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CHARACTERIZATIONS OF WEIGHTED COMPACTNESS OF COMMUTATORS VIA $\mathrm{CMO}(\mathbb{R}^n)$

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ABSTRACT. In this paper, the authors show that a function $b \in \mathrm{BMO}(\mathbb{R}^n)$ is in $\mathrm{CMO}(\mathbb{R}^n)$ if and only if the Riesz transform commutator $[b, R_i]$ is compact on $L^p_w(\mathbb{R}^n)$ for $i \in \{1, 2, \cdots, n\}$, $p \in (1, \infty)$, and $w \in A_p(\mathbb{R}^n)$, and if and only if the fractional integral commutator $[b, I_\alpha]$ is compact from $L^p_{w^p}(\mathbb{R}^n)$ to $L^q_{w^q}(\mathbb{R}^n)$, where $\alpha \in (0, n)$, $p, q \in (1, \infty)$ with $\frac{1}{p} = \frac{1}{q} + \frac{\alpha}{n}$ and $w \in A_{p,\,q}(\mathbb{R}^n)$.

1. Introduction and statement of main results

For $i \in \{1, 2, \cdots, n\}$, let R_i be the i-th Riesz transform on $\mathbb{R}^n$; that is,

$$R_i(f)(x) := \mathrm{p.\,v.}\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\int_{\mathbb{R}^n}\frac{x_i - y_i}{|x-y|^{n+1}}f(y)\,dy,$$

where x_i and y_i are the i-th elements of x and y, respectively. The equivalent characterization of compactness of commutator

$$[b, T]f := bTf - T(bf)$$

with singular integral operator T was initialized by Uchiyama in [22], where he refined the result of Coifman et al. [8] on the L^p-boundedness of commutators with the symbol b in the space $\mathrm{BMO}(\mathbb{R}^n)$ to compactness, showing that the Riesz transform commutator $[b, R_i]$ is compact on $L^p(\mathbb{R}^n)$, $p \in (1, \infty)$, if and only if $b \in \mathrm{CMO}(\mathbb{R}^n)$, which is the closure in $\mathrm{BMO}(\mathbb{R}^n)$ of the space $\mathcal{D}$, the space of C^∞ functions with compact supports. In [23], Wang showed that the fact $b \in \mathrm{CMO}(\mathbb{R}^n)$ is also sufficient and necessary for the compactness of the commutator $[b, I_\alpha]$ with fractional integral operator I_α from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $\alpha \in (0, n)$, $p, q \in (1, \infty)$ with $\frac{1}{p} = \frac{1}{q} + \frac{\alpha}{n}$, and

$$I_\alpha f(x) := \int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}\,dy. \tag{1.1}$$

Since then, many authors have focused on the compactness of commutators with singular integrals and fractional integrals on variant function spaces; see, for example, [2–7, 11, 15–17, 23] and the references therein. It is known that the compactness

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of the commutator has extensive applications in many fields of mathematics, such as in the study of $\bar{\partial}$-Neumann problem on forms [21, Chapter 12, Section 8] or in the L^p-theory of quasiregular mappings in [11]; see also [1, 7, 18].

Recently, equivalent characterizations of two-weight norm inequalities for Riesz transform commutators and fractional integral commutators were established in [9] and [10], respectively. It is easy to see from [9, Theorem 1.2] and [10, Theorem 1.1] that a function b is in $\mathrm{BMO}(\mathbb{R}^n)$ if and only if $[b, R_i]$ is bounded on the weighted Lebesgue space $L^p_w(\mathbb{R}^n)$ for any $i \in \{1, 2, \cdots, n\}$, $p \in (1, \infty)$, and $w \in A_p(\mathbb{R}^n)$, and if and only if $[b, I_\alpha]$ is bounded from $L^p_{u^p}(\mathbb{R}^n)$ to $L^q_{u^q}(\mathbb{R}^n)$ for any $p,\, q \in (1, \infty)$ such that $\frac{1}{p} = \frac{1}{q} + \frac{\alpha}{n}$ and $u \in A_{p,\,q}(\mathbb{R}^n)$, where $A_p(\mathbb{R}^n)$ and $A_{p,\,q}(\mathbb{R}^n)$ were introduced by Muckenhoupt and Muckenhoupt–Wheeden [19] (see Definition 1.1 below). Moreover,

$$
\begin{aligned}
(1.2)\quad \sum_{i=1}^{n} \|[b, R_i] : L^p_w(\mathbb{R}^n) \to L^p_w(\mathbb{R}^n) \quad \| &\sim \quad \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \\
&\sim \quad \|[b, I_\alpha] : L^p_{u^p}(\mathbb{R}^n) \to L^q_{u^q}(\mathbb{R}^n)\|,
\end{aligned}
$$

where, and in what follows, by C we denote a positive constant that may change at each occurrence, and we write $f \lesssim g$ or $g \gtrsim f$ if $f \le Cg$, and $f \sim g$ if $f \lesssim g \lesssim f$. The purpose of this paper is to study the equivalent characterizations of compactness of commutators $[b, R_i]$ and $[b, I_\alpha]$ on weighted Lebesgue spaces. To this end, we first recall some necessary notions and notation.

Definition 1.1. Let $p, q \in (1, \infty)$. A non-negative function $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is called a Muchenhoupt A_p weight (or $w \in A_p(\mathbb{R}^n)$) if

$$
[w]_{A_p} := \sup_Q \langle w \rangle_Q \langle w^{1-p'} \rangle_Q^{p-1} < \infty,
$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$, $w(Q) := \int_Q w(y)\, dy$, and $\langle w \rangle_Q := \frac{1}{|Q|} w(Q)$. w is called an $A_{p,\,q}$ weight(or $w \in A_{p,\,q}(\mathbb{R}^n)$) if

$$
[w]_{A_{p,\,q}} := \sup_Q \langle w^q \rangle_Q \langle w^{-p'} \rangle_Q^{q/p'} < \infty.
$$

The class $A_{p,\,q}(\mathbb{R}^n)$ was first introduced by Muckenhoupt–Wheeden in [19] to study the weighted norm inequalities of fractional integral I_α. It is known that if $w \in A_{p,\,q}(\mathbb{R}^n)$, then $w^p \in A_p(\mathbb{R}^n)$, $w^q \in A_q(\mathbb{R}^n)$, and $w^{-p'} \in A_{p'}(\mathbb{R}^n)$, where $\frac{1}{p} + \frac{1}{p'} = 1$; see [10, 19].

Our main results of this paper are stated as follows:

Theorem 1.2. *Let $i \in \{1, 2, \cdots, n\}$, $p \in (1, \infty)$, $w \in A_p(\mathbb{R}^n)$, and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then $b \in \mathrm{CMO}(\mathbb{R}^n)$ if and only if the Riesz transform commutator $[b, R_i]$ is compact on $L^p_w(\mathbb{R}^n)$.*

Theorem 1.3. *Let $\alpha \in (0, n)$, $p,\, q \in (1, \infty)$ with $\frac{1}{p} = \frac{1}{q} + \frac{\alpha}{n}$, $w \in A_{p,\,q}(\mathbb{R}^n)$, and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then $b \in \mathrm{CMO}(\mathbb{R}^n)$ if and only if the commutator $[b, I_\alpha]$ is compact from $L^p_{w^p}(\mathbb{R}^n)$ to $L^q_{w^q}(\mathbb{R}^n)$.*

We present the proof of Theorem 1.2 in Section 2 and the proof of Theorem 1.3 in Section 3. We point out that the basic properties of $A_p(\mathbb{R}^n)$ and $A_{p,\,q}(\mathbb{R}^n)$ play important roles in the proofs of Theorems 1.2 and 1.3, respectively. Besides, for a given cube Q, a sequence of triadic cubes related to Q is constructed in Lemmas 2.3 and 3.2, which is useful in the proofs of Theorems 1.2 and 1.3.

