{1-x}\Big)P_{n-k}^{(\beta_k,\alpha;M_k,N_k)}(x)\,(1-x)^k\,(P_k^{(\gamma,\beta;\lambda_1,\lambda_2)})^{j)}\Big(\frac{y}{1-x}\Big)\\
&=P_{n-k}^{(\beta_k,\alpha;M_k,N_k)}(x)\,(1-x)^k\sum_{j=0}^{m} a_j\Big(\frac{y}{1-x}\Big)(P_k^{(\gamma,\beta;\lambda_1,\lambda_2)})^{j)}\Big(\frac{y}{1-x}\Big)=0,
\end{aligned}$$

and the result holds. □

Remark 3.2. Observe that the coefficients $a_j(t)$ may depend on k, and eventually on the other parameters γ, β, λ_1, and λ_2. In particular, if $a_j(t)$ is a polynomial of degree at most j in one variable, then $(1-x)^j a_j\big(\frac{y}{1-x}\big)$ is a polynomial of degree at most j in two variables. Moreover, in the cases studied in [9], all the coefficients are independent of k except for $a_0(t) = a_0$. Therefore, the eigenvalues of the partial differential equation may depend on the partial degree k of the Krall-type polynomial.

In [9], the authors looked for differential equations of the form

$$\begin{aligned}
&M\sum_{i=0}^{\infty}\hat{a}_i(t)\,u^{i)}(t)+N\sum_{i=0}^{\infty}\hat{b}_i(t)\,u^{i)}(t)+M\,N\sum_{i=0}^{\infty}\hat{c}_i(t)\,u^{i)}(t)\\
&\quad+(1-t)\,t\,u''(t)+[(b+1)(1-t)-(a+1)t]u'(t)+n(n+a+b+1)u(t)=0,
\end{aligned}$$

satisfied by Jacobi-type polynomials $\{P_n^{(a,b,M,N)}(t)\}_{n\geqslant 0}$, where $\hat{a}_i(t)$, $\hat{b}_i(t)$, and $\hat{c}_i(t)$, for $i \geqslant 1$, are independent of n, the degree of the polynomial solution, and $\hat{a}_0(t)$, $\hat{b}_0(t)$, and $\hat{c}_0(t)$ are independent of t. They proved that, for $M^2 + N^2 > 0$, Jacobi-type polynomials satisfy a unique differential equation of infinite order, except for $a \in \{0, 1, 2, \ldots\}$ or $b \in \{0, 1, 2, \ldots\}$. In particular, they showed that the order equals

$$\begin{cases}
2\,b+4 & \text{if} \quad M>0, \quad N=0, \quad \text{and} \quad b\in\{0,1,2,\ldots\},\\
2\,a+4 & \text{if} \quad M=0, \quad N>0, \quad \text{and} \quad a\in\{0,1,2,\ldots\},\\
2\,a+2\,b+6 & \text{if} \quad M>0, \quad N>0, \quad \text{and} \quad a,b\in\{0,1,2,\ldots\}.
\end{cases}$$

In this way, Proposition 3.1 and [9] provide a unique finite order partial differential equation for Krall-type polynomials on the triangle when $\lambda_1^2+\lambda_2^2 > 0$ in the following form:

(1) For $\lambda_1 > 0, \lambda_2 = 0$, and $\beta \in \{0, 1, 2, \ldots\}$, it follows that the polynomials $Q_{n,k}^{(\sigma;\lambda)}(x, y)$ are eigenfunctions of a differential operator in y of order $2\beta+4$ with eigenvalue depending only on k.
(2) For $\lambda_1 = 0, \lambda_2 > 0$, and $\gamma \in \{0, 1, 2, \ldots\}$, the polynomials $Q_{n,k}^{(\sigma;\lambda)}(x, y)$ are again eigenfunctions of a partial differential operator in y of order $2\gamma + 4$ with eigenvalue depending only on k.
(3) For $\lambda_1 > 0, \lambda_2 > 0$, and $\gamma, \beta \in \{0, 1, 2, \ldots\}$, then $Q_{n,k}^{(\sigma;\lambda)}(x, y)$ are eigenfunctions of a partial differential operator in y of order $2\beta + 2\gamma + 6$ with eigenvalue depending on k.

Observe that above finite partial differential equations for Krall-type polynomials are unique in this form, only have derivatives in the second variable, and the eigenvalue depends only on the second index k.

