UNIVERSALITY AT AN ENDPOINT FOR ORTHOGONAL POLYNOMIALS WITH GERONIMUS-TYPE WEIGHTS

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ABSTRACT. We provide a new closed form expression for the Geronimus polynomials on the unit circle and use it to obtain new results and formulas. Among our results is a universality result at an endpoint of an arc for polynomials orthogonal with respect to a Geronimus-type weight on an arc of the unit circle. The key tool is a formula of McLaughlin for the nth power of a 2×2 matrix, which we use to derive convenient formulas for Geronimus polynomials.

1. INTRODUCTION

Let μ be a probability measure whose support is an infinite and compact subset of the unit circle $\partial \mathbb{D}$ in the complex plane. Let $\{\Phi_n(z;\mu)\}_{n=0}^\infty$ be the sequence of monic orthogonal polynomials for the measure μ and let $\{\varphi_n(z;\mu)\}_{n=0}^\infty$ be the sequence of orthonormal polynomials. It is well known that corresponding to this measure is a sequence of Verblunsky coefficients $\{\alpha_n\}_{n=0}^\infty \in \mathbb{D}^{\mathbb{N}_0}$ so that

$$(1) \qquad \Phi_{n+1}(z;\mu) = z\Phi_n(z;\mu) - \bar{\alpha}_n \Phi_n^*(z;\mu), \qquad n \in \mathbb{N}_0,$$

where $\Phi_n^*(z;\mu) := z^n \overline{\Phi_n(1/\bar{z};\mu)}$. The formula (1) is often called the *Szegő recursion* (see [26, Section 1.5]). The relationship between infinitely supported probability measures on the unit circle and sequences of Verblunsky coefficients is a bijection (see [26, Section 1.7]) and there is a substantial literature describing the relationship between the sequence and the corresponding measure (see [26, 27] and the references therein).

Our focus in this work will be on the so-called *Geronimus polynomials*, which are orthogonal with respect to the measure corresponding to the sequence of Verblunsky coefficients $\{\alpha, \alpha, \alpha, \ldots\}$ for some $\alpha \in \mathbb{D}$. The measure of orthogonality in this case is supported on an arc of the unit circle whose length depends on α and possibly a mass point outside this arc, whose weight depends on α. The Geronimus polynomials have been studied before (see [3, 8–11, 21, 22]) and there is a known closed form expression for them (see also [26, Section 1.6]). This formula was later used by Lubinsky and Nguyen in [18] to obtain a universality result for certain polynomial reproducing kernels at an interior point of the arc supporting the measure of orthogonality. Our goal will be to provide a new closed form expression for the Geronimus polynomials, which will enable us to prove several new results and formulas, including a universality result at the endpoint of the arc supporting the measure of orthogonality.

The key tool in our analysis comes from matrix theory. The Szegő recursion can be written

$$\begin{pmatrix} \Phi_{n+1}(z;\mu) \\ \Phi_{n+1}^*(z;\mu) \end{pmatrix} = \begin{pmatrix} z & -\bar{\alpha}_n \\ -\alpha_n z & 1 \end{pmatrix} \begin{pmatrix} \Phi_n(z;\mu) \\ \Phi_n^*(z;\mu) \end{pmatrix};$$

(see [26, Section 3.2]). The 2×2 matrix in this relation is called the nth *transfer matrix* for μ. If the Verblunsky coefficients form a constant sequence, then one can recover the polynomial $\Phi_n(z;\mu)$ in a straightforward way by using the following formula for the nth power of a 2×2 matrix.

Theorem 1.1 (Mc Laughlin, [20]). *Let A be a 2×2 matrix given by*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If R denotes the trace of A and D denotes its determinant, then

$$A^n = \begin{pmatrix} y_n - dy_{n-1} & by_{n-1} \\ cy_{n-1} & y_n - ay_{n-1} \end{pmatrix},$$

where

(2) $$y_n = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-m}{m} R^{n-2m}(-D)^m.$$

This simple result is all that we require to prove our new formula, which appears as Theorem 3.1. Before we can state our results and formulas in Section 3, we review some notation and terminology in the next section. Finally, in Section 4, we state and prove our universality result.

2. PRELIMINARIES

In this section we discuss some notation, formulas, and terminology that we will use throughout Sections 3 and 4. Many of the topics we discuss here are part of a rich theory that is too long to discuss in full detail. Therefore, we will focus only on the specific formulas that we will need for our proofs.

2.1. **Chebyshev polynomials.** The formula that we will obtain for the Geronimus polynomials involves the Chebyshev polynomials of the second kind, which are orthonormal with respect to the measure $\frac{2}{\pi}\sqrt{1-x^2}dx$ on the interval $[-1,1]$. We denote this sequence of polynomials by $\{U_n\}_{n=0}^{\infty}$ and note that these polynomials are given by the formula

(3) $$U_n(x) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n-j}{j}(2x)^{n-2j}$$

(see [1, page 37]). We also recall from [1, page 37] that

(4) $$U_n(x) = \frac{(x+\sqrt{x^2-1})^{n+1} - (x-\sqrt{x^2-1})^{n+1}}{2\sqrt{x^2-1}}.$$

We will make one use of the Chebyshev polynomials of the first kind, which are orthogonal with respect to the measure $\frac{1}{\pi\sqrt{1-x^2}}dx$ on the interval $[-1,1]$. We will denote this sequence of polynomials by $\{T_n\}_{n=0}^{\infty}$ and define them by the formula

(5) $$T_n(x) = U_n(x) - xU_{n-1}(x)$$

(see [1, page 37]).