Throughout the paper, we denote by C, c, and $\widetilde{C}$ positive constants which are independent of the main parameters, but they may vary from line to line. Constants with subscripts, such as C_1 and $\widetilde{C}_1$, do not change in different occurrences. For a given cube $Q := Q(x_Q, r_Q)$, x_Q denotes its center and r_Q its side-length. For any $t \in (0, \infty)$, $y \in \mathbb{R}^n$, and cube $Q := Q(x, r)$ with $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, $tQ := Q(x, tr)$ and $Q + \{y\} := \{x + y : \ x \in Q\}$.

2. The proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2. To begin with, we recall that the kernel $R_i(y, z)$ of the Riesz transform R_i for each $i \in \{1, 2, \cdots, n\}$ is a standard Calderón-Zygmund kernel, satisfying that there exists a positive constant C such that

i) for any $y, z \in \mathbb{R}^n$ with $y \neq z$,

$$|R_i(y, z)| \le C \frac{1}{|y - z|^n}; \tag{2.1}$$

ii) for any y, y_0, $z \in \mathbb{R}^n$ with $|y_0 - z| \le |y_0 - y|/2$,

$$|R_i(y, y_0) - R_i(y, z)| + |R_i(y_0, y) - R_i(z, y)| \le C \frac{|y_0 - z|}{|y_0 - y|^{n+1}}. \tag{2.2}$$

We now recall the following compactness of $[b, T]$ for a general Calderón-Zygmund operator T in [7], which implies the necessity of Theorem 1.2 immediately.

Theorem 2.1. *Let $w \in A_p(\mathbb{R}^n)$ with $p \in (1, \infty)$, let $b \in \mathrm{CMO}(\mathbb{R}^n)$, and let T be a Calderón-Zygmund singular integral operator. Then the commutator $[b, T]$ is compact on $L^p_w(\mathbb{R}^n)$.*

For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and cube $Q \subset \mathbb{R}^n$, let

$$M(f, Q) := \frac{1}{|Q|} \int_Q |f(y) - \langle f \rangle_Q| \, dy.$$

Next we come to an equivalent characterization of CMO($\mathbb{R}^n$) in [22].

Lemma 2.2. *Let $f \in \mathrm{BMO}(\mathbb{R}^n)$. Then $f \in \mathrm{CMO}(\mathbb{R}^n)$ if and only if f satisfies the following three conditions:*

(i)

$$\lim_{a \to 0^+} \sup_{|Q| = a} M(f, Q) = 0,$$

(ii)

$$\lim_{a \to \infty} \sup_{|Q| = a} M(f, Q) = 0,$$

(iii) *for each cube Q,*

$$\lim_{x \to \infty} M(f, Q + \{x\}) = 0.$$

Before giving the proof of Theorem 1.2, we first establish a lemma for the upper and lower bounds of integrals of $[b, R_i] f_j$ on certain cubes. To this end, we recall the median value in [12–14, 20]. For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and cube $Q \subset \mathbb{R}^n$, let $\alpha_Q(f)$ be a real number such that

$$\inf_c \frac{1}{|Q|} \int_Q |f(x) - c| \, dx$$

is attained. Moreover, it is known that $\alpha_Q(f)$ satisfies that

$$|\{x \in Q : f(x) > \alpha_Q(f)\}| \le |Q|/2 \tag{2.3}$$

and

$$|\{x \in Q : f(x) < \alpha_Q(f)\}| \le |Q|/2; \tag{2.4}$$

see [14, p. 30]. By the choice of $\alpha_Q(f)$, it is easy to see that for any cube $Q \subset \mathbb{R}^n$,

$$M(f,Q) \sim \frac{1}{|Q|}\int_Q |f(y) - \alpha_Q(f)|\, dy. \tag{2.5}$$

Before we present the proof of Theorem 1.2, we first establish the following technical lemma on the construction of sequences $\{f_j\}_j$ of functions uniformly bounded in $L^p_w(\mathbb{R}^n)$ for given sequences of cubes $\{Q_j\}_j$, which adapts Uchiyama's idea in [22] to our weighted cases. More precisely, for each j, the function $[b, R_i]f_j$ has certain lower bounds on a sequence of triadic cubes $\{Q_j^k\}_k$ (see Lemma 2.3 below for the definition) constructed by Q_j, and upper bounds on $\{3^{k+1}Q_j \setminus 3^k Q_j\}_k$. We remark that the basic properties of $A_p(\mathbb{R}^n)$ play an important role in the proof of Lemma 2.3. Besides, the geometric properties of $\{Q_j^k\}_k$ turn out to be quite useful in the proof of Lemma 2.3.

Lemma 2.3. *Let $i \in \{1, 2, \cdots, n\}$, $w \in A_p(\mathbb{R}^n)$ for $p \in (1,\infty)$, and $b \in \mathrm{BMO}(\mathbb{R}^n)$ satisfying $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} = 1$. Assume that there exist $\delta \in (0,\infty)$ and a sequence $\{Q_j\}_j := \{Q(x^j, r_j)\}_j$ of cubes such that for each $j \in \mathbb{N}$,*

$$M(b, Q_j) > \delta. \tag{2.6}$$

Then there exist functions $\{f_j\}_j \subset L^p_w(\mathbb{R}^n)$, positive constants $K_0 \in \mathbb{N}$ large enough, $\widetilde{C}_0$, $\widetilde{C}_1$, and $\widetilde{C}_2$ such that for any integers $j \in \mathbb{N}$ and $k \ge K_0$, $\|f_j\|_{L^p_w(\mathbb{R}^n)} \le \widetilde{C}_0$,

$$\int_{Q_j^k} |[b, R_i]\, f_j(y)|^p\, w(y)\, dy \ge \widetilde{C}_1 \delta^p \frac{w(3^k Q_j)}{3^{knp} w(Q_j)}, \tag{2.7}$$

where $\vec{e_i} = (0, \cdots, 0, 1, 0, \cdots, 0)$ is the i-th unit vector and $Q_j^k := 3^{k-1}Q_j + 3^{k-1}r_j\{\vec{e_i}\}$, and

$$\int_{3^{k+1}Q_j \setminus 3^k Q_j} |[b, R_i]\, f_j(y)|^p\, w(y)\, dy \le \widetilde{C}_2 \frac{w(3^k Q_j)}{3^{knp} w(Q_j)}. \tag{2.8}$$

Proof. For each j, define the function f_j as follows:

$$f_j^1 := \chi_{Q_{j,1}} - \chi_{Q_{j,2}} := \chi_{\{x \in Q_j : b(x) > \alpha_{Q_j}(b)\}} - \chi_{\{x \in Q_j : b(x) < \alpha_{Q_j}(b)\}},\ f_j^2 := a_j \chi_{Q_j} \tag{2.9}$$

and

$$f_j := [w(Q_j)]^{-1/p} \left(f_j^1 - f_j^2\right),$$

where a_j is a constant such that

$$\int_{\mathbb{R}^n} f_j(x)\, dx = 0. \tag{2.10}$$

Then by the definition of a_j, (2.3), and (2.4), we see that $|a_j| \le 1/2$. Moreover, we also have that $\operatorname{supp}(f_j) \subset Q_j$ and that for any $y \in Q_j$,

$$f_j(y)\left[b(y) - \alpha_{Q_j}(b)\right] \ge 0. \tag{2.11}$$

On the other hand, since $|a_j| \le 1/2$, we see that for any $y \in (Q_{j,1} \cup Q_{j,2})$,

$$|f_j(y)| \sim [w(Q_j)]^{-1/p}. \tag{2.12}$$

Moreover, we have that $\|f_j\|_{L^p_w(\mathbb{R}^n)} \lesssim 1$.