If we consider the special case when the second family of univariate orthogonal polynomials involved in (2.6) are classical Jacobi polynomials, then, as a consequence of Proposition 3.1, we obtain a second order partial differential equation for Krall-type orthogonal polynomials on the triangle.

Let $\alpha, \beta, \gamma > -1$. For $0 \leqslant k \leqslant n$, the bivariate polynomials

$$Q_{n,k}^{(\sigma;\lambda)}(x,y) = P_{n-k}^{(\beta_k,\alpha;M_k,N_k)}(x)(1-x)^k P_k^{(\gamma,\beta)}\Big(\frac{y}{1-x}\Big) \tag{3.1}$$

are orthogonal with respect to the inner product (2.4) with the constants given by (2.7) and (2.8), for the particular case when the parameters λ_1 and λ_2 vanish, that is,

$$\begin{aligned}\langle f,g\rangle_\sigma^\lambda = \omega_{(\alpha,\beta,\gamma)} &\iint_\Delta f(x,y)g(x,y)x^\alpha y^\beta (1-x-y)^\gamma dxdy \\ &+ \lambda_3 \omega_{(\gamma,\beta)} \int_0^1 f(0,y)g(0,y)(1-y)^\gamma y^\beta dy \\ &+ \lambda_4 f(1,0)g(1,0).\end{aligned}$$

In such a case, the parameters appearing in (3.1) take the form

$$\beta_k = 2k+\beta+\gamma+1, \qquad M_k = \frac{\omega_{(\beta_k,\alpha)}}{\omega_{(\beta_0,\alpha)}}\lambda_3, \qquad N_k = \lambda_4\, \delta_{k,0}.$$

Using Proposition 3.1 and the second order differential equation for classical Jacobi polynomials $\{P_k^{(\gamma,\beta)}(t)\}_{k\geqslant 0}$ given in [17], Krall-type polynomials on the triangle (3.1) satisfy the next second order partial differential equation,

$$\mathcal{D}(Q_{n,k}^{(\sigma;\lambda)})(x,y) = -k(k+\beta+\gamma+1)\, Q_{n,k}^{(\sigma;\lambda)}(x,y), \tag{3.2}$$

where

$$\mathcal{D}(f)(x,y) = (1-x-y)y\frac{\partial^2 f}{\partial y^2} + [(\beta+1)(1-x-y) - (\gamma+1)y]\,\frac{\partial f}{\partial y}. \tag{3.3}$$

Now, we will use different tools in order to obtain a fourth order partial differential equation for Krall-type polynomials on the triangle in such a way that partial derivatives with respect to both x and y variables will appear. This will be done when both parameters α and λ_4 vanish. We consider this particular case since we use results given in [10] for the univariate Jacobi-type polynomials.

In the rest of the section, we will consider $\alpha = \lambda_4 = 0$. Observe that in this case

$$M_k = \lambda_3\,\frac{\beta_k+1}{\beta_0+1} = \lambda_3\,\frac{2k+\beta+\gamma+2}{\beta+\gamma+2} \neq 0.$$

Define the second order partial differential operators $\mathcal{L}_1$ and $\mathcal{L}_2$, acting on functions of two variables:

$$\begin{aligned}\mathcal{L}_1(f)(x,y) =& \frac{\lambda_3}{\beta_0+1}\Big[-(1-x)x\,\frac{\partial^2 f}{\partial x^2} + 2xy\,\frac{\partial^2 f}{\partial x \partial y} - xy\,\frac{\partial^2 f}{\partial y^2} \\ & + (\beta+\gamma+2)x\frac{\partial f}{\partial x} - (\beta+1)x\frac{\partial f}{\partial y} + \frac{\beta_0+1}{\lambda_3}\,f\Big], \\ \mathcal{L}_2(f)(x,y) =& -(1-x)x\,\frac{\partial^2 f}{\partial x^2} + 2xy\,\frac{\partial^2 f}{\partial x \partial y} - xy\,\frac{\partial^2 f}{\partial y^2} \\ & +(-2+(\beta+\gamma+4)x)\frac{\partial f}{\partial x} + (2y-(\beta+1)x)\frac{\partial f}{\partial y} + \frac{(\beta_0+1)(\lambda_3+1)}{\lambda_3} f.\end{aligned}$$