2.2. **Second kind polynomials.** We have already mentioned that to every sequence of complex numbers $\{\alpha_0, \alpha_1, \alpha_2, \ldots\} \in \mathbb{D}^{\mathbb{N}_0}$ there corresponds a unique probability measure μ on the unit circle having infinite support. The sequence $\{\alpha_n\}_{n=0}^{\infty}$ generates the sequence of orthogonal polynomials $\{\Phi_n(z; \mu)\}_{n=0}^{\infty}$ via the Szegő recursion. One can similarly generate a sequence of monic polynomials from the Szegő recursion using the sequence $\{-\alpha_0, -\alpha_1, -\alpha_2, \ldots\}$, and the resulting polynomials are what we call the *second kind polynomials* for the measure μ and we denote them by $\{\Psi_n(z; \mu)\}_{n=0}^{\infty}$ as in [26, 27]. The polynomials $\{\Psi_n(z; \mu)\}_{n=0}^{\infty}$ are also orthogonal with respect to a probability measure on the unit circle, which is in the family of Aleksandrov measures for the measure μ. We will not make use of this particular fact, so we refer the reader to [26, Section 1.3.9] for details.

2.3. **Wall polynomials.** Corresponding to every probability measure on the unit circle is a Schur function f, which maps $\mathbb{D}$ to itself. When the support of μ is finite, this map is a Blaschke product, but when the support is infinite, there is a canonical pair of sequences of polynomials $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ such that A_n/B_n converges to f uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$ (see [26, Section 1.3.8]). These polynomials are called the *Wall polynomials* for the measure μ and the Pintér-Nevai formulas (see [26, Theorem 3.2.10] or [23]) tell us that

$$A_n(z) = \frac{\Psi_{n+1}^*(z; \mu) - \Phi_{n+1}^*(z; \mu)}{2z},$$
$$B_n(z) = \frac{\Psi_{n+1}^*(z; \mu) + \Phi_{n+1}^*(z; \mu)}{2}.$$

2.4. **Paraorthogonal polynomials.** Suppose μ is a probability measure on the unit circle having $\{\Phi_n(z; \mu)\}_{n=0}^{\infty}$ as its monic orthogonal polynomials. For each $\beta \in \partial \mathbb{D}$ and each $n \in \mathbb{N}_0$, one defines the paraorthogonal polynomial $\Phi_{n+1}^{(\beta)}(z; \mu)$ by

$$\Phi_{n+1}^{(\beta)}(z; \mu) := z\Phi_n(z; \mu) - \bar{\beta}\Phi_n^*(z; \mu).$$

We also define

$$\Psi_{n+1}^{(\beta)}(z; \mu) := z\Psi_n(z; \mu) - \bar{\beta}\Psi_n^*(z; \mu)$$

for each $n \in \mathbb{N}_0$. Paraorthogonal polynomials were introduced in [13] and have the property that all of their zeros are simple and lie on the unit circle. Paraorthogonal polynomials arising from Geronimus polynomials have been previously considered in [4].

2.5. **Regularity.** If μ is a probability measure on the unit circle with orthonormal polynomials $\{\varphi_n(z; \mu)\}_{n=0}^{\infty}$, let κ_n denote the leading coefficient of φ_n. Following the terminology from [29], we will say that the measure μ is *regular* if

$$\lim_{n \to \infty} \kappa_n^{1/n} = \frac{1}{\text{cap}(\text{supp}(\mu))},$$

where $\text{cap}(K)$ is the logarithmic capacity of the compact set K. Regularity is a complicated notion and we will not discuss the technical details here. We mention

that a measure μ whose support is an arc Γ of the unit circle is regular if and only if

$$(6) \qquad \lim_{n \to \infty} \left(\sup_{\substack{\deg(P) \le n \\ P \not\equiv 0}} \left[\frac{\|P\|_{L^\infty(\Gamma)}}{\|P\|_{L^2(\mu)}} \right]^{1/n} \right) = 1$$

(see [29, Theorem 3.2.3(v)]). Regularity indicates that the measure μ has sufficient density that a polynomial cannot have an exponentially small L^2-norm without having an exponentially small L^∞-norm.

With these preliminaries in hand, we can now proceed to state and prove our new results.

3. GERONIMUS POLYNOMIALS

For any $\alpha \in \mathbb{D}$, let $\rho = \sqrt{1 - |\alpha|^2}$, and let μ_α be the probability measure on the unit circle whose Verblunsky coefficients satisfy $\alpha_n = \alpha$ for all $n \in \mathbb{N}_0$. This measure is supported on the arc $\{e^{i\theta} : 2\arcsin(|\alpha|) \le \theta \le 2\pi - 2\arcsin(|\alpha|)\}$ and possibly one point outside this arc (see [26, Section 1.6]). Our first result is a new formula for the polynomials $\varphi_n(z; \mu_\alpha)$ and $\varphi_n^*(z; \mu_\alpha)$.