Observe that

$$[b, R_i]f = R_i\left([b - \alpha_{Q_j}(b)]f\right) - \left[b - \alpha_{Q_j}(b)\right] R_i(f). \tag{2.13}$$

Then for any integer $k \ge K_0$, we have that

$$3^k Q_j \subset 5Q_j^k \subset 3^{k+2} Q_j. \tag{2.14}$$

Since $w \in A_p(\mathbb{R}^n)$, we see that for any cube $Q \subset \mathbb{R}^n$ and $t > 1$, $w(tQ) \lesssim t^{np} w(Q)$. From this and (2.14), we deduce that

$$w\left(Q_j^k\right) \sim w\left(3^k Q_j\right), \tag{2.15}$$

where the implicit constants depend on p, n but not on k, j.

Now we prove the inequality (2.7). By (2.12), (2.10), and (2.2), we see that for any $y \in \mathbb{R}^n \setminus 3Q_j$,

$$|R_i(f_j)(y)| = \left| \int_{Q_j} \left[R_i(y,z) - R_i(y,x^j)\right] f_j(z)\,dz \right| \lesssim \frac{r_j [w(Q_j)]^{-1/p} |Q_j|}{|x^j - y|^{n+1}}. \tag{2.16}$$

Moreover, from the well known John-Nirenberg inequality and $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} = 1$, we deduce that for each $k \in \mathbb{N}$ and $Q \subset \mathbb{R}^n$,

$$\begin{aligned} &\int_{3^{k+1}Q} |b(y) - \alpha_Q(b)|^p\,dy \\ &\quad \lesssim \int_{3^{k+1}Q} \left|b(y) - \alpha_{3^{k+1}Q}(b)\right|^p dy + \left|3^{k+1}Q\right| \left|\alpha_{3^{k+1}Q}(b) - \alpha_Q(b)\right|^p \\ &\quad \lesssim k^p \left|3^k Q\right|. \end{aligned} \tag{2.17}$$

Since $w \in A_p(\mathbb{R}^n)$, there exists $\epsilon \in (0, \infty)$ such that the reverse Hölder inequality

$$\left[\frac{1}{|Q|}\int_Q w(x)^{1+\epsilon}\,dx\right]^{\frac{1}{1+\epsilon}} \lesssim \frac{1}{|Q|}\int_Q w(x)\,dx$$

holds for any cube $Q \subset \mathbb{R}^n$. By this fact, the Hölder inequality, the John-Nirenberg inequality, (2.17), and (2.16), we see that there exists a positive constant $\widetilde{C}_3$ such that for any $k \in \mathbb{N}$ with $k \ge 2$,

$$\begin{aligned} &\int_{Q_j^k} \left|\left[b(y) - \alpha_{Q_j}(b)\right] R_i(f_j)(y)\right|^p w(y)\,dy \\ &\quad \lesssim \frac{1}{3^{kp(n+1)} w(Q_j)} \int_{Q_j^k} \left|b(y) - \alpha_{Q_j}(b)\right|^p w(y)\,dy \\ &\quad \lesssim \frac{1}{3^{kp(n+1)}} \frac{|3^k Q_j|}{w(Q_j)} \left\{\frac{1}{|3^{k+1}Q_j|} \int_{3^{k+1}Q_j} \left|b(y) - \alpha_{Q_j}(b)\right|^{p(1+\epsilon)'} dy\right\}^{\frac{1}{(1+\epsilon)'}} \\ &\qquad \times \left\{\frac{1}{|3^{k+1}Q_j|} \int_{3^{k+1}Q_j} w(y)^{1+\epsilon}\,dy\right\}^{\frac{1}{1+\epsilon}} \\ &\quad \le \widetilde{C}_3 \frac{k^p}{3^{kp(n+1)}} \frac{w(3^k Q_j)}{w(Q_j)}. \end{aligned} \tag{2.18}$$

Next, observe that $y_i > z_i$ and $y_i - z_i \sim |y-z|$ for any $y \in Q_j^k$ and $z \in Q_j$. By (2.11), (2.12), (2.5), (2.6), and the fact that $b - \alpha_{Q_j}(b)=0$ on $Q_j \setminus (Q_{j,1} \cup Q_{j,2})$, we have that for $y \in Q_j^k$,

$$\begin{aligned} \left|R_i\left[(b-\alpha_{Q_j}(b))f_j\right](y)\right| &= \int_{Q_{j,1}\cup Q_{j,2}} |R_i(y,z)| \left|\left[b(z)-\alpha_{Q_j}(b)\right] f_j(z)\right| dz \\ &\sim [w(Q_j)]^{-1/p} \int_{Q_j} \frac{|b(z)-\alpha_{Q_j}(b)|}{|y-z|^n}\, dz \\ &\gtrsim \delta\, [w(Q_j)]^{-1/p} \frac{1}{3^{kn}}. \end{aligned}$$

From this and (2.15), we deduce that there exists a positive constant $\widetilde{C}_4$ depending on n, p but not on k, j, δ, such that

$$\int_{Q_j^k} \left|R_i\left[(b-\alpha_{Q_j}(b))f_j\right](y)\right|^p w(y)\, dy \geq \delta^p \widetilde{C}_4 \frac{w(3^k Q_j)}{w(Q_j)} \frac{1}{3^{pkn}}. \tag{2.19}$$

Take $K_0 \in \mathbb{N}$ large enough such that for any integer $k \geq K_0$,

$$\widetilde{C}_4 \frac{\delta^p}{2^{p-1}} - \widetilde{C}_3 \frac{k^p}{3^{kp}} \geq \widetilde{C}_4 \frac{\delta^p}{2^p}.$$

By (2.13), (2.19), and (2.18), we conclude that for any integer $k \geq K_0$,

$$\begin{aligned} &\int_{Q_j^k} |[b,R_i]f_j(y)|^p\, w(y)\, dy \\ &\quad\geq \left[\frac{1}{2^{p-1}} \int_{Q_j^k} \left|R_i\left[(b-\alpha_{Q_j}(b))f_j\right](y)\right|^p w(y)\, dy \right. \\ &\qquad \left. - \int_{Q_j^k} \left|\left[b(y)-\alpha_{Q_j}(b)\right] R_i(f_j)(y)\right|^p w(y)\, dy\right] \\ &\quad\geq \left(\widetilde{C}_4 \frac{\delta^p}{2^{p-1}} - \widetilde{C}_3 \frac{k^p}{3^{kp}}\right) \frac{w(3^k Q_j)}{w(Q_j)} \frac{1}{3^{pkn}} \\ &\quad\geq \widetilde{C}_4 \frac{\delta^p}{2^p} \frac{w(3^k Q_j)}{w(Q_j)} \frac{1}{3^{pkn}}. \end{aligned}$$

This shows the inequality (2.7).