Theorem 3.3. *The polynomials*

$$Q_{n,k}^{(\sigma;\lambda)}(x,y) = P_{n-k}^{(\beta_k,0;M_k,0)}(x)(1-x)^k P_k^{(\gamma,\beta)}\Big(\frac{y}{1-x}\Big) \tag{3.4}$$

satisfy the forth order partial differential equation

$$\mathcal{L}(Q_{n,k}^{(\sigma;\lambda)})(x,y) = \eta_{n,k}^{(\sigma;\lambda)} Q_{n,k}^{(\sigma;\lambda)}(x,y), \tag{3.5}$$

where $\mathcal{L}(f)(x,y) = \mathcal{L}_1(\mathcal{L}_2(f))(x,y)$, *and*

$$\begin{aligned}\eta_{n,k}^{(\sigma;\lambda)} =& \frac{\beta_0+1}{\lambda_3}\Big[1+(n-k)(n-k+\beta_k)\frac{\lambda_3}{\beta_0+1}\Big] \\ & \times\Big[1+\Big((n-k)(n-k+\beta_k)+\beta_n+1\Big)\frac{\lambda_3}{\beta_0+1}\Big].\end{aligned} \tag{3.6}$$

In order to prove this theorem, we are going to use a similar construction as in [11] for finding a fourth order partial differential equation for Krall-type orthogonal polynomials in two variables. Let us define the differential operators acting over functions of one variable:

$$\begin{aligned}L_1(f)(t) =& \frac{M}{a+1}\Big[-(1-t)\,t\,f''(t) + (a+1)\,t\,f'(t) + \frac{a+1}{M}f(t)\Big], \\ L_2(f)(t) =& -(1-t)t\,f''(t) + ((a+3)t-2)\,f'(t) + (M+1)\frac{a+1}{M}\,f(t), \\ L_3(f)(x,y) =& \frac{(1-x-y)y}{(1-x)^2}f''\Big(\frac{y}{1-x}\Big) + \frac{(\beta+1)(1-x)-(\beta+\gamma+2)y}{1-x}f'\Big(\frac{y}{1-x}\Big) \\ & + k(k+\beta+\gamma+1)f\Big(\frac{y}{1-x}\Big).\end{aligned}$$

Let us observe that $L_1(f)$ and $L_2(f)$ are univariate functions, but $L_3(f)$ is a bivariate function.

In [11], up to a change of variable, it was proved that

$$L_1(P_n^{(a,0)})(t) = P_n^{(a,0;M,0)}(t), \tag{3.7}$$

$$L_2(P_n^{(a,0;M,0)})(t) = \eta_n^{(a,0;M,0)} P_n^{(a,0)}(t), \tag{3.8}$$

where

$$\eta_n^{(a,0;M,0)} = \frac{a+1}{M}\Big[1+M\frac{n(n+a)}{a+1}\Big]\Big[1+M\frac{n(n+a)}{a+1}+M\frac{2n+a+1}{a+1}\Big].$$

Next, we relate these three operators with $\mathcal{L}_1$ and $\mathcal{L}_2$ in the following lemma, which can be proved by straightforward computations.

Lemma 3.4. *Let f and g be real-valued functions of one variable, 2-times differentiable, and let*

$$h(x,y) = f(x)(1-x)^k g\Big(\frac{y}{1-x}\Big).$$

Then, taking $a = \beta_k$ and $M = M_k$ in the definitions of L_1 and L_2, the following identities hold:

$$\mathcal{L}_1(h)(x,y) = L_1(f)(x)(1-x)^k g\Big(\frac{y}{1-x}\Big) - \frac{\lambda_3}{\beta_0+1}\, x\, f(x)(1-x)^{k-1} L_3(g)(x,y),$$
$$\mathcal{L}_2(h)(x,y) = L_2(f)(x)(1-x)^k g\Big(\frac{y}{1-x}\Big) - x\, f(x)(1-x)^{k-1} L_3(g)(x,y).$$