Theorem 3.1. *For any $\alpha \in \mathbb{D}$ and $n \in \mathbb{N}_0$, it holds that*

$$\varphi_n(z; \mu_\alpha) = z^{n/2} \left(U_n \left(\frac{z+1}{2\rho\sqrt{z}} \right) - \frac{1 + \bar{\alpha}}{\rho\sqrt{z}} U_{n-1} \left(\frac{z+1}{2\rho\sqrt{z}} \right) \right),$$

$$\varphi_n^*(z; \mu_\alpha) = z^{n/2} \left(U_n \left(\frac{z+1}{2\rho\sqrt{z}} \right) - \frac{\sqrt{z}(1 + \alpha)}{\rho} U_{n-1} \left(\frac{z+1}{2\rho\sqrt{z}} \right) \right),$$

where $U_{-1} = 0$.

Proof. Since the Verblunsky coefficients for the Geronimus polynomials are all the same, we have

$$\begin{pmatrix} \Phi_n(z; \mu_\alpha) \\ \Phi_n^*(z; \mu_\alpha) \end{pmatrix} = \begin{pmatrix} z & -\bar{\alpha} \\ -\alpha z & 1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

Therefore, Theorem 1.1 implies

$$(7) \qquad\qquad \Phi_n(z; \mu_\alpha) = y_n - (1 + \bar{\alpha})y_{n-1},$$

$$(8) \qquad\qquad \Phi_n^*(z; \mu_\alpha) = y_n - z(1 + \alpha)y_{n-1},$$

where for any choice of $\sqrt{z}$ we have

$$y_n(z) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-m}{m} (z+1)^{n-2m} (-\rho^2 z)^m = \rho^n z^{n/2} U_n \left(\frac{z+1}{2\rho\sqrt{z}} \right).$$

If we plug this into (7) and (8) and note that the leading coefficient of φ_n is ρ^{-n} (see [26, Equation 1.5.22]), we get the desired formulas. $\qquad\square$

Let us explore some elementary consequences of Theorem 3.1. First notice that we can find the generating function for the polynomials $\{\varphi_n(z; \mu_\alpha)\}_{n=0}^\infty$.

Corollary 3.2. *The polynomials $\{\varphi_n(z; \mu_\alpha)\}_{n=0}^\infty$ satisfy*

$$\sum_{n=0}^\infty \varphi_n(z; \mu_\alpha)t^n = \frac{\rho - t - t\bar{\alpha}}{\rho - t(z+1) + \rho z t^2}$$

whenever this series converges.

Remark. For a related result, see [3, Section 2].

Proof. This is an immediate consequence of Theorem 3.1 and the fact that

$$\sum_{n=0}^{\infty} U_n(x)t^n = \frac{1}{1 - 2xt + t^2}$$

whenever this series converges (see [12, Equation 4.5.23]).   □

As a second application, we can use Theorem 3.1 to find convenient formulas for the Wall polynomials for the measure μ_α.

Corollary 3.3. *For all $n \in \mathbb{N}$, the Wall polynomials A_n and B_n for the measure μ_α are given by*

$$A_n(z) = \alpha \rho^n z^{n/2} U_n\left(\frac{z+1}{2\rho\sqrt{z}}\right),$$

$$B_n(z) = \rho^{n+1} z^{(n+1)/2}\left[U_{n+1}\left(\frac{z+1}{2\rho\sqrt{z}}\right) - \frac{\sqrt{z}}{\rho}U_n\left(\frac{z+1}{2\rho\sqrt{z}}\right)\right].$$

Proof. This is an immediate consequence of Theorem 3.1 and the Pintér-Nevai formulas.   □

As an additional application, we have the following relation for the first and second kind paraorthogonal polynomials.

Corollary 3.4. *For every $\alpha \in \mathbb{D}$ and every $n \in \mathbb{N}$ it holds that*

$$\Phi_n^{(1)}(1; \mu_\alpha) + \Psi_n^{(1)}(1; \mu_\alpha) = 0.$$

We can also use Theorem 3.1 to provide new proofs of some existing results. For instance, we can apply Corollary 3.3 and send $n \to \infty$ to find the Schur function for the measure μ_α. Indeed, by [25, Theorem 1], we know that

$$(9) \qquad \lim_{n\to\infty} \frac{U_{n+1}(x)}{U_n(x)} = x + \sqrt{x^2 - 1}, \qquad x \notin [-1, 1].$$

If we apply Corollary 3.3 and (9) with $x = \frac{z+1}{2\rho\sqrt{z}}$ we conclude

$$\lim_{n\to\infty} \frac{A_n(z)}{B_n(z)} = \frac{2\alpha}{1 - z + \sqrt{(z+1)^2 - 4\rho^2 z}}, \qquad |z| < 1,$$

which agrees with the formula given for f in [26, Section 1.6]. We can also use Theorem 3.1 to deduce the ratio asymptotic behavior of the orthonormal Geronimus polynomials. If we apply the formula from Theorem 3.1 and (9) with $x = \frac{z+1}{2\rho\sqrt{z}}$, we see that

$$\lim_{n\to\infty} \frac{\varphi_{n+1}(z; \mu_\alpha)}{\varphi_n(z; \mu_\alpha)} = \frac{z + 1 + \sqrt{(z+1)^2 - 4\rho^2 z}}{2\rho}, \qquad z \notin \mathrm{supp}(\mu_\alpha).$$

This result is not new and follows from the stronger results in [11, Theorem 1], but Theorem 3.1 provides us with an easy proof.