Now we show the inequality (2.8). From $\operatorname{supp}(f_j) \subset Q_j$, (2.1), (2.5), and (2.12), we deduce that for any $y \in \mathbb{R}^n \setminus 3Q_j$,

$$\begin{aligned} \left|R_i\left[(b-\alpha_{Q_j}(b))f_j\right](y)\right| &\lesssim [w(Q_j)]^{-1/p} \int_{Q_j} \frac{|b(z)-\alpha_{Q_j}(b)|}{|y-z|^n}\, dz \\ &\lesssim [w(Q_j)]^{-1/p} \frac{|Q_j|}{|y-x^j|^n}, \end{aligned}$$

from which together with (2.18) (still holds with Q_j^k replaced by $3^{k+1}Q_j \setminus 3^kQ_j$), it follows that for any $k \geq K_0$,

$$\begin{aligned}
&\int_{3^{k+1}Q_j\setminus 3^kQ_j} |[b,R_i]f_j(y)|^p\, w(y)\, dy \\
&\quad\lesssim \int_{3^{k+1}Q_j\setminus 3^kQ_j} \left|R_i\left([b-\alpha_{Q_j}(b)]f_j\right)(y)\right|^p w(y)\, dy \\
&\quad\quad + \int_{3^{k+1}Q_j\setminus 3^kQ_j} \left|\left[b-\alpha_{Q_j}(b)\right] R_i(f_j)(y)\right|^p w(y)\, dy \\
&\quad\lesssim \frac{w(3^{k+1}Q_j)}{3^{kpn}w(Q_j)} + \frac{k^p}{3^{kp(n+1)}}\frac{w(3^kQ_j)}{w(Q_j)} \\
&\quad\lesssim \frac{w(3^kQ_j)}{3^{kpn}w(Q_j)}.
\end{aligned}$$

We then finish the proof of Lemma 2.3. $\square$

Proof of Theorem 1.2. As we mentioned before, since the necessity of Theorem 1.2 follows from Theorem 2.1 directly, we only need to show that if for $p \in (1,\infty)$ and $i \in \{1,2,\cdots,n\}$, $[b,R_i]$ is compact on $L^p_w(\mathbb{R}^n)$, then $b \in \mathrm{CMO}(\mathbb{R}^n)$. To this end, we employ the idea in [22] via a contradiction argument. The approach is as follows: we will show that if we assume that $[b,R_i]$ is compact on $L^p_w(\mathbb{R}^n)$ and $b \notin \mathrm{CMO}(\mathbb{R}^n)$, then b fails to satisfy at least one of (i)-(iii) in Lemma 2.2, and by Lemma 2.3, one can further construct sequences $\{f_j\}_j$ of functions uniformly bounded in $L^p_w(\mathbb{R}^n)$ such that $\{[b,R_i]f_j\}_j$ has no convergent subsequence, which contradicts the compactness assumption on $[b,R_i]$.

Without loss of generality, we assume that $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} = 1$. Observe that if $b \notin \mathrm{CMO}(\mathbb{R}^n)$, b does not satisfy at least one of (i)-(iii) in Lemma 2.2. We now consider the following three cases.

Case 1. b does not satisfy (i) in Lemma 2.2. In this case, there exist $\delta \in (0,\infty)$ and a sequence $\{Q_j\}_{j=1}^\infty$ of cubes satisfying (2.6) and that $|Q_j| \to 0$ as $j \to \infty$. Let f_j, $\widetilde{C}_2$, $\widetilde{C}_1$, K_0 be as in Lemma 2.3 and $C_1 := 3^{K_1} > C_2 := 3^{K_0}$ for some $K_1 \in \mathbb{N}$ large enough such that

$$C_3 := C_4\delta^p C_2^{(\sigma-p)n} > 2C_5C_1^{(p_0-p)n},$$

where $p_0 \in (1,p)$ such that $w \in A_{p_0}(\mathbb{R}^n)$, $\sigma \in (0,\infty)$ such that for any cube Q and measurable set $E \subset Q$,

$$\frac{w(E)}{w(Q)} \lesssim \frac{|E|^\sigma}{|Q|^\sigma}, \tag{2.20}$$

and C_4 and C_5 are positive constants depending only on $\widetilde{C}_1$, $\widetilde{C}_2$, p, n, p_0, and w. Since $|Q_j| \to 0$ as $j \to \infty$, we may choose a subsequence $\{Q_{j_\ell}^{(1)}\}$ of $\{Q_j\}$ such that

$$\frac{|Q_{j_{\ell+1}}^{(1)}|}{|Q_{j_\ell}^{(1)}|} < \frac{1}{C_1^n}. \tag{2.21}$$

For fixed ℓ, $m \in \mathbb{N}$, denote

$$\mathcal{J} := C_1Q_{j_\ell}^{(1)} \setminus C_2Q_{j_\ell}^{(1)},\ \mathcal{J}_1 := \mathcal{J} \setminus C_1Q_{j_{\ell+m}}^{(1)},\ \text{and } \mathcal{J}_2 := \mathbb{R}^n \setminus C_1Q_{j_{\ell+m}}^{(1)}.$$

Note that

$$\mathcal{J}_1 \subset C_1 Q_{j_\ell}^{(1)} \cap \mathcal{J}_2 \text{ and } \mathcal{J}_1 = \mathcal{J} \cap \mathcal{J}_2.$$

We then have

(2.22)

$$\begin{aligned}
&\| [b, R_i] (f_{j_\ell}) - [b, R_i] (f_{j_{\ell+m}})\|_{L^p_w(\mathbb{R}^n)} \\
&\quad \geq \left(\int_{\mathcal{J}_1} \left| [b, R_i] (f_{j_\ell})(y) - [b, R_i] (f_{j_{\ell+m}})(y) \right|^p w(y)\, dy \right)^{1/p} \\
&\quad \geq \left(\int_{\mathcal{J}_1} |[b, R_i] (f_{j_\ell})(y)|^p w(y)\, dy \right)^{1/p} - \left(\int_{\mathcal{J}_2} \left| [b, R_i] (f_{j_{\ell+m}})(y) \right|^p w(y)\, dy \right)^{1/p} \\
&\quad = \left(\int_{\mathcal{J} \cap J_2} |[b, R_i] (f_{j_\ell})(y)|^p w(y)\, dy \right)^{1/p} - \left(\int_{\mathcal{J}_2} \left| [b, R_i] (f_{j_{\ell+m}})(y) \right|^p w(y)\, dy \right)^{1/p} \\
&\quad =: \mathrm{F}_1 - \mathrm{F}_2.
\end{aligned}$$

We first consider the term F_1. Assume that $E_{j_\ell} := \mathcal{J} \setminus \mathcal{J}_2 \neq \emptyset$ first. Then $E_{j_\ell} \subset C_1 Q_{j_{\ell+m}}^{(1)}$. Hence, by (2.21), we have

$$|E_{j_\ell}| \leq \left| C_1 Q_{j_{\ell+m}}^{(1)} \right| = C_1^n \left| Q_{j_{\ell+m}}^{(1)} \right| < \left| Q_{j_\ell}^{(1)} \right|. \tag{2.23}$$