Proof. Observe that

$$\begin{aligned}\frac{\partial h}{\partial x} =& f'(x)(1-x)^k g\Big(\frac{y}{1-x}\Big) - kf(x)(1-x)^{k-1} g\Big(\frac{y}{1-x}\Big)\\ &+ f(x)y(1-x)^{k-2} g'\Big(\frac{y}{1-x}\Big),\\ \frac{\partial h}{\partial y} =& f(x)(1-x)^{k-1} g'\Big(\frac{y}{1-x}\Big),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 h}{\partial x^2} =& f''(x)(1-x)^k g\Big(\frac{y}{1-x}\Big) - 2kf'(x)(1-x)^{k-1} g\Big(\frac{y}{1-x}\Big)\\ &+ 2f'(x)y(1-x)^{k-2} g'\Big(\frac{y}{1-x}\Big) + k(k-1)f(x)(1-x)^{k-2} g\Big(\frac{y}{1-x}\Big)\\ &- (k-2)f(x)y(1-x)^{k-3} g\Big(\frac{y}{1-x}\Big) - kf(x)y(1-x)^{k-3} g'\Big(\frac{y}{1-x}\Big)\\ &+ f(x)y^2(1-x)^{k-4} g''\Big(\frac{y}{1-x}\Big),\\ \frac{\partial^2 h}{\partial x \partial y} =& f'(x)(1-x)^{k-1} g'\Big(\frac{y}{1-x}\Big) + (1-k)f(x)(1-x)^{k-2} g'\Big(\frac{y}{1-x}\Big)\\ &+ f(x)y(1-x)^{k-3} g''\Big(\frac{y}{1-x}\Big),\\ \frac{\partial^2 h}{\partial y^2} =& f(x)(1-x)^{k-2} g''\Big(\frac{y}{1-x}\Big).\end{aligned}$$

Then, gathering the second order terms, we get

$$\begin{aligned}-(1-x)x\frac{\partial^2 h}{\partial x^2} + 2xy\frac{\partial^2 h}{\partial x \partial y} - xy\frac{\partial^2 h}{\partial y^2} =& -(1-x)xf''(x)(1-x)^k g\Big(\frac{y}{1-x}\Big)\\ &+2kxf'(x)(1-x)^k g\Big(\frac{y}{1-x}\Big) - \frac{xy(1-x-y)}{(1-x)^2} f(x)(1-x)^{k-1} g''\Big(\frac{y}{1-x}\Big)\\ &+\frac{xy(2-k)}{1-x} f(x)(1-x)^{k-1} g'\Big(\frac{y}{1-x}\Big) - k(k-1)xf(x)(1-x)^{k-1} g\Big(\frac{y}{1-x}\Big)\\ &+(k-2)xyf(x)(1-x)^{k-2} g\Big(\frac{y}{1-x}\Big).\end{aligned}$$

We observe that the terms with $f'(x)g'(y/(1-x))$ cancel, and adding the first order terms in the definition of $\mathcal{L}_1$, the result follows.

The second identity can be obtained in the same way. $\square$

Now, the proof of Theorem 3.3 is a direct consequence of the next proposition.

Proposition 3.5. *For $0 \leqslant k \leqslant n$, Krall-type polynomials* (3.4) *satisfy*

$$\mathcal{L}_1(P_{n,k}^{(0,\beta,\gamma)})(x,y) = Q_{n,k}^{(\sigma;\lambda)}(x,y),$$
$$\mathcal{L}_2(Q_{n,k}^{(\sigma;\lambda)})(x,y) = \eta_{n,k}^{(\sigma;\lambda)} P_{n,k}^{(0,\beta,\gamma)}(x,y),$$

where $\eta_{n,k}^{(\sigma;\lambda)}$ is given in (3.6).

Proof. Evaluating on $\frac{y}{1-x}$ the differential equation for Jacobi polynomials $P_k = P_k^{(\gamma,\beta)}$, which reads as

$$\frac{(1-x-y)y}{(1-x)^2} P_k''\Big(\frac{y}{1-x}\Big) + \frac{(\beta+1)(1-x)-(\beta+\gamma+2)y}{1-x} P_k'\Big(\frac{y}{1-x}\Big)$$
$$+k(k+\beta+\gamma+1)P_k\Big(\frac{y}{1-x}\Big) = 0,$$

we get that $L_3(P_k^{(\gamma,\beta)})(x,y) = 0$. The result follows using the definition of classical orthogonal polynomials on the triangle $P_{n,k}^{(0,\beta,\gamma)}(x,y)$ given in (2.1), the explicit expression (3.4) of $Q_{n,k}^{(\sigma;\lambda)}(x,y)$, Lemma 3.4, and relations (3.7)–(3.8). $\square$

We remark that we have obtained two partial differential operators $\mathcal{D}$ and $\mathcal{L}$, defined in (3.3) and (3.5), respectively, of order 2 and 4 for which the bivariate orthogonal polynomials defined in (3.4) are eigenfunctions. The operators $\mathcal{D}$ and $\mathcal{L}$ commute, they are algebraically independent, and they generate an algebra of differential operators which have the orthogonal polynomials as eigenfunctions.