Theorem 3.1 also provides a new proof of the following fact, which appears in [7, Equation 5]. To state it, we recall the polynomials $\{T_n\}_{n\in\mathbb{N}}$ from Section 2.1.

Corollary 3.5. *The pair $(X, Y) = (T_n(z), U_{n-1}(z))$ solves the Pell equation*

$$X^2 - (z^2 - 1)Y^2 = 1.$$

Proof. We recall [26, Proposition 3.2.2], which tells us that for any measure μ with Verblunsky coefficients $\{\alpha_j\}_{j=0}^{\infty}$ it holds that

$$\Psi_n^*(z;\mu)\Phi_n(z;\mu) + \Phi_n^*(z;\mu)\Psi_n(z;\mu) = 2z^n \prod_{j=0}^{n-1}(1 - |\alpha_j|^2).$$

Applying this formula with $\mu = \mu_\alpha$, we find

$$U_n\left(\frac{z+1}{2\rho\sqrt{z}}\right)^2 + U_{n-1}\left(\frac{z+1}{2\rho\sqrt{z}}\right)^2 - \frac{z+1}{\rho\sqrt{z}}U_n\left(\frac{z+1}{2\rho\sqrt{z}}\right)U_{n-1}\left(\frac{z+1}{2\rho\sqrt{z}}\right) = 1.$$

Therefore, by invoking (5) we find that for any $w \in \mathbb{C}$ it holds that

$$T_n(w)^2 + U_{n-1}(w)^2 = (U_n(w) - wU_{n-1}(w))^2 + U_{n-1}(w)^2 = 1 + w^2 U_{n-1}(w)^2$$

as desired. $\square$

One can also use Theorem 3.1 to prove more substantial new results that require more detailed calculation and analysis. The next section is devoted to just such a result, namely, a universality result at the endpoint of the arc supporting the measure of orthogonality.

4. UNIVERSALITY

Let μ be a probability measure with infinite support on the unit circle. The degree n polynomial reproducing kernel $K_n(z, w; \mu)$ is given by

$$K_n(z, w; \mu) := \sum_{m=0}^{n} \varphi_m(z;\mu)\overline{\varphi_m(w;\mu)}$$

and is the reproducing kernel for the space of polynomials of degree at most n in $L^2(\mu)$. One is often interested in calculating the following limit (if it exists):

$$(10) \qquad \lim_{n\to\infty} \frac{K_n(z_0 + \sigma_1(n), z_0 + \sigma_2(n); \mu)}{K_n(z_0, z_0; \mu)},$$

where $\sigma_j(n) \to 0$ as $n \to \infty$ in a specific way for $j = 1, 2$. If this limit exists and is the same for a large class of measures μ, then we call the corresponding result a *universality* result. Some universality results when the point z_0 is the endpoint of an interval supporting the measure of orthogonality can be found in [5, 15–17], but all of these results assume that the measure is supported on a compact subset of the real line. Our main result in this section is Theorem 4.1, which considers measures supported on an arc of the unit circle. Before we can state it, we need to define some notation. If J_s denotes the Bessel function of the first kind of order s, then we set

$$\mathbb{J}_{1/2}^*(a, b) := \begin{cases} \dfrac{J_{1/2}(\sqrt{a})\sqrt{b}J_{1/2}'(\sqrt{b}) - J_{1/2}(\sqrt{b})\sqrt{a}J_{1/2}'(\sqrt{a})}{2a^{1/4}b^{1/4}(a-b)}, & a \neq b, \\ \dfrac{1}{4\sqrt{a}}\left(J_{1/2}^2(\sqrt{a}) - J_{3/2}(\sqrt{a})J_{-1/2}(\sqrt{a})\right) & a = b, \end{cases}$$

as in [5, 16, 17]. As noted in [16], the function $\mathbb{J}_{1/2}^*$ is entire. Now we can state our main result about universality after recalling the notion of regularity from Section 2.5. For the remainder of this section, we identify the unit circle with the interval $[0, 2\pi)$.

Theorem 4.1. *Fix $\alpha \in (-1, 0)$ and let μ be a probability measure on the arc $\{e^{i\theta} : 2\arcsin(|\alpha|) \leq \theta \leq 2\pi - 2\arcsin(|\alpha|)\}$ of the form $h(\theta)w(\theta)\frac{d\theta}{2\pi} + d\tilde{\mu}$ where*

$$\operatorname{supp}(\tilde{\mu}) \subseteq [2\arcsin(|\alpha|) + \tilde{\epsilon}, 2\pi - 2\arcsin(|\alpha|)]$$

for some $\tilde{\epsilon} > 0$ and

$$w(\theta) = \begin{cases} \frac{\sqrt{1-\alpha^2-\cos^2(\theta/2)}}{(1+\alpha)\sin(\theta/2)}, & 2\arcsin(|\alpha|) < \theta < 2\pi - 2\arcsin(|\alpha|), \\ 0, & o.w., \end{cases}$$

and $h(\theta)$ is continuous at $2\arcsin(|\alpha|)$ and $h(2\arcsin(|\alpha|)) > 0$. Assume also that μ is regular. Then uniformly for a, b in compact subsets of the complex plane, it holds that

$$\lim_{n \to \infty} \frac{K_n(e^{i(\theta_\alpha - \frac{a}{n^2})}, e^{i(\theta_\alpha - \frac{b}{n^2})}; \mu)}{K_n(e^{i\theta_\alpha}, e^{i\theta_\alpha}; \mu)} = \frac{\mathbb{J}_{1/2}^*(\frac{\alpha a}{\rho}, \frac{\alpha \bar{b}}{\rho})}{\mathbb{J}_{1/2}^*(0, 0)},$$

where $\theta_\alpha = 2\arcsin(|\alpha|)$ and $\rho = \sqrt{1 - \alpha^2}$.