Now for $k \in \mathbb{N}$, let

$$Q_{j_\ell, k}^{(1)} := 3^{k-1} Q_{j_\ell}^{(1)} + 3^{k-1} r_{j_\ell}^{(1)} \{\vec{e_i}\},$$

where $r_{j_\ell}^{(1)}$ is the side-length of $Q_{j_\ell}^{(1)}$. Then we see that

$$\left| Q_{j_\ell, k}^{(1)} \right| = 3^{(k-1)n} \left| Q_{j_\ell}^{(1)} \right| > |E_{j_\ell}|.$$

From this fact it follows that there are at most two of $\{Q_{j_\ell, k}^{(1)}\}_{k=K_0}^{K_1-2}$ intersecting E_{j_ℓ}. This together with $C_1 = 3^{K_1} \gg C_2 = 3^{K_0}$, (2.20), (2.14), and (2.7), implies that

$$\begin{aligned}
\mathrm{F}_1^p &\geq \sum_{k=K_0,\, Q_{j_\ell, k}^{(1)} \cap E_{j_\ell} = \emptyset}^{K_1-2} \int_{Q_{j_\ell, k}^{(1)}} |[b, R_i] (f_{j_\ell})(y)|^p w(y)\, dy \\
&\gtrsim \delta^p \sum_{k=K_0,\, Q_{j_\ell, k}^{(1)} \cap E_{j_\ell} = \emptyset}^{K_1-2} \frac{w(3^k Q_{j_\ell}^{(1)})}{3^{knp} w(Q_{j_\ell}^{(1)})} \\
&\gtrsim \delta^p \sum_{k=K_0,\, Q_{j_\ell, k}^{(1)} \cap E_{j_\ell} = \emptyset}^{K_1-2} \frac{1}{3^{kn(p-\sigma)}} \\
&\geq C_4 \delta^p C_2^{(\sigma-p)n} = C_3.
\end{aligned}$$

If $E_{j_\ell} := \mathcal{J} \setminus \mathcal{J}_2 = \emptyset$, the inequality above still holds.

On the other hand, since $w \in A_{p_0}(\mathbb{R}^n)$, for any cube $Q \subset \mathbb{R}^n$ and $t > 1$, $w(tQ) \lesssim t^{np_0} w(Q)$. From this and (2.8), we deduce that

$$\begin{aligned}
\mathrm{F}_2^p &\le \sum_{k=K_1}^{\infty} \int_{3^{k+1}Q^{(1)}_{j_{\ell+m}} \setminus 3^k Q^{(1)}_{j_{\ell+m}}} \left|[b, R_i]\,(f_{j_{\ell+m}})(y)\right|^p w(y)\,dy \\
&\lesssim \sum_{k=K_1}^{\infty} \frac{w(3^k Q_j)}{3^{knp} w(Q_j)} \\
&\lesssim \sum_{k=K_1}^{\infty} \frac{1}{3^{k(p-p_0)n}} \\
&\le C_5 C_1^{(p_0-p)n} < C_3/2.
\end{aligned}$$

By these two inequalities and (2.22), we get

$$\left\|[b, R_i](f_{j_\ell}) - [b, R_i](f_{j_{\ell+m}})\right\|_{L^p_w(\mathbb{R}^n)} \gtrsim C_3^{1/p}.$$

Thus, $[b, R_i]$ is not compact on $L^p_w(\mathbb{R}^n)$. Therefore, b satisfies condition (i).

Case 2. b violates (ii) in Lemma 2.2. In this case, we also have that there exist $\delta \in (0, \infty)$ and a sequence $\{Q_j\}$ of cubes satisfying (2.6) and that $|Q_j| \to \infty$ as $j \to \infty$. We take a subsequence $\{Q^{(2)}_{j_\ell}\}$ of $\{Q_j\}$ such that

$$\frac{|Q^{(2)}_{j_\ell}|}{|Q^{(2)}_{j_{\ell+1}}|} < \frac{1}{C_1^n}. \tag{2.24}$$

We can use a similar method as in Case 1 and redefine our sets in a reversed order. That is, for fixed ℓ and m, let

$$\widetilde{\mathcal{J}} := C_1 Q^{(2)}_{j_{\ell+m}} \setminus C_2 Q^{(2)}_{j_{\ell+m}},\ \widetilde{\mathcal{J}_1} := \widetilde{\mathcal{J}} \setminus C_1 Q^{(2)}_{j_\ell}, \quad \text{and} \quad \widetilde{\mathcal{J}_2} := \mathbb{R}^n \setminus C_1 Q^{(2)}_{j_\ell}.$$

Then we have that

$$\widetilde{\mathcal{J}_1} \subset (C_1 Q^{(2)}_{j_{\ell+m}} \cap \widetilde{\mathcal{J}_2}) \quad \text{and} \quad \widetilde{\mathcal{J}_1} = \widetilde{\mathcal{J}} \cap \widetilde{\mathcal{J}_2}.$$

As in Case 1, by Lemma 2.3 and (2.24), we see that $[b, R_i]$ is not compact on $L^p_w(\mathbb{R}^n)$. This contradiction implies that b satisfies (ii) of Lemma 2.2.

Case 3. Condition (iii) in Lemma 2.2 does not hold for b. Then there exist $Q := Q(x_0, r_0) \subset \mathbb{R}^n$ and $\delta > 0$ such that for any $N > 1$ large enough, there exists $x_N \in \mathbb{R}^n$ such that $|x_N| > N$ and $M(b, Q + \{x_N\}) > \delta$. We claim that there exists a sequence $\{Q^{(3)}_j\}_j$ of cubes such that for any j,

$$M(b,\, Q^{(3)}_j) > \delta, \tag{2.25}$$

and for any $\ell \ne m$,

$$C_1 Q^{(3)}_\ell \cap C_1 Q^{(3)}_m = \emptyset. \tag{2.26}$$

In fact, for $N_1 > 1$ large enough, there exists a cube $Q^{(3)}_1 := Q + \{x_{N_1}\}$ such that (2.25) holds. Now assume that for $j \in \mathbb{N}$, $Q^{(3)}_m$, $m = 1, 2, \cdots, j$, are chosen to satisfy (2.25) and (2.26). Let $R_j > 0$ be large enough such that $\bigcup_{m=1}^{j} C_1 Q^{(3)}_m \subset R_j Q$. Take $N_j > \frac{\sqrt{n}}{2}(R_j + C_1) r_0$. According to our assumption in this case, there exists $x_{N_j} \in \mathbb{R}^n$ such that $|x_{N_j}| > N_j$ and $M(b, Q + \{x_{N_j}\}) > \delta$. Let $Q^{(3)}_{j+1} := Q + \{x_{N_j}\}$.

Then $C_1 Q_{j+1}^{(3)} \cap R_j Q = \emptyset$, and hence (2.26) holds. Repeating this procedure, we obtain $\{Q_j^{(3)}\}_j$ as desired.