Next, we observe that the operator $\mathcal{L}$ contains terms without derivatives. Then, the fourth order partial differential equation (3.5) can be written as

$$\mathcal{L}^{(\lambda_3)}(Q_{n,k}^{(\sigma;\lambda)})(x,y) = \hat{\eta}_{n,k}^{(\lambda_3)}\, Q_{n,k}^{(\sigma;\lambda)}(x,y), \tag{3.9}$$

where $\mathcal{L}^{(\lambda_3)}(f)(x,y) = \mathcal{L}_1(\mathcal{L}_2(f))(x,y) - \mathcal{L}_1(\mathcal{L}_2(1))\, f(x,y)$ is a partial differential operator without zero order part, and the eigenvalue takes the form

$$\hat{\eta}_{n,k}^{(\lambda_3)} = \eta_{n,k}^{(\sigma;\lambda)} - \frac{(\beta_0+1)(\lambda_3+1)}{\lambda_3}. \tag{3.10}$$

Now, we take limit when λ_3 goes to 0, and define

$$\mathcal{L}^{(0)} = \lim_{\lambda_3\to 0} \mathcal{L}^{(\lambda_3)},$$

which is a second order partial differential operator such that the classical orthogonal polynomials on the triangle (2.1) with $\alpha = 0$, are eigenfunctions with eigenvalue

$$\hat{\eta}_{n,k}^{(0)} = \lim_{\lambda_3\to 0} \hat{\eta}_{n,k}^{(\lambda_3)} = 2[n(n+\beta+\gamma+2) - k(k+\beta+\gamma+1)].$$

On the other hand, the second order partial differential operator $\mathcal{D}$ is independent of λ_3, and it also has the classical orthogonal polynomials on the triangle as eigenfunctions, and the corresponding eigenvalues are the same as in (3.2).

Let us recall that classical orthogonal polynomials on the triangle are eigenfunctions of a second order partial differential operator (see [14]) with eigenvalue independent of k, that is,

$$\mathcal{S}^{(\alpha,\beta,\gamma)}(P_{n,k}^{(\alpha,\beta,\gamma)})(x,y) = -n(n+\alpha+\beta+\gamma+2)P_{n,k}^{(\alpha,\beta,\gamma)}(x,y),$$

where

$$\begin{aligned}\mathcal{S}^{(\alpha,\beta,\gamma)} =& x(1-x)\frac{\partial^2}{\partial x^2} - 2xy\frac{\partial^2}{\partial x \partial y} + y(1-y)\frac{\partial^2}{\partial y^2}\\ &+ [(\alpha+1) - (\alpha+\beta+\gamma+3)x]\frac{\partial}{\partial x} + [(\beta+1) - (\alpha+\beta+\gamma+3)y]\frac{\partial}{\partial y}.\end{aligned}$$

In [10, p. 464], the author claims that the algebra $\mathcal{A}$ of all differential operators which have classical bivariate orthogonal polynomials on the triangle (2.1) as eigenfunctions has algebraic dimension two. In fact, it is easy to see that

$$\mathcal{L}^{(0)} = 2(\mathcal{D} - \mathcal{S}^{(0,\beta,\gamma)}),$$

that is, $\mathcal{L}^{(0)} \in \mathcal{A}$, for $\alpha = 0$.

In the more general case with $\lambda_3 > 0$, we have found two algebraically independent differential operators $\mathcal{D}$ and $\mathcal{L}^{(\lambda_3)}$ having orthogonal polynomials (3.4) as eigenfunctions. However, it is not guaranteed that they generate the whole algebra of partial differential operators having orthogonal polynomials (3.4) as eigenfunctions. This will be the aim of further research.

Nevertheless, it can be easily checked that there is not a fourth order partial differential operator in the algebra generated by $\mathcal{D}$ and $\mathcal{L}^{(\lambda_3)}$ with eigenvalues independent of k. In [8], it was pointed out that in the multivariate case, to get fourth order partial differential equations independent of k, we have to deal with Sobolev inner products.

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