Remark. Notice that the limiting kernel in Theorem 4.1 is the same as in the real line case from [5, Theorem 1.4].

Before we proceed with the proof of Theorem 4.1, we present a proof of the following fact about the kernel $\mathbb{J}_{1/2}^*$.

Proposition 4.2. *The function $\mathbb{J}_{1/2}^*(t, \bar{t})$ is non-vanishing as a function of $t \in \mathbb{C}$.*

Proof. When t is real, we use the fact that

$$J_{1/2}(z) = \frac{\sqrt{2}\sin(z)}{\sqrt{\pi z}}, \qquad J_{-1/2}(z) = \frac{\sqrt{2}\cos(z)}{\sqrt{\pi z}},$$

$$J_{3/2}(z) = \frac{\sqrt{2}(\sin(z) - z\cos(z))}{\sqrt{\pi z^3}}$$

(see [14, pages 16 & 17]) to see that

$$\mathbb{J}_{1/2}^*(t, t) = \frac{1}{2\pi t}\left(1 - \frac{\sin(2\sqrt{t})}{2\sqrt{t}}\right), \qquad t \in \mathbb{R},$$

which is non-zero for all $t \in \mathbb{R}$. Using similar formulas, we find that when $t \notin \mathbb{R}$, we have

$$(11) \qquad \mathbb{J}_{1/2}^*(t, \bar{t}) = \frac{\frac{\cos(\sqrt{\bar{t}})\sin(\sqrt{t})}{\sqrt{t}} - \frac{\cos(\sqrt{t})\sin(\sqrt{\bar{t}})}{\sqrt{\bar{t}}}}{\pi(t - \bar{t})} = \frac{\operatorname{Im}\frac{\cos(\sqrt{\bar{t}})\sin(\sqrt{t})}{\sqrt{t}}}{\pi \operatorname{Im} t}.$$

Since we are assuming $t \notin \mathbb{R}$, we may assume $\operatorname{Re}[\sqrt{t}] \neq 0$. We will show that (11) is never zero when $\operatorname{Re}[\sqrt{t}] > 0, \operatorname{Im}[\sqrt{t}] > 0$ and the other cases can be deduced by using the symmetry of this expression.

Suppose $\sqrt{t} = \frac{1}{2}(x + iy)$ and $\sqrt{\bar{t}} = \frac{1}{2}(x - iy)$ (our choice of $\sqrt{\bar{t}}$ does not matter because the cosine function and the sinc function are both even). Using basic trigonometric identities, we can rewrite the numerator of (11) as

$$(12) \qquad \operatorname{Im}\left[\frac{\sin(x) + i\sinh(y)}{x + iy}\right] = \frac{x\sinh(y) - y\sin(x)}{x^2 + y^2}.$$

If we assume that $x > 0$, then notice that the denominator of (12) is positive. Furthermore, notice that the numerator of (12) is 0 when $y = 0$ and its partial derivative with respect to y is positive. This shows that (12) is positive when x

and y are positive. Similar calculations for negative values of x or y show (11) is non-zero when $t \notin \mathbb{R}$. $\square$

The proof of Theorem 4.1 will follow the method pioneered by Lubinsky, which consists of first proving the result in one particular case (when $h \equiv 1$ and $\tilde{\mu} = 0$) and then using localization techniques and the assumed regularity of the measure to prove the more general case (see [17]).

4.1. **A model case.** Fix $\alpha \in (-1, 0)$ and define $\theta_\alpha = 2 \arcsin(|\alpha|)$. The measure μ_α from Section 3 is of the form given in Theorem 4.1 with $h \equiv 1$ and $\tilde{\mu} = 0$ (see [26, Section 1.6]). Let us write $z = e^{iw}$, where we allow w to be complex. Since U_n is even or odd (depending on the parity of n) our choice of $\sqrt{z}$ will not effect our calculations, so we will write $\sqrt{z} = e^{iw/2}$. Theorem 3.1 then gives

$$(13) \quad \varphi_n(z; \mu_\alpha) = e^{inw/2} \left(U_n \left(\frac{\cos(w/2)}{\rho} \right) - \frac{1 + \bar{\alpha}}{\rho e^{iw/2}} U_{n-1} \left(\frac{\cos(w/2)}{\rho} \right) \right),$$

$$(14) \quad \varphi_n^*(z; \mu_\alpha) = e^{inw/2} \left(U_n \left(\frac{\cos(w/2)}{\rho} \right) - \frac{e^{iw/2}(1 + \alpha)}{\rho} U_{n-1} \left(\frac{\cos(w/2)}{\rho} \right) \right).$$

We will apply (13) and (14) with $w = \theta_\alpha + t/n^2$ for various values of t. We begin with the following lemma.

Lemma 4.3. *The collection of functions*

$$\left\{ \frac{K_n(e^{i(\theta_\alpha - \frac{a}{n^2})}, e^{i(\theta_\alpha - \frac{b}{n^2})}; \mu_\alpha)}{n^3} \right\}_{n \in \mathbb{N}}$$

is a normal family on $\mathbb{C}^2$ in the variables a and b.