Now we define

$$\widetilde{\widetilde{\widetilde{\mathcal{J}}}}_1 := C_1 Q_\ell^{(3)} \setminus C_2 Q_\ell^{(3)} \quad \text{and} \quad \widetilde{\widetilde{\widetilde{\mathcal{J}}}}_2 := \mathbb{R}^n \setminus C_1 Q_{\ell+m}^{(3)}.$$

Note that $\widetilde{\widetilde{\widetilde{\mathcal{J}}}}_1 \subset \widetilde{\widetilde{\widetilde{\mathcal{J}}}}_2$. Thus, similar to the estimates of F_1 and F_2 in Case 1, for any ℓ, m, we get

$$\begin{aligned}
&\| [b, R_i] (f_\ell) - [b, R_i] (f_{\ell+m}) \|_{L_w^p(\mathbb{R}^n)} \\
&\geq \left\{ \int_{\widetilde{\widetilde{\widetilde{\mathcal{J}}}}_1} |[b, R_i] (f_\ell)(y) - [b, R_i] (f_{\ell+m})(y)|^p \, w(y) dy \right\}^{1/p} \\
&\geq \left\{ \int_{\widetilde{\widetilde{\widetilde{\mathcal{J}}}}_1} |[b, R_i] (f_\ell)(y)|^p \, w(y) dy \right\}^{1/p} - \left\{ \int_{\widetilde{\widetilde{\widetilde{\mathcal{J}}}}_2} |[b, R_i] (f_{\ell+m})(y)|^p \, w(y) dy \right\}^{1/p} \\
&\gtrsim C_3^{1/p}.
\end{aligned}$$

This contradicts the compactness of $[b, R_i]$ on $L_w^p(\mathbb{R}^n)$, so b also satisfies condition (iii) in Lemma 2.2.

To sum up, we see that if $[b, R_i]$ is compact on $L_w^p(\mathbb{R}^n)$, then b satisfies (i)-(iii) of Lemma 2.2. This via Lemma 2.2 implies that $b \in \mathrm{CMO}(\mathbb{R}^n)$ and hence finishes the proof of Theorem 1.2. □

3. Compactness of fractional integral commutators

In this section, we study the compactness of $[b, I_\alpha]$. To this end, we first recall that a metric space (X, d) is totally bounded if for every $\delta > 0$, there exists a finite number of open balls of radius δ whose union is the space X, and a metric space (X, d) is compact if and only if it is complete and totally bounded; see, for example, [7]. Moreover, we also recall the following weighted Fréchet-Kolmogorov theorem obtained in [7].

Lemma 3.1. *For $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$, a subset $\mathcal{F}$ of $L_w^p(\mathbb{R}^n)$ is totally bounded (or relatively compact) if the following statements hold:*

(a) *$\mathcal{F}$ is uniformly bounded, i.e., $\sup_{f \in \mathcal{F}} \|f\|_{L_w^p(\mathbb{R}^n)} < \infty$.*

(b) *$\mathcal{F}$ uniformly vanishes at infinity; i.e., for every $\epsilon > 0$, there exists some positive constant N such that for every $f \in \mathcal{F}$,*

$$\int_{|x|>N}^{\infty} |f(x)|^p w(x) \, dx < \epsilon^p.$$

(c) *$\mathcal{F}$ is uniformly equicontinuous; i.e., for every $\epsilon > 0$, there exists some positive constant ρ such that for every $f \in \mathcal{F}$ and $y \in \mathbb{R}^n$ with $|y| < \rho$,*

$$\int_{\mathbb{R}^n} |f(x+y) - f(x)|^p w(x) \, dx < \epsilon^p.$$

The following lemma is an analogue of Lemma 2.3, which is suitable for I_α.

Lemma 3.2. *Assume that $w \in A_{p,\,q}(\mathbb{R}^n)$ with $p, q \in (1, \infty)$ such that $\frac{1}{p} = \frac{1}{q} + \frac{\alpha}{n}$, $b \in \mathrm{BMO}(\mathbb{R}^n)$ satisfies $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} = 1$, and there exist $\delta \in (0, \infty)$ and a sequence*

$\{Q_j\}_j := \{Q(x^j, r_j)\}_j$ *of cubes satisfying* (2.6). *Then there exist functions* $\{\widetilde{f}_j\}_j \subset L^p_{w^p}(\mathbb{R}^n)$, *positive constants* $k_0 \in \mathbb{N}$ *large enough,* $\widetilde{c}_0$, $\widetilde{c}_1$, *and* $\widetilde{c}_2$ *such that for any integers* $j \in \mathbb{N}$ *and* $k \geq k_0$, $\|\widetilde{f}_j\|_{L^p_{w^p}(\mathbb{R}^n)} \leq \widetilde{c}_0$,

$$\int_{Q_j^k} \left|[b, I_\alpha]\, \widetilde{f}_j(y)\right|^q w^q(y)\, dy \geq \widetilde{c}_1 \delta^q r_j^{q\alpha} \frac{w^q(3^k Q_j)}{3^{kq(n-\alpha)}[w^p(Q_j)]^{\frac{q}{p}}}, \tag{3.1}$$

and

$$\int_{3^{k+1}Q_j \setminus 3^k Q_j} \left|[b, I_\alpha]\, \widetilde{f}_j(y)\right|^q w^q(y)\, dy \leq \widetilde{c}_2 r_j^{q\alpha} \frac{w^q(3^k Q_j)}{3^{kq(n-\alpha)}[w^p(Q_j)]^{\frac{q}{p}}}. \tag{3.2}$$

Proof. The proof of Lemma 3.2 is similar to that of Lemma 2.3, and we only present the argument briefly. Firstly, we define $\widetilde{f}_j := [w^p(Q_j)]^{-\frac{1}{p}}(f_j^1 - f_j^2)$, where f_j^1 and f_j^2 are as in (2.9). Then $\widetilde{f}_j$ satisfies (2.10), (2.11), and $\|\widetilde{f}_j\|_{L^p_{w^p}(\mathbb{R}^n)} \lesssim 1$.

Now we recall that $w \in A_{p,\,q}(\mathbb{R}^n)$ implies that $w^q \in A_q(\mathbb{R}^n)$. Then by the fact that for any $y \in Q_j^k$,

$$|I_\alpha(\widetilde{f}_j)(y)| \lesssim \frac{r_j^{n+1}}{|x^j - y|^{n-\alpha+1}[w^p(Q_j)]^{\frac{1}{p}}},$$

and the Hölder inequality and reverse Hölder inequality, we see that

$$\int_{Q_j^k} \left|\left[b(y) - \alpha_{Q_j}(b)\right] I_\alpha(\widetilde{f}_j)(y)\right|^q w^q(y)\, dy \lesssim k^q r_j^{q\alpha} \frac{w^q(3^k Q_j)}{3^{kq(n-\alpha+1)}[w^p(Q_j)]^{\frac{q}{p}}}. \tag{3.3}$$

On the other hand, from (2.6), we deduce that

$$\left|I_\alpha\left[(b - \alpha_{Q_j}(b))\widetilde{f}_j\right](y)\right| \gtrsim \delta \frac{r_j^\alpha}{[w^p(Q_j)]^{\frac{1}{p}}} \frac{1}{3^{k(n-\alpha)}},$$

and hence

$$\int_{Q_j^k} \left|I_\alpha\left[(b - \alpha_{Q_j}(b))\widetilde{f}_j\right](y)\right|^q w^q(y)\, dy \gtrsim \delta^q r_j^{q\alpha} \frac{w^q(3^k Q_j)}{3^{kq(n-\alpha)}[w^p(Q_j)]^{\frac{q}{p}}}.$$