Proof. By Montel's Theorem and the Cauchy-Schwarz inequality, it suffices to show that the collection

$$\left\{ \frac{K_n(e^{i(\theta_\alpha - \frac{a}{n^2})}, e^{i(\theta_\alpha - \frac{a}{n^2})}; \mu_\alpha)}{n^3} \right\}_{n \in \mathbb{N}}$$

is uniformly bounded in compact subsets of $\mathbb{C}$ (as a function of a). To do so, we use (4) to see that

$$U_n \left(\frac{1}{\rho} \cos \left(\frac{\theta_\alpha}{2} - \frac{a}{2n^2} \right) \right) = U_n \left(1 - \frac{\alpha a}{2\rho n^2} + o(n^{-2}) \right) = O(n)$$

as $n \to \infty$ and hence (13) implies $|\varphi_n(e^{i(\theta_\alpha - \frac{a}{n^2})}; \mu_\alpha)| = O(n)$ as $n \to \infty$ uniformly for a in compact subsets of $\mathbb{C}$. It follows that

$$K_n(e^{i(\theta_\alpha - \frac{a}{n^2})}, e^{i(\theta_\alpha - \frac{a}{n^2})}; \mu_\alpha) = O \left(\sum_{m=1}^{n} m^2 \right) = O(n^3)$$

uniformly for a in compact subsets of $\mathbb{C}$. This is the desired conclusion. $\square$

To prove Theorem 4.1 in the case $\mu = \mu_\alpha$ for $\alpha \in (-1, 0)$, we take $a \neq \bar{b}$ and apply the Christoffel-Darboux formula

$$K_n(z, \zeta; \mu) = \frac{\overline{\varphi_{n+1}^*(\zeta; \mu)} \varphi_{n+1}^*(z; \mu) - \overline{\varphi_{n+1}(\zeta; \mu)} \varphi_{n+1}(z; \mu)}{1 - z\bar{\zeta}}$$

(see [26, Theorem 2.2.7]). For ease of notation, we define $\theta_n(z) := \frac{1}{2}\theta_\alpha - \frac{z}{2n^2}$ for all complex numbers z. Using Theorem 3.1, we find

$$K_n(e^{i(\theta_\alpha - \frac{a}{n^2})}, e^{i(\theta_\alpha - \frac{b}{n^2})}; \mu_\alpha) = (1 - e^{i(\bar{b}-a)/n^2})^{-1}$$

$$\times \left[e^{i(\bar{b}-a)\frac{n+1}{2n^2}} \left(U_{n+1}\left(\frac{\cos(\theta_n(a))}{\rho} \right) - \frac{e^{i\theta_n(a)}(1+\alpha)}{\rho} U_n\left(\frac{\cos(\theta_n(a))}{\rho} \right) \right) \right.$$

$$\times \left(U_{n+1}\left(\frac{\cos(\theta_n(\bar{b}))}{\rho} \right) - \frac{e^{-i\theta_n(\bar{b})}(1+\alpha)}{\rho} U_n\left(\frac{\cos(\theta_n(\bar{b}))}{\rho} \right) \right)$$

$$- e^{i(\bar{b}-a)\frac{n+1}{2n^2}} \left(U_{n+1}\left(\frac{\cos(\theta_n(a))}{\rho} \right) - \frac{(1+\alpha)}{\rho e^{i\theta_n(a)}} U_n\left(\frac{\cos(\theta_n(a))}{\rho} \right) \right)$$

$$\times \left. \left(U_{n+1}\left(\frac{\cos(\theta_n(\bar{b}))}{\rho} \right) - \frac{(1+\alpha)}{\rho e^{-i\theta_n(\bar{b})}} U_n\left(\frac{\cos(\theta_n(\bar{b}))}{\rho} \right) \right) \right]$$

$$= \frac{2ie^{i(\bar{b}-a)\frac{n+1}{2n^2}}(1+\alpha)}{\rho(1 - e^{i(\bar{b}-a)/n^2})}$$

$$\times \left[- U_{n+1}\left(\frac{\cos(\theta_n(\bar{b}))}{\rho} \right) U_n\left(\frac{\cos(\theta_n(a))}{\rho} \right) \sin(\theta_n(a)) \right.$$

$$\text{(15)} \qquad + U_{n+1}\left(\frac{\cos(\theta_n(a))}{\rho} \right) U_n\left(\frac{\cos(\theta_n(\bar{b}))}{\rho} \right) \sin(\theta_n(\bar{b}))$$

$$\left. + U_n\left(\frac{\cos(\theta_n(\bar{b}))}{\rho} \right) U_n\left(\frac{\cos(\theta_n(a))}{\rho} \right) \frac{1+\alpha}{\rho} \sin\left(\frac{\bar{b}-a}{2n^2} \right) \right].$$

Using basic angle addition formulas, we find

$$\frac{1}{\rho}\cos\left(\frac{\theta_\alpha}{2} + \frac{t}{2n^2} \right) = \cos\left(\frac{t}{2n^2} \right) + \frac{\alpha}{\rho}\sin\left(\frac{t}{2n^2} \right) = 1 + \frac{\alpha t}{2\rho n^2} + O(n^{-4})$$

$$\sin\left(\frac{\theta_\alpha}{2} + \frac{t}{2n^2} \right) = -\alpha\cos\left(\frac{t}{2n^2} \right) + \rho\sin\left(\frac{t}{2n^2} \right) = -\alpha + \frac{t\rho}{2n^2} + O(n^{-4}).$$