Taking $k \in \mathbb{N}$ large enough we see that (3.1) holds. Moreover, from (3.3) and the fact that for any $y \in 3^{k+1}Q_j \setminus 3^k Q_j$,

$$\left|I_\alpha\left[(b - \alpha_{Q_j}(b))\widetilde{f}_j\right](y)\right| \lesssim \frac{r_j^n}{|x^j - y|^{n-\alpha}[w^p(Q_j)]^{\frac{1}{p}}},$$

we have that (3.2) holds. □

Proof of Theorem 1.3. Sufficiency: Assume that $b \in \mathrm{BMO}(\mathbb{R}^n)$ with $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} = 1$ and $[b, I_\alpha]$ is compact from $L^p_{w^p}(\mathbb{R}^n)$ to $L^q_{w^q}(\mathbb{R}^n)$. As in the proof of Theorem 1.2, to show that $b \in \mathrm{CMO}(\mathbb{R}^n)$, we first assume that b does not satisfy (i) in Lemma 2.2. Then there exist $\delta \in (0, \infty)$ and a sequence $\{Q_j\}_{j=1}^\infty$ of cubes satisfying (2.6) and that $|Q_j| \to 0$ as $j \to \infty$. Since $w \in A_{p,\,q}(\mathbb{R}^n)$ implies that $w^p \in A_p(\mathbb{R}^n)$, let $\widetilde{f}_j$, $\widetilde{c}_1$, $\widetilde{c}_2$ be as in Lemma 3.2 and let $c_1 := 3^{k_1} > c_2 := 3^{k_0}$ for some $k_1 \in \mathbb{N}$ large enough such that

$$c_3 := c_4 \delta^q c_2^{\sigma n - q(n-\alpha)} > 2c_5 c_1^{q_0 n - q(n-\alpha)},$$

where $q_0 \in (1, (1 - \frac{\alpha}{n})q)$ such that $w^q \in A_{q_0}(\mathbb{R}^n)$ (see [10]), σ is as in (2.20), and c_4 is a positive constant depending only on $\widetilde{c}_1$, $\widetilde{c}_2$, p, n, α, q, q_0, and w. Since

$|Q_j|\to 0$ as $j\to\infty$, we may choose a subsequence $\{Q^{(1)}_{j_\ell}\}$ of $\{Q_j\}$ satisfying (2.21). For fixed $k,\ell,\,m\in\mathbb{N}$, let $Q^{(1)}_{j_\ell,\,k},\mathcal{J},\mathcal{J}_1,\mathcal{J}_2$ be as in the proof of Theorem 1.2. Then we have

$$\begin{aligned}
&\|\,[b,I_\alpha]\,(\widetilde{f}_{j_\ell})-[b,I_\alpha]\,(\widetilde{f}_{j_{\ell+m}})\|_{L^q_{w^q}(\mathbb{R}^n)}\\
&\quad\geq\left(\int_{\mathcal{J}\cap\mathcal{J}_2}\left|[b,I_\alpha]\,(\widetilde{f}_{j_\ell})(y)\right|^q w^q(y)\,dy\right)^{1/q}\\
&\qquad-\left(\int_{\mathcal{J}_2}\left|[b,I_\alpha]\,(\widetilde{f}_{j_{\ell+m}})(y)\right|^q w(y)\,dy\right)^{1/q}\\
&\quad=:\mathrm{G}_1-\mathrm{G}_2.
\end{aligned}$$

From the Hölder inequality and the fact that $\frac{1}{p}=\frac{1}{q}+\frac{\alpha}{n}$, it follows that

$$[w^p(Q_j)]^{\frac{q}{p}}\leq w^q(Q_j)r_j^{q\alpha},$$

which together with $w^q\in A_q(\mathbb{R}^n)$ and (3.1) further implies that

$$G_1^q\gtrsim\sum_{k=k_0,\,Q^{(1)}_{j_\ell,\,k}\cap\mathcal{J}\cap\mathcal{J}_2=\emptyset}^{k_1-2}\delta^q r_j^{q\alpha}\frac{1}{3^{kq(n-\alpha)}}\frac{3^{k\sigma n}w^q(Q^{(1)}_{j_\ell,\,k})}{[w^p(Q^{(1)}_{j_\ell,\,k})]^{\frac{q}{p}}}\geq c_3.$$

Moreover, by $w\in A_{p,\,q}(\mathbb{R}^n)$, the fact that $\frac{1}{p}=\frac{1}{q}+\frac{\alpha}{n}$, and the observation that

$$|Q_j|^q=\left[\int_{Q_j}w^{-1}(x)w(x)\,dx\right]^q\leq[w^{-p'}(Q_j)]^{\frac{q}{p'}}[w^p(Q)]^{\frac{q}{p}},$$

we see that

$$w^q(Q_j)\lesssim|Q_j|^{1+\frac{q}{p'}}[w^{-p'}(Q_j)]^{-\frac{q}{p'}}\lesssim r_j^{-q\alpha}[w^p(Q_j)]^{\frac{q}{p}}.$$

This implies that

$$G_2^q\lesssim\sum_{k=k_1}^{\infty}r_j^{q\alpha}\frac{1}{3^{kq(n-\alpha)}}\frac{3^{kq_0n}w^q(Q_j)}{[w^p(Q_j)]^{\frac{q}{p}}}<c_3/2.$$

By the estimates for G_1 and G_2, we conclude that

$$\|\,[b,I_\alpha]\,(\widetilde{f}_{j_\ell})-[b,I_\alpha]\,(\widetilde{f}_{j_{\ell+m}})\|_{L^q_{w^q}(\mathbb{R}^n)}\gtrsim c_3^{1/q}.$$

Thus, $\{[b,I_\alpha]\widetilde{f}_j\}_j$ is not relatively compact in $L^q_{w^q}(\mathbb{R}^n)$, which implies that $[b,I_\alpha]$ is not compact from $L^p_{w^p}(\mathbb{R}^n)$ to $L^q_{w^q}(\mathbb{R}^n)$. Therefore, b satisfies condition (i). The arguments for (ii) and (iii) are similar and omitted. Therefore, b satisfies (i)-(iii) of Lemma 2.2, which shows that $b\in\mathrm{CMO}(\mathbb{R}^n)$.

Necessity: By a standard argument, it suffices to show that for any $b\in\mathcal{D}$, $[b,I_\alpha]$ is compact from $L^p_{w^p}(\mathbb{R}^n)$ to $L^q_{w^q}(\mathbb{R}^n)$. Moreover, we use some idea in [16]; see also [5] and [7]. Take $\varphi\in\mathcal{D}$ supported in the ball $B(0,1)$ such that $\varphi(x)\equiv 1$ on $B(0,\frac{1}{2})$ and $0\leq\varphi(x)\leq 1$. For every $\eta>0$ small enough, let

$$I^\eta_\alpha(x,y):=I_\alpha(x,y)\left[1-\varphi\left(\frac{x-y}{\eta}\right)\right].$$

Then we have

(i) $I^\eta_\alpha(x,y)=I_\alpha(x,y)$ if $|x-y|\geq\eta$,
(ii) $0\leq I^\eta_\alpha(x,y)\leq\frac{1}{|x-y|^{n-\alpha}}$,

(iii) $I_\alpha^\eta(x,y)=0$ if $|x-y|<\frac{\eta}{2}$.