If μ^* is the measure of orthogonality for the polynomials $\{U_n\}_{n\geq 0}$, then

$$K_n(x, y; \mu^*) = \frac{\overline{U_n(y)}U_{n+1}(x) - U_n(x)\overline{U_{n+1}(y)}}{2(x - \bar{y})}$$

(see [28, Section 3]). Letting $x = \frac{1}{\rho}\cos\left(\frac{\theta_\alpha}{2} - \frac{a}{2n^2} \right)$ and $y = \frac{1}{\rho}\cos\left(\frac{\theta_\alpha}{2} - \frac{b}{2n^2} \right)$, we find that the first two terms in (15) can be rewritten

$$2\alpha\left(\frac{\alpha(a - \bar{b})}{2\rho n^2} \right) K_n\left(1 - \frac{\alpha a(1 + o(1))}{2\rho n^2}, 1 - \frac{\alpha b(1 + o(1))}{2\rho n^2}; \mu^* \right) (1 + o(1))$$

as $n \to \infty$. By [5, Theorem 1.4], we see that we can rewrite this as

$$\text{(16)} \qquad 2\alpha\left(\frac{\alpha(a - \bar{b})}{2\rho n^2} \right) K_n(1, 1; \mu^*)(1 + o(1)) \frac{\mathbb{J}^*_{1/2}(\frac{\alpha a}{\rho}, \frac{\alpha\bar{b}}{\rho})}{\mathbb{J}^*_{1/2}(0, 0)}$$

as $n \to \infty$. Using the fact that $U_n(1) = n+1$ (see [1, page 37]) we find $K_n(1, 1; \mu^*) = \frac{1}{6}(n+1)(n+2)(2n+3)$, so (16) simplifies to

$$n \frac{\alpha^2(a - \bar{b})\mathbb{J}^*_{1/2}(\frac{\alpha a}{\rho}, \frac{\alpha \bar{b}}{\rho})}{3\rho\mathbb{J}^*_{1/2}(0, 0)} + o(n)$$

as $n \to \infty$.

To estimate the last term in (15), we use (4) as in the proof of Lemma 4.3 to see that

$$U_n\left(\frac{1}{\rho}\cos\left(\frac{\theta_\alpha}{2} + \frac{t}{2n^2}\right)\right) = U_n\left(1 + \frac{\alpha t}{2\rho n^2} + o(n^{-2})\right) = O(n)$$

as $n \to \infty$. Since $\sin\left(\frac{\bar{b}-a}{2n^2}\right) = O(n^{-2})$ as $n \to \infty$, we see that the last term in (15) is $O(1)$ as $n \to \infty$. Combining all that we have learned so far yields

$$(17) \qquad K_n(e^{i(\theta_\alpha - \frac{a}{n^2})}, e^{i(\theta_\alpha - \frac{b}{n^2})}; \mu_\alpha) = o(n^3) + n^3\left(\frac{2(1+\alpha)\alpha^2\mathbb{J}^*_{1/2}(\frac{\alpha a}{\rho}, \frac{\alpha \bar{b}}{\rho})}{3\rho^2\mathbb{J}^*_{1/2}(0, 0)}\right)$$

as $n \to \infty$ when $a \neq \bar{b}$. By continuity and Lemma 4.3 we may extend this formula to the case $a = \bar{b}$ and deduce that the error term can be estimated uniformly for a and b in compact subsets of $\mathbb{C}$. Setting $a = b = 0$, we find

$$K_n(e^{i\theta_\alpha}, e^{i\theta_\alpha}; \mu_\alpha) = o(n^3) + n^3\left(\frac{2(1+\alpha)\alpha^2}{3\rho^2}\right)$$

as $n \to \infty$ (see also [6, Theorem 1.2]). We have thus proven

$$\lim_{n\to\infty} \frac{K_n(e^{i(\theta_\alpha - \frac{a}{n^2})}, e^{i(\theta_\alpha - \frac{b}{n^2})}; \mu_\alpha)}{K_n(e^{i\theta_\alpha}, e^{i\theta_\alpha}; \mu_\alpha)} = \frac{\mathbb{J}^*_{1/2}(\frac{\alpha a}{\rho}, \frac{\alpha \bar{b}}{\rho})}{\mathbb{J}^*_{1/2}(0, 0)},$$

where the convergence is uniform for a and b in compact subsets of $\mathbb{C}$. This proves the desired result in the case $\mu = \mu_\alpha$ when $\alpha \in (-1, 0)$.

4.2. The general case.

To prove the general case, we will use the following theorem, which is due to Bourgade and appears in a more general form as [2, Theorem 3.10].

Theorem 4.4 ([2]). *Let μ be as in Theorem 4.1 and let μ_α be as in Section 4.1. If*

$$(18) \qquad \lim_{r\to 0^+} \limsup_{n\to\infty} \frac{K_n(e^{i(\theta_\alpha - \frac{t}{n^2})}, e^{i(\theta_\alpha - \frac{t}{n^2})}; \mu_\alpha)}{K_{n-\lfloor rn\rfloor}(e^{i(\theta_\alpha - \frac{t}{n^2})}, e^{i(\theta_\alpha - \frac{t}{n^2})}; \mu_\alpha)} = 1$$

uniformly for t in compact subsets of $\mathbb{R}$, then uniformly for a and b in compact subsets of $(-\infty, 0]$, it holds that

$$\lim_{n\to\infty} \left| \frac{h(\theta_\alpha)K_n(e^{i(\theta_\alpha - \frac{a}{n^2})}, e^{i(\theta_\alpha - \frac{b}{n^2})}; \mu) - K_n(e^{i(\theta_\alpha - \frac{a}{n^2})}, e^{i(\theta_\alpha - \frac{b}{n^2})}; \mu_\alpha)}{K_n(e^{i(\theta_\alpha - \frac{a}{n^2})}, e^{i(\theta_\alpha - \frac{a}{n^2})}; \mu)} \right| = 0.$$

Remark. Note that [2, Theorem 3.10] includes a mutual regularity condition for the two measures in question, but the regularity of μ and μ_α immediately implies that this condition is satisfied.

The fact that μ_α satisfies the condition (18) is a direct consequence of the calculations in Section 4.1. All that remains then is to show that the conclusion of Theorem 4.4 gives us the conclusion that we want. For this purpose, the following lemma is essential.

Lemma 4.5. *Let μ be as in the statement of Theorem 4.1 and let μ_α be as in Section 4.1. For any $a \in \mathbb{C}$ it holds that*

$$\lim_{n\to\infty} \frac{K_n(e^{i(\theta_\alpha - \frac{a}{n^2})}, e^{i(\theta_\alpha - \frac{a}{n^2})}; \mu_\alpha)}{K_n(e^{i(\theta_\alpha - \frac{a}{n^2})}, e^{i(\theta_\alpha - \frac{a}{n^2})}; \mu)} = h(\theta_\alpha)$$

and the convergence is uniform for a in compact subsets of $\mathbb{C}$.

The proof of Lemma 4.5 is very much analogous to the proof of [24, Lemma 2.8], so we will not present the details here. It is based on Christoffel functions and relies heavily on ideas from the proof of [19, Theorem 7].

Corollary 4.6. *Let μ be as in the statement of Theorem 4.1. Then*

$$(19) \qquad \lim_{n\to\infty} \frac{K_n(e^{i\theta_\alpha}, e^{i\theta_\alpha}; \mu)}{K_n(e^{i(\theta_\alpha - \frac{a}{n^2})}, e^{i(\theta_\alpha - \frac{a}{n^2})}; \mu)} = \frac{\mathbb{J}^*_{1/2}(0,0)}{\mathbb{J}^*_{1/2}(\frac{\alpha a}{\rho}, \frac{\alpha \bar{a}}{\rho})} > 0$$

and the convergence is uniform on compact subsets of $\mathbb{C}$. Furthermore, the collection

$$\left\{ \frac{K_n(e^{i(\theta_\alpha - \frac{a}{n^2})}, e^{i(\theta_\alpha - \frac{b}{n^2})}; \mu)}{K_n(e^{i\theta_\alpha}, e^{i\theta_\alpha}; \mu)} \right\}_{n\in\mathbb{N}}$$

is a normal family on $\mathbb{C}^2$ in the variables a and b.

Remark. By Proposition 4.2, the right-hand side of (19) is a well-defined positive real number.

Proof. The limit (19) follows from Lemma 4.5 and the fact that the limit holds when $\mu = \mu_\alpha$. The statement about normality follows from Montel's Theorem, the Cauchy-Schwarz inequality, and the uniformity in the limit (19). $\square$

Proof of Theorem 4.1. We have already seen that we may apply Theorem 4.4. By applying Corollary 4.6, we may rewrite the conclusion of Theorem 4.4 as

$$\lim_{n\to\infty} \left| \frac{K_n(e^{i(\theta_\alpha - \frac{a}{n^2})}, e^{i(\theta_\alpha - \frac{b}{n^2})}; \mu)}{K_n(e^{i\theta_\alpha}, e^{i\theta_\alpha}; \mu)} - \frac{K_n(e^{i(\theta_\alpha - \frac{a}{n^2})}, e^{i(\theta_\alpha - \frac{b}{n^2})}; \mu_\alpha)}{h(\theta_\alpha) K_n(e^{i\theta_\alpha}, e^{i\theta_\alpha}; \mu)} \right| = 0.$$

By applying Lemma 4.5, we may rewrite this as

$$\lim_{n\to\infty} \left| \frac{K_n(e^{i(\theta_\alpha - \frac{a}{n^2})}, e^{i(\theta_\alpha - \frac{b}{n^2})}; \mu)}{K_n(e^{i\theta_\alpha}, e^{i\theta_\alpha}; \mu)} - \frac{K_n(e^{i(\theta_\alpha - \frac{a}{n^2})}, e^{i(\theta_\alpha - \frac{b}{n^2})}; \mu_\alpha)}{K_n(e^{i\theta_\alpha}, e^{i\theta_\alpha}; \mu_\alpha)} \right| = 0,$$

and hence when a and b are real, the desired convergence follows from the calculations in Section 4.1. The desired uniform convergence on compact subsets of $\mathbb{C}^2$ follows from the statement about normal families in Corollary 4.6. $\square$

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