Let

$$[b,I_\alpha^\eta]f(x):=\int_{\mathbb{R}^n}[b(x)-b(y)]I_\alpha^\eta(x,y)f(y)\,dy.$$

Arguing as in [7, Lemma 7], we see that for any $\eta>0$,

$$|[b,I_\alpha]f(x)-[b,I_\alpha^\eta]f(x)|\lesssim\eta\|\nabla b\|_{L^\infty(\mathbb{R}^n)}M_\alpha f(x),$$

where $M_\alpha f(x)$ is the fractional maximal function defined by

$$M_\alpha f(x):=\sup_{Q\ni x}\frac{1}{|Q|^{1-\frac{\alpha}{n}}}\int_Q|f(y)|\,dy.$$

By the boundedness of M_α from $L^p_{w^p}(\mathbb{R}^n)$ to $L^q_{w^q}(\mathbb{R}^n)$ (see [19]), we see that

$$\lim_{\eta\to0}\|[b,I_\alpha]-[b,I_\alpha^\eta]\|_{L^p_{w^p}(\mathbb{R}^n)\to L^q_{w^q}(\mathbb{R}^n)}=0.$$

It suffices to show that for fixed $b\in\mathcal{D}$ and $\eta>0$ small enough, $[b,I_\alpha^\eta]$ is compact from $L^p_{w^p}(\mathbb{R}^n)$ to $L^q_{w^q}(\mathbb{R}^n)$. To this end, we only need to show that for every bounded subset $\mathcal{F}\subset L^p_{w^p}(\mathbb{R}^n)$, $[b,I_\alpha^\eta]\mathcal{F}$ is a relatively compact subset of $L^q_{w^q}(\mathbb{R}^n)$. Equivalently, we only need to show that $[b,I_\alpha^\eta]\mathcal{F}$ satisfies the conditions (a)—(c) in Lemma 3.1. Recall that I_α is bounded from $L^p_{w^p}(\mathbb{R}^n)$ to $L^q_{w^q}(\mathbb{R}^n)$ (see [19]). Since $b\in\mathcal{D}$, we have that for any $f\in\mathcal{F}$,

$$|[b,I_\alpha^\eta]f(x)|\le 2\|b\|_{L^\infty(\mathbb{R}^n)}I_\alpha(|f|)(x),$$

and hence $[b,I_\alpha^\eta]\mathcal{F}$ satisfies (a) in Lemma 3.1. Next, since $b\in\mathcal{D}$, without loss of generality, we assume that $\operatorname{supp}(b)\subset Q:=Q(x_Q,r_Q)$. By the Hölder inequality, there exists $N>>|x_Q|+r_Q+100$ such that for any $f\in\mathcal{F}$ and $x\in\mathbb{R}^n$ with $|x|>N$,

$$\begin{aligned}|[b,I_\alpha^\eta]f(x)| &= \left|\int_{\mathbb{R}^n}I_\alpha^\eta(x,y)b(y)f(y)\,dy\right|\\ &\lesssim \|b\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^p_{w^p}(\mathbb{R}^n)}[w^{-p'}(Q)]^{\frac{1}{p'}}\frac{1}{|x-x_Q|^{n-\alpha}}.\end{aligned}$$

This together with $w\in A_{p,q}(\mathbb{R}^n)$ (and hence $w^q\in A_{q_0}(\mathbb{R}^n)$ with $q_0\in(1,(1-\frac{\alpha}{n})q)$) and $\frac{1}{p}=\frac{1}{q}+\frac{\alpha}{n}$ implies that

$$\begin{aligned}&\int_{|x|>N}|[b,I_\alpha^\eta]f(x)|^q\,w^q(x)\,dx\\ &\lesssim\|b\|^q_{L^\infty(\mathbb{R}^n)}\|f\|^q_{L^p_{w^p}(\mathbb{R}^n)}\sum_{k=\lfloor\log_2\frac{N}{2}\rfloor}^{\infty}\frac{w^q(2^kQ)}{(2^kr_Q)^{q(n-\alpha)}}\langle w^q\rangle_Q^{-1}|Q|^{\frac{q}{p'}}\\ &\lesssim N^{nq_0-q(n-\alpha)},\end{aligned}$$

where $\lfloor a\rfloor$ for $a\in\mathbb{R}$ means the largest integer i no more than a. Thus, (b) in Lemma 3.1 holds for $[b,I_\alpha^\eta]\mathcal{F}$.

It remains to prove $[b, I_\alpha^\eta]\mathcal{F}$ also satisfies (c). Let $z \in \mathbb{R}^n$ with $|z| < \eta/8$ small enough. Then for any $x \in \mathbb{R}^n$,

$$\begin{aligned}
&[b, I_\alpha^\eta]f(x) - [b, I_\alpha^\eta]f(x+z)\\
&\quad = [b(x) - b(x+z)] \int_{\mathbb{R}^n} I_\alpha^\eta(x,y) f(y)\, dy\\
&\qquad + \int_{\mathbb{R}^n} [I_\alpha^\eta(x,y) - I_\alpha^\eta(x+z,y)][b(x+z) - b(y)] f(y)\, dy\\
&\quad =: \sum_{j=1}^{2} \mathrm{L}_i(x).
\end{aligned}$$

Observe that

$$|\mathrm{L}_1(x)| \leq |z| \|\nabla b\|_{L^\infty(\mathbb{R}^n)} \int_{|x-y|\geq \frac{\eta}{2}} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy \leq |z| \|\nabla b\|_{L^\infty(\mathbb{R}^n)} I_\alpha(|f|)(x).$$

To estimate $\mathrm{L}_2(x)$, we first see that if $|x-y| < \eta/4$, then $I_\alpha^\eta(x,y) = 0$ and $I_\alpha^\eta(x+z,y) = 0$. Moreover, when $|x-y| \geq \eta/4$, we have

$$|I_\alpha^\eta(x,y) - I_\alpha^\eta(x+z,y)| \lesssim \frac{|z|}{|x-y|^{n-\alpha+1}}.$$

Then we see that

$$\begin{aligned}
|\mathrm{L}_2(x)| \quad &\leq \int_{|x-y|\geq \frac{\eta}{4}} |I_\alpha^\eta(x+z,y) - I_\alpha^\eta(x,y)|\, |b(x+z) - b(y)| |f(y)|\, dy\\
&\lesssim |z| \|b\|_{L^\infty(\mathbb{R}^n)} \int_{|x-y|\geq \frac{\eta}{4}} \frac{1}{|x-y|^{n-\alpha+1}} |f(y)|\, dy\\
&\sim |z| \|b\|_{L^\infty(\mathbb{R}^n)} \sum_{k=0}^{\infty} \int_{\frac{\eta}{4}\cdot 2^k \leq |x-y| < \frac{\eta}{4}\cdot 2^{k+1}} \frac{|f(y)|}{|x-y|^{n-\alpha+1}}\, dy\\
&\lesssim \frac{|z|}{\eta} \|b\|_{L^\infty(\mathbb{R}^n)} M_\alpha f(x).
\end{aligned}$$

Recall that I_α and M_α are bounded from $L^p_{w^p}(\mathbb{R}^n)$ to $L^q_{w^q}(\mathbb{R}^n)$. Combining the estimates of $\mathrm{L}_1(x)$ and $\mathrm{L}_2(x)$ and letting $|z| \to 0$, we see that $[b, I_\alpha^\eta]\mathcal{F}$ satisfies the condition (c) in Lemma 3.1. Hence, $[b, I_\alpha^\eta]$ is a compact operator. This finishes the proof of Theorem 1.3." ☐